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# Finsler spaces with $(\mu, \beta)$-metric

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*Dedicated to Professor Dr. Lajos Tamássy on his 80th birthday*

**Abstract.** We consider in the present paper a class of Finsler metrics $L = L(\mu, \beta)$ obtained by a deformation of a (locally) Minkowski metric $\mu(x, y)$ by a linear one form $\beta(x, y)$. So we obtain a Randers–Minkowski metric. More general, we study the Finsler spaces with a fundamental metric $L = L(\mu, \beta)$, by analogy with the $(\alpha, \beta)$-metrics, emphasizing especially the variational problem, geodesics, canonical spray, connections, etc. These metrics are the generalization of Numata's Finsler metrics of scalar curvature.

## Introduction

The study of special Finsler spaces is a very important topic in Finsler geometry because it leads to new classes of Finsler spaces with remarkable geometrical properties. Among the most studied special Finsler spaces are the Finsler spaces with $(\alpha, \beta)$ metrics having the fundamental function

$$F(x, y) = \tilde{F}(\alpha(x, y), \beta(x, y)),$$

where $\alpha^2 = a_{ij}(x)dx^i dx^j$ is a Riemannian metric on $M$, and $\beta = b_i(x)dx^i$ is a linear 1-form. As particular cases we have the Randers, Kropina and

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Matsumoto classes of Finsler metrics given by ([1], [4]):

$$\alpha+\beta, \quad \frac{\alpha^2}{\beta}, \quad \frac{\alpha^2}{\alpha-\beta}.$$

These metrics are deformations of the Riemannian metric $a_{ij}(x)$ by the linear one form $\beta$. Recently, R. Miron and B. T. Hassan studied the variational problem of these spaces ([6]).

On the other hand, in 1978, S. Numata ([3]) gave an interesting example of a Finsler space of scalar curvature, namely

$$F(x,y)=\mu(x,y)+\beta(x,y),$$

where $\mu(x,y)$ is a locally Minkowski metric and $\beta(x,y)$ is a linear one form. In other words, there is a preferential system of coordinates $(x,y)$ on $M$ such that Numata's metric reads $F(x,y)=\mu(y)+\beta(x,y)$, where $\mu=\mu(y)$ and $\beta(x,y)=b_i(x)y^i$. This metric is of scalar curvature if and only if $\beta(x,y)$ is the gradient of a function, i.e. $\frac{\partial b_i(x)}{\partial x^j}-\frac{\partial b_j(x)}{\partial x^i}=0$.

Motivated by this example and by analogy with the Finsler spaces with $(\alpha,\beta)$ metrics we consider here the deformations of a locally Minkowski metric $\mu(x,y)$ by a linear one form $\beta$. We will call these metrics $(\mu,\beta)$-*metrics*. This idea is also based on [5] where a particular deformation of a general Finsler metric is considered.

If we consider the metric

$$\mu(x,y)=\eta_{ij}y^iy^j, \quad i,j\in\{1,2,\dots,n\},$$

i.e. the metric used in the theory of relativity of the space-time manifold, we obtain a special case being a good model for a theory of gravitation and electromagnetism. This is of course a particular case of Randers space. But in general, if the metric $\mu(x,y)$ is a Finsler locally Minkowski metric, for example the one in [3], then this space is not a Randers space anymore. Therefore the set of Finsler spaces with $(\alpha,\beta)$ metrics and the set of Finsler spaces with $(\mu,\beta)$ metrics are distinct, yet their intersection is nonvanishing.

This is a very important aspect to pay attention to this new class of Finsler spaces. Even though the formalism used in the study of $(\mu,\beta)$-metrics is quite similar to the one used for $(\alpha,\beta)$-metrics, the geometrical objects are completely different, and although some formulas seem familiar from $(\alpha,\beta)$-metrics, the proof is different. It is therefore very important to notice the similarities and the differences with respect to $(\alpha,\beta)$-metrics because these are not at all trivial.

