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A new proof on Lévy's random domain

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. In this note I give a new analytical proof of LÉVY's result [3], [6] on the distribution of the directed random domain $\int[\xi_1(s)d\xi_2 - \xi_2(s)d\xi_1]$ for complex AR (autoregression) processes. In LÉVY's book [6] this statement is given for Brownian motion processes. In statistical literature this result is known, but the proof of it is very sophisticated (see [2], [3], [5], [7], [9]).

1. In the statistical inference of stochastic processes with continuous time we have only a few examples for exact distributions of statistics. Even these results are connected with the Brownian motion (see, e.g., [6], [7]). Using NOVIKOV's method ([9], [8]) I gave a new proof for the estimator of the periodical component of a complex valued AR process [3]. This theorem was stated by KOLMOGOROV in the early sixties [5]. The so called damping parameter was discussed in [1]. The real valued AR process was investigated later [4], [2].

Let the complex valued process $\xi(t) = \xi_1(t) + i\xi_2(t)$ be the solution of the following differential (stochastic) equation

$$d\xi(t) = -\gamma\xi(t)dt + dw(t)\,, \tag{1}$$

where

$$\gamma = \lambda - i\omega, \quad \lambda, \omega > 0, \qquad w(t) = w_1(t) + iw_2(t)\,,$$

where $w_1(t),\ w_2(t)$ are independent Brownian motion processes (standard) $\mathbb{E}\,w_j(t) = 0$, $\mathbb{E}(w_j(t))^2 = t$ $(j = 1, 2)$. If $\xi(t)$ is the stationary solution of

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(1) then

$$\mathbb{E}\xi(t)\overline{\xi(t+\tau)} = \frac{1}{\lambda} e^{-\lambda\tau - i\omega\tau}, \quad \tau > 0\,.$$

Let the process $\theta(t)$ be defined by

$$\xi(t) = |\xi(t)| e^{i\theta(t)}.$$

We assume that $\eta(t)$ is another complex valued process with different (λ_2, ω_2) parameters, and with the same Brownian motion $w(t)$

$$d\eta(t) = -\gamma_2 \eta(t) dt + dw(t), \qquad \gamma_2 = \lambda_2 - i\omega_2, \quad \lambda_2, \omega_2 > 0\,. \tag{2}$$

Let $\mathbb{P}_{\lambda,\omega}$, $\mathbb{P}_{\lambda_2,\omega_2}$ be the measures generated by the processes $\xi(t)$ and $\eta(t)$, respectively. Then the Radon–Nikodym derivative on the interval $[0, t]$ has the following form

$$\begin{aligned} \frac{d\mathbb{P}_{\lambda,\omega}}{d\mathbb{P}_{\lambda_2,\omega_2}}(\xi) = \frac{\lambda}{\lambda_2} \exp\Big\{ -\frac{\lambda^2 + \omega^2 - (\lambda_2^2 + \omega_2^2)}{2} s_\xi^2(t) - \\ -\frac{\lambda - \lambda_2}{2} \left[|\xi(0)|^2 + |\xi(t)|^2 \right] + (\lambda - \lambda_2)t + (\omega - \omega_2) r_\xi(t) \Big\}\,, \end{aligned} \tag{3}$$

where we use the following functionals of the process $\xi(t)$

$$\begin{aligned} s_\xi^2(t) &= \int_0^t \left[\xi_1^2(s) + \xi_2^2(s) \right] ds = \int_0^t |\xi(s)|^2 ds, \\ r_\xi(t) &= \int_0^t [\xi_1 d\xi_2 - \xi_2 d\xi_1] = \int_0^t |\xi(s)|^2 d\theta(s)\,. \end{aligned} \tag{4}$$