In the present paper we are going to study the variational problem for Finsler spaces with $(\mu,\beta)$ metric considering the integral action of the regular Lagrangian

$$L(\mu(x,y),\beta(x,y)) = \tilde{F}^2(\mu(x,y),\beta(x,y)) = F^2(x,y),$$

and we are going to determine the canonical 1-spray of the space, the canonical nonlinear connection, etc.

## 1. Finsler spaces with $(\mu,\beta)$-metric

A Finsler space $F^n = (M, F(x,y))$ is called one with $(\mu,\beta)$ *metric* if its fundamental function $F(x,y)$ is written in the form

$$F(x,y) = \tilde{F}(\mu(x,y),\beta(x,y)), \tag{1.1}$$

where

$$\mu(x,y) = \{a_{ij}(x,y)y^i y^j\}^{\frac{1}{2}} \tag{1.2}$$

is a locally Minkowski Finsler metric, and

$$\beta(x,y) = b_i(x)y^i \tag{1.2'}$$

is a "1-form field".

In general, the function $\tilde{F}$ is defined on an open subset of the manifold $\widetilde{TM} = TM \setminus \{0\}$, where $(TM,\pi,M)$ is the tangent bundle of a smooth real $n$-dimensional manifold $M$, and $(x^i,y^i)$ are the local coordinates of the points $u=(x,y)\in TM$, induced on $TM$ by the local coordinates of the points $x=\pi(u)\in M$.

It is obvious that $\mu > 0$ on the open set mentioned of $\widetilde{TM}$, and at every point $u \in \widetilde{TM}$ there exists a local system of coordinates with the property

$$\mu(x,y) = \mu(y) \tag{1.3}$$

since $\mu(x,y)$ is a locally Minkowski Finsler metric.

The function $F(x,y)$ being positively 1-homogeneous with respect to $y^i$ we have

$$\tilde{F}(\mu(x,ty),\beta(x,ty)) = t\tilde{F}(\mu(x,y),\beta(x,y)), \quad \forall t \in R_+, \tag{1.4}$$

and from

$$\mu(x,ty) = t\mu(x,y), \quad \beta(x,ty) = t\beta(x,y), \quad t \in R_+,$$

follows

$$\tilde{F}(t\mu(x,y),t\beta(x,y)) = t\tilde{F}(\mu(x,y),\beta(x,y)), \quad \forall t \in R_+. \tag{1.4'}$$

Therefore the function $\tilde{F}(\mu,\beta)$ is positively 1-homogeneous in the arguments $(\mu,\beta)$. In other words, we have

$$\tilde{F}(t\mu,t\beta) = t\tilde{F}(\mu,\beta). \tag{1.4"}$$

This fact is very important because it allows us to use a formalism similar to the $(\alpha,\beta)$-metrics. However, the geometrical objects are different in the sense that the Riemannian metric $\alpha$ is replaced by a locally Minkowski one.

Denoting

$$L(\mu,\beta) = \tilde{F}^2(\mu,\beta) \tag{1.5}$$

and taking into account the 2-homogeneity of the Lagrangian $L(\mu,\beta)$, we have

$$\begin{gathered} \mu L_\mu + \beta L_\beta = 2L, \quad \mu L_{\mu\mu} + \beta L_{\mu\beta} = L_\mu, \\ \mu L_{\mu\beta} + \beta L_{\beta\beta} = L_\beta, \quad \mu^2 L_{\mu\mu} + 2\mu\beta L_{\mu\beta} + \beta^2 L_{\beta\beta} = 2L \end{gathered} \tag{1.6}$$

where we put

$$L_\mu = \frac{\partial L}{\partial \mu},\ L_\beta = \frac{\partial L}{\partial \beta},\ L_{\mu\mu} = \frac{\partial^2 L}{\partial\mu\partial\mu},\ L_{\mu\beta} = \frac{\partial^2 L}{\partial\mu\partial\beta},\ L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta\partial\beta}. \tag{1.6'}$$

From (1.2) we obtain:

$$a_{ij}(x,y) = \frac{1}{2}\frac{\partial^2\mu^2}{\partial y^i \partial y^j}, \quad p_i = \frac{1}{2}\frac{\partial \mu^2}{\partial y^i}, \tag{1.7}$$

and

$$\frac{\partial \mu}{\partial y^i} = \frac{1}{\mu} p_i = \frac{1}{\mu} a_{ij} y^j$$
$$\frac{\partial \beta}{\partial y^i} = b_i(x). \tag{1.7'}$$

The following lemma holds:

**Lemma 1.1.** *The covector fields $p_i(x, y)$ and $b_i(x)$ are linearly independent on the manifold $\widetilde{TM}$.*

Proof. Let us consider the relation

$$f(x, y) p_i(x, y) + g(x, y) b_i(x) = 0.$$

By contraction with $y^i$, one deduces

$$f(x, y) \mu^2(x, y) + g(x, y) \beta(x, y) = 0.$$

By differentiation with respect to $\beta$ we obtain $g = 0$. Then, from $f\mu^2 = 0$, $\mu > 0$ we obtain $f = 0$. It follows that $p_i$ and $b_i$ are linearly independent. □

Let us determine the fundamental tensor $g_{ij}(x, y)$ of a Finsler space with $(\mu, \beta)$-metric

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{1.8}$$

or

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(\mu(x, y), \beta(x, y))}{\partial y^i \partial y^j}. \tag{1.8'}$$

If we denote

$$y_i = g_{ij}(x, y) y^j = \frac{1}{2} \frac{\partial L(\mu, \beta)}{\partial y^i} = \frac{1}{2} \left( L_\mu \frac{\partial \mu}{\partial y^i} + L_\beta \frac{\partial \beta}{\partial y^i} \right), \tag{1.9}$$

then we obtain

$$y_i = \rho p_i + \rho_1 b_i, \tag{1.10}$$

where

$$\rho = \frac{1}{2\mu} L_\mu, \quad \rho_1 = \frac{1}{2} L_\beta. \tag{1.11}$$

*Remark.* The vector fields $p_i$ and $b_i$ being linearly independent, it follows that the $y_i$ are uniquely represented in the form (1.10).

By taking the derivative of (1.11) with respect to $y^i$ we obtain

$$\frac{\partial \rho}{\partial y^i} = \rho_{-2} p_i + \rho_{-1} b_i, \quad \frac{\partial \rho_1}{\partial y^i} = \rho_{-1} p_i + \rho_0 b_i, \tag{1.12}$$

where

$$\rho_0 = \frac{1}{2} L_{\beta\beta}, \quad \rho_{-1} = \frac{1}{2\mu} L_{\mu\beta}, \quad \rho_{-2} = \frac{1}{2\mu^2}\left(L_{\mu\mu} - \frac{1}{\mu} L_\mu\right). \tag{1.13}$$

The functions $\rho_1, \rho, \rho_0, \rho_{-1}, \rho_{-2}$ are called the invariants of the spaces $F^n$ with $(\mu, \beta)$-metric. The subscripts 1, 0, $-1$, $-2$ show the degree of homogeneity of these invariants.

Now we give an important result:

**Theorem 1.1.** *The fundamental tensor field of the Finsler space with $(\mu, \beta)$-metric is given by*

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}(b_i p_j + b_j p_i) + \rho_{-2} p_i p_j. \tag{1.14}$$

This result can be proved by direct calculation, or it can be noticed that it follows from a similar result in the general case of $\beta$-change metrics (see [9] for details).

*Examples.* 1. Let $\mu(y)$ be given by

$$\mu(y) = \{(y^1)^m + (y^2)^m + \cdots + (y^n)^m\}^{\frac{1}{m}}, \quad m \geq 3. \tag{1.15}$$

In the two-dimensional case, its fundamental tensor is ([10])

$$g_{ij} = \begin{pmatrix} \dfrac{(y^1)^{m-2}[(m-1)(y^2)^m + (y^1)^m]}{\mu^{2m-2}} & \dfrac{-2(y^1y^2)^{m-1}}{\mu^{2m-2}} \\[2ex] \dfrac{-2(y^1y^2)^{m-1}}{\mu^{2m-2}} & \dfrac{(y^2)^{m-2}[(m-1)(y^1)^m + (y^2)^m]}{\mu^{2m-2}} \end{pmatrix}.$$