Statement. *If $\xi(t)$ is the solution of (1) and it is stationary, $\mathbb{E}\xi(t) = 0$, $\mathbb{D}^2\xi_i(t) = \frac{1}{2\lambda}$ $(i = 1, 2)$, then, $(\Lambda = \sqrt{\lambda^2 + 2a})$,*

$$\begin{aligned} \mathbb{E}\exp\{-a s_\xi^2(t)\} = \\ = 4\lambda\Lambda e^{\lambda t} \left[(\lambda + \Lambda)^2 \exp(\Lambda t) - (\lambda - \Lambda)^2 \exp(-\Lambda t) \right]^{-1}\,. \end{aligned} \tag{5}$$

If $\xi(0) = x + iy$ and $\xi(t)$ is the solution of (1) then we have

$$\begin{aligned} \mathbb{E}\exp\{-a s_\xi^2(t)\} = \\ = \exp\left\{ \lambda t - (x^2 + y^2) a [\Lambda \coth \Lambda t + \lambda]^{-1} \right\} \left[\cosh \Lambda t + \frac{\lambda}{\Lambda} \sinh \Lambda t \right]^{-1}\,. \end{aligned} \tag{6}$$

The proof can be seen in [2], [7], [8], [9] (formulae (3.3.5), (3.6.5), (4.2.15), (4.3.9) in [2]). Our main result is the following

Theorem. *If $\xi(t)$ is the stationary solution of* (1) *then the statistic*

$$\left(r_\xi(t) - \omega \cdot s_\xi^2(t)\right)\left(s_\xi^2(t)\right)^{-1/2} = \zeta(t)$$

has a standard normal (0,1) distribution.

Proof. Let us consider the expectation

$$\begin{aligned} v(c) &= \mathbb{E}_{\lambda,\omega} \exp\left\{-(cr_\xi(t) - c\omega s_\xi^2(t))\right\} = \\ &= \mathbb{E} \exp\left\{-c\int_0^t |\xi(s)|^2 d\theta(s) + c\omega \int_0^t |\xi(s)|^2 ds\right\}, \end{aligned}$$

which can be rewritten by the help of the Radon–Nikodym transformation, assuming $\lambda_2 = \lambda,\ \omega_2 = \omega - c$, in the form

$$\begin{aligned} v(c) &= \mathbb{E} \exp\left\{-\frac{\omega^2 - (\omega - c)^2 - 2c\omega}{2} \int_0^t |\eta(s)|^2 ds\right\} = \\ &= \mathbb{E} \exp\left\{\frac{c^2}{2}\int_0^t |\eta(s)|^2 ds\right\}. \end{aligned}$$

Under the condition $s_\eta^2 = \text{const} = \sigma^2$ we get

$$v(c) = \exp\left\{\frac{c^2\sigma^2}{2}\right\},$$

and this is the moment generating function of a normally distributed random variable and gives that ζ has a normal distribution with (0,1) parameters, independently of $s_\eta^2(t)$. Then the unconditional distribution of the random variable ζ is also normal. This proves the theorem.

2. Now we consider the second order AR process with compex roots. We assume that $\xi(t)$ is the solution of the following stochastic differential equation, where $w(t)$ is a standard Brownian motion

$$d\xi'(t) + [A_1\xi'(t) + A_2\xi(t)]dt = dw(t). \tag{7}$$

We take the process $\eta(t)$ with the same brownian motion and different parameters, which will be defined later

$$d\eta'(t) + [a_1\xi'(t) + a_2\xi(t)]dt = dw(t). \tag{8}$$

The Radon–Nikodym derivative has the following form, in the stationary case, (see (4.5.6) or (4.7.45a) in [2])

$$(9)\quad \frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(x(t)) = \frac{A_1\sqrt{A_2}}{a_1\sqrt{a_2}} \exp\Big\{ -\frac{A_2^2-a_2^2}{2}\int_0^t x^2(s)ds + + (A_2-a_2)\int_0^t (x'(s))^2 ds - \frac{A_1^2-a_1^2}{2}\int_0^t (x'(s))^2 ds + + \frac{A_1-a_1}{2}t - \frac{A_1-a_1}{2}\left[(x'(t))^2+(x'(0))^2\right] - - \frac{A_2-a_2}{2}\left[x(t)x'(t)-x(0)x'(0)\right] + + \frac{A_1A_2-a_1a_2}{2}\left[(x(t))^2+(x(0))^2\right]\Big\}$$