We also have

$$\det(g_{ij}) = \frac{(m-1)(y^1y^2)^{m-1}}{\mu^{2m-4}}.$$

Using this function, P. L. ANTONELLI and H. SHIMADA [2] constructed the celebrated Ecological metric $\bar{F} := e^{\alpha_i x^i}\mu$ as the conformal change of the $m$-th root. This metric is well known for its applications in Biology and Ecology. The theory of $(\mu, \beta)$ spaces gives a new possibility. We can deform the Minkowski metric $\mu$ by a *1-form* $b$ and obtain a new metric $F^* = F^*(\mu, \beta)$. The applications of this kind of metrics in Biology and Physics will be considered in another paper.

2. Other concrete examples of Finsler spaces with $(\mu, \beta)$-metric can be constructed as follows:

$$\begin{aligned}
&L(\mu,\beta) = (\mu+\beta)^2, && \mu\text{-Randers type}\\
&L(\mu,\beta) = \left(\frac{\mu^2}{\beta}\right)^2,\ \beta \neq 0, && \mu\text{-Kropina type}\\
&L(\mu,\beta) = \left(\frac{\mu^2}{(\mu-\beta)}\right)^2,\ \beta \neq 0, && \mu\text{-Matsumoto type}\\
&L(\mu,\beta) = (\mu^2+\beta^2), && \mu\text{-“Riemannian type”}.
\end{aligned} \tag{1.16}$$

The indicatrix of a $\mu$- Randers type metric was used by P. L. ANTONELLI to constrain the exponentially independent population densities ([1]).

We remark that for the fundamental function $L(\mu, \beta) = (\mu^2 + \beta^2)$, the fundamental tensor is

$$g_{ij} = \frac{1}{2}\,\frac{\partial^2 \mu^2}{\partial y^i \partial y^j} + \frac{1}{2}\,\frac{\partial^2 \beta^2}{\partial y^i \partial y^j} = a_{ij}(y) + b_i(x) b_j(x).$$

Therefore this is not a Riemannian metric, like in the case of $(\alpha, \beta)$ metrics, but a deformation in the sense of BEIL [8] of the considered Minkowski metric.

It will be interesting to study the classes described in (1.16) using as $\mu$ the expression (1.15), or taking $\mu$ to be the fundamental function of a locally Minkowski Finsler space.

## 2. Variational problem

Let $F^n = (M, F(x,y))$ be a Finsler space with $(\mu,\beta)$-metric, and let us put $F(x,y) = \tilde{F}(\mu,\beta)$, $\tilde{F}^2(\mu,\beta) = L(\mu,\beta)$.

The function $L(\mu(x,y),\beta(x,y))$ being a differentiable Lagrangian on $\widetilde{TM}$, we can study the variational problem for the spaces with $(\mu,\beta)$-metric using the integral of action of this Lagrangian.

Let $c : t \in [0,1] \mapsto c(t) \in M$, $x^i = x^i(t)$, $t \in [0,1]$ be a parametrized smooth curve on the manifold $M$, having its image in a domain of a local chart of $M$. The integral of action of the regular Lagrangian

$$F^2(x,y) = L(\mu(x,y),\beta(x,y))$$

is given by the functional

$$I(c) = \int_0^1 F^2\left(x, \frac{dx}{dt}\right) dt. \tag{2.1}$$

The variational problem for $I(c)$ leads to the following Euler–Lagrange equations ([6]):

$$E_i(F^2) := \frac{\partial F^2}{\partial x^i} - \frac{d}{dt}\frac{\partial F^2}{\partial y^i} = 0,\ y^i = \frac{dx^i}{dt}, \tag{2.2}$$

and the energy of the Lagrangian $F^2(x,y)$ is given by

$$\mathcal{E}_{F^2} = y^i \frac{\partial F^2}{\partial y^i} - F^2 = F^2(x,y).$$

We have the following law of conservation of energy:

*Along the integral curves of Euler–Lagrange equations the energy $\mathcal{E}_{F^2}$ is constant, i.e. we have*

$$\frac{d\mathcal{E}_{F^2}}{dt} = \frac{dF^2}{dt} = 0$$

*on every solution curve $c$ of the differential equations* (2.2).

By applying the operator $E_i$ to $L(\mu(x,y),\beta(x,y))$ we obtain

**Proposition 2.1.** *The covector field $E_i(L)$ is expressed by*

$$E_i(L) = L_\mu E_i(\mu) + L_\beta E_i(\beta) - \left\{\frac{dL_\mu}{dt}\frac{\partial\mu}{\partial y^i} + \frac{dL_\beta}{dt}\frac{\partial\beta}{\partial y^i}\right\}. \tag{2.3}$$

Indeed, from

$$E_i(L(\mu,\beta)) = \frac{\partial L(\mu,\beta)}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L(\mu,\beta)}{\partial y^i}\right)$$

it follows

$$\begin{aligned}\frac{\partial L}{\partial x^i} &= L_\mu \frac{\partial \mu}{\partial x^i} + L_\beta \frac{\partial \beta}{\partial x^i}\\ \frac{\partial L}{\partial y^i} &= L_\mu \frac{\partial \mu}{\partial y^i} + L_\beta \frac{\partial \beta}{\partial y^i}\\ \frac{d}{dt}\frac{\partial L}{\partial y^i} &= \frac{dL_\mu}{dt}\frac{\partial L}{\partial y^i} + \frac{dL_\beta}{dt}\frac{\partial \beta}{\partial y^i} + L_\mu \frac{d}{dt}\frac{\partial \mu}{\partial y^i} + L_\beta \frac{d}{dt}\frac{\partial \beta}{\partial y^i}.\end{aligned} \tag{2.4}$$

Now it is easy to see that (2.3) holds.

The relation between $E_i(\mu)$ and $E_i(\mu^2)$ is given by

$$2\mu E_i(\mu) = E_i(\mu^2) + 2\frac{d\mu}{dt}\frac{\partial \mu}{\partial y^i}. \tag{2.5}$$

Using (2.3) and (2.5) we can prove

**Proposition 2.2.** *The Euler–Lagrange equations* (2.2) *are equivalent to the following system of differential equations:*

$$\begin{gathered}E_i(\mu^2) + 2\frac{\rho_1}{\rho}E_i(\beta) + 2\frac{d\mu}{dt}\frac{\partial\mu}{\partial y^i} = \frac{1}{\rho}\left[L_\mu \frac{d}{dt}\frac{\partial \mu}{\partial y^i} + L_\beta \frac{d}{dt}\frac{\partial \beta}{\partial y^i}\right],\\ y^i = \frac{dx^i}{dt}.\end{gathered} \tag{2.6}$$

For simplicity, let us consider now the curve $c$ parametrized by the arc length $s$ given by $ds^2 = \mu^2\big(x(t), \frac{dx}{dt}\big)$. In this case we have $\mu^2\big(x, \frac{dx}{ds}\big) = 1$, and

$$\frac{d\mu}{ds} = 0. \tag{2.7}$$

Now we can show

**Proposition 2.3.** *In the canonical parametrization along the solution curves $c$ of the Euler–Lagrange equations* (2.6) *we have:*

$$\frac{d\mu}{ds} = 0, \quad \frac{d\beta}{ds} = 0, \quad \frac{dL_\mu}{ds} = 0, \quad \frac{dL_\beta}{ds} = 0. \tag{2.8}$$

Proof. Indeed, the first relation is already given by (2.7). The law of conservation gives $\frac{dL}{ds} = 0 = L_\mu \frac{d\mu}{ds} + L_\beta \frac{d\beta}{ds} = L_\beta \frac{d\beta}{ds} = 0$ which implies $\frac{d\beta}{ds} = 0$. From here $\frac{dL_\mu}{ds} = L_{\mu\mu} \frac{d\mu}{ds} + L_{\mu\beta} \frac{d\beta}{ds} = 0$ and analogously $\frac{dL_\beta}{ds} = 0$. $\square$

Let

$$F_{ij}(x) = \frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j} \tag{2.9}$$

be the electromagnetic tensor field of the $(\mu, \beta)$-metric.