If[1] A_1 is the only unkown parameter, then the sufficient statistic is

$$\int_0^t (x'(s))^2 ds,\quad (x'(t))^2+(x'(0))^2,\quad (x(t))^2+(x(0))^2\,.$$

For A_2 the sufficient statistic is

$$\int_0^t (x(s))^2 ds,\quad \int_0^t (x'(s))^2 ds,\quad x(0)x'(0)-x(t)x'(t),\quad (x(t))^2+(x(0))^2\,.$$

a) The moment generating function of the functional $\int_0^t (x'(s))^2 ds$ can be calculated in the following way. Let be $a_2 = A_2$ and $-2c = A_1^2 - a_1^2$, then in the usual way we get (using (7)–(9))

$$(10)\quad \mathbb{E}_{A_1,A_2}\exp\left\{-c\int_0^t (x'(s))^2 ds\right\} = \mathbb{E}\exp\left\{-c\int_0^t (\xi'(s))^2 ds\right\} = = \frac{A_1}{a_1}\mathbb{E}\exp\Big\{ -c\int_0^t (\eta'(s))^2 ds - \frac{A_1^2-a_1^2}{2}\int_0^t (\eta'(s))^2 ds + \frac{A_1-a_1}{2}t - -A_2\frac{A_1-a_1}{2}[\eta^2(t)+\eta^2(0)] - \frac{A_1-a_1}{2}\left[(\eta'(t))^2+(\eta'(0))^2\right]\Big\}.$$

As the integral functionals disappear in the exponent we get

$$\mathbb{E}\exp\left\{-c\int_0^t (\xi'(s))^2 ds\right\} = \frac{A_1}{a_1}\exp\left\{\frac{A_1-a_1}{2}t\right\}\cdot \cdot\,\mathbb{E}\exp\left\{-\frac{A_1-a_1}{2}\left[\eta'(0)^2 + A_2\eta(0)^2 + \eta'(t)^2 + A_2\eta(t)^2\right]\right\}.$$

[1] We assume that the roots of the characteristic equation $x^2+A_1x+A_2=0$ are complex. The "natural parameters" are: $\lambda = \frac{A_1}{2}$, $\omega = \sqrt{A_2 - \frac{A_1^2}{4}}$, $(x_{1,2} = -\lambda \pm i\omega)$.

The expectation can be calculated as the random variables $\eta'(0)^2 + A_2\eta(0)^2$ and $\eta'(t)^2 + A_2\eta(t)^2$ are Chi-square distributed (but correlated).

b) The moment generating function of the functional $\int_0^t \xi(s)^2 ds$ depends on the functional $\int_0^t (\xi(s)')^2 ds$, too. Let us put $a_1 = A_1$ and $-c = \frac{1}{2}(A_2 + a_2)(A_2 - a_2)$. Then we have

$$\begin{aligned}(11)\qquad &\mathbb{E}_{A_1,A_2} \exp\left\{-c\int_0^t x^2(s)ds - \frac{A_2 - a_2}{2}\int_0^t (x'(s))^2 ds\right\}\\ &= \mathbb{E}\exp\left\{-c\int_0^t (\xi(s))^2 ds - \frac{A_2 - a_2}{2}\int_0^t (\xi'(s))^2 ds\right\}\end{aligned}$$

and using (7)–(9)

$$\begin{aligned}(12)\qquad &\mathbb{E}\exp\left\{-c\int_0^t (\xi(s))^2 ds - \frac{A_2 - a_2}{2}\int_0^t (\xi'(s))^2 ds\right\} =\\ &= \sqrt{\frac{A_2}{a_2}}\mathbb{E}\exp\left\{-c\int_0^t (\eta(s))^2 ds - \frac{A_2 - a_2}{2}\int_0^t (\eta'(s))^2 ds - \right.\\ &\quad