Taking into account the expression of $E_i(\beta)$, we obtain

$$E_i(\beta) = F_{ij}(x) \frac{dx^j}{dt}. \tag{2.10}$$

Using (2.6), (2.8) and (2.9) we have:

**Theorem 2.1.** *In the canonical parametrization the Euler–Lagrange equations of a Finsler space with $(\mu, \beta)$ metric are as follows:*

$$E_i(\mu^2) + 2\frac{\rho_1}{\rho} F_{ij} y^j = 0, \quad y^j = \frac{dx^j}{ds}. \tag{2.11}$$

Consider the function $\mu$ given in a preferential system of coordinates. So we have:

$$\mu^2(y) = a_{ij}(y) y^i y^j$$

and

$$E_i(\mu^2) = \frac{\partial \mu^2}{\partial x^i} - \frac{d}{ds} \frac{\partial \mu^2}{\partial y^i} = -\frac{d}{ds} \frac{\partial \mu^2}{\partial y^i} = -2a_{ij}(y) \frac{dy^j}{ds}.$$

It follows:

**Theorem 2.2.** *In the canonical parametrization the Euler–Lagrange equations of the Finsler spaces with $(\mu, \beta)$ metric are given by the following Lorentz equations:*

$$\frac{d^2 x^i}{ds^2} = \frac{\rho_1}{\rho} a^{ir}(y) F_{rj}(x) \frac{dx^j}{ds}. \tag{2.12}$$

In particular, we have

**Corollary 2.1.** *If the 1-form $\beta = b_i(x) dx^i$ is closed then the Euler–Lagrange equations of the Finsler space with $(\mu, \beta)$ metric are given by*

$$\frac{d^2 x^i}{dt^2} = 0. \tag{2.13}$$

*Remark.* If $\beta$ is closed, then the $(\mu,\beta)$-*space* is a Finsler space of scalar curvature (Numata's theorem [3]).

## 3. Canonical nonlinear connection and the metrical $N$-connection

Consider the Lorentz equations (2.12) writen in the form

$$\frac{d^2x^i}{ds^2} - \frac{\rho_1}{\rho}a^{ir}(y)F_{rj}(x)\frac{dx^j}{ds} = 0, \quad y^i = \frac{dx^i}{ds}. \tag{3.1}$$

These equations are the differential equations of a spray with the coefficients

$$G^i(x,y) = -\frac{1}{2}\frac{\rho_1}{\rho}a^{ir}(y)F_{rj}(x)y^j, \tag{3.2}$$

where the functions $G^i$ are 2-homogeneous with respect to $y^i$.

Based on the general theory, it determines a nonlinear connection with the coefficients

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{3.3}$$

The connection $N$ will be called the canonical (or Lorentz) nonlinear connection of the Finsler space with $(\mu,\beta)$-metric.

**Proposition 3.1.** *The canonical nonlinear connection $N$ of the Finsler space with $(\mu,\beta)$ metric has the coefficients:*

$$N^i_j = -\frac{1}{2}\left[\frac{\partial}{\partial y^j}\left(\frac{\rho_1}{\rho}\right)a^{ir}(y)y^s + \frac{\rho_1}{\rho}\left(\frac{\partial a^{ir}}{\partial y^j}y^s + a^{ir}\delta^s_j\right)\right]F_{rs}(x). \tag{3.4}$$

We recall that the functions $N^i_j$ are 1-homogeneous with respect to $y^i$.

**Corollary 3.1.** *If $F_{ij}(x) = 0$ then $N^i_j = 0$.*

The Berwald connection determined by $N$ has the coefficients

$$B^i_{jk} = \frac{\partial N^i_j}{\partial y^k} = -\frac{1}{2}\left[\frac{\partial^2\left(\frac{\rho_1}{\rho}\right)}{\partial y^j\partial y^k}a^{ir}(y)y^s + \frac{\partial\left(\frac{\rho_1}{\rho}\right)}{\partial y^j}\left(\frac{\partial a^{ir}(y)}{\partial y^k}y^s + a^{ir}(y)\delta^s_k\right)\right.$$

$$+\frac{\partial\left(\frac{\rho_1}{\rho}\right)}{\partial y^k}\left(\frac{\partial a^{ir}(y)}{\partial y^j}y^s + a^{ir}(y)\delta^s_j\right)$$
$$\left.+\frac{\rho_1}{\rho}\left(\frac{\partial^2 a^{ir}}{\partial y^j \partial y^k}y^s + \frac{\partial a^{ir}}{\partial y^j}\delta^s_k + \frac{\partial a^{ir}}{\partial y^k}\delta^s_j\right)\right] F_{rs}(x).$$

We recall that its torsion is $t^i_{jk} = B^i_{jk} - B^i_{kj} = 0$.

Now, the geometric theory of Finsler spaces with $(\mu, \beta)$-metrics can be constructed by analogy with the one of Finsler spaces with $(\alpha, \beta)$-metrics. However, one should pay attention to the fact that sometimes even the formulas involved are formally identical, in fact they are different only as geometrical objects.

By means of the nonlinear connection $N = (N^i_j)$ given in (3.4) one obtains the direct decomposition $TTM = HTM \oplus VTM$, where the horizontal subbundle $HTM$ and the vertical subbundle $VTM$ are spanned by the local adapted bases $\frac{\delta}{\delta x^i}$, and $\frac{\partial}{\partial y^i}$, respectively. Next, Matsumoto's axiom leads to a unique Cartan connection $D\Gamma(N) = (L^k_{ij}, C^k_{ij})$, with the coefficients given by the usual formulas (see [7]).

It should be interesting to calculate explicitely these coefficients for concrete examples of $(\mu, \beta)$-metrics in order to depict their geometric particularities. The metric $F = \mu + \beta$, where $\mu$ is the $m$-th root metric, is a good place to start with.

Taking into account that the canonical nonlinear connection is 1-homogeneous with respect to $y^i$, it follows that the 1-forms $\delta y^i$ are 1-homogeneous. Therefore we can consider the following *homogeneous lift* of the fundamental tensor of the Finsler spaces with $(\mu, \beta)$-metric:

$$\mathbb{G} = g_{ij}(x,y)dx^i \otimes dx^j + \left(\frac{\rho}{\rho_1}\right)^2 g_{ij}(x,y)\delta y^i \otimes \delta y^j. \tag{3.5}$$

Obviously we have $\rho_1 = \frac{1}{2}L_\beta \neq 0$.

The metric $\mathbb{G}$ is a Riemannian structure 0-homogeneous with respect to $y^i$, globally defined on $\widetilde{TM}$ if $N$ has this property, that depends only on the fundamental function $F(x,y)$ of the Finsler space with $(\mu, \beta)$-metric. The distributions $N$ and $V$ are orthogonal with respect to $\mathbb{G}$.

Consider the almost complex structure $\mathbb{F} : \mathfrak{X}(\widetilde{TM}) \to \mathfrak{X}(\widetilde{TM})$ defined in the adapted basis by

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\rho_1}{\rho}\frac{\partial}{\partial y^i}, \quad \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\rho}{\rho_1}\frac{\delta}{\delta x^i}, \quad (i = 1, \dots, n).$$

From the general theory it follows that $\mathbb{F}$ is an almost complex structure, globally defined on $(\widetilde{TM})$ if the canonical nonlinear connection $N$ has this property. One can easily see that $\mathbb{F}$ is a 1-homogeneous tensor field with respect to $y^i$, locally given by

$$\mathbb{F} = -\frac{\rho_1}{\rho}\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\rho}{\rho_1}\frac{\delta}{\delta x^i} \otimes \delta y^i.$$

For a Finsler space $F^n = (M, F(x, y))$ with $(\mu, \beta)$-metric the pair $(\mathbb{G}, \mathbb{F})$ is an almost hermitian structure depending only on the fundamental function $F(x, y)$. The almost symplectic structure associated to $(\mathbb{G}, \mathbb{F})$ is

$$\theta = \frac{\rho}{\rho_1} g_{ij} \delta y^i \wedge dx^j.$$

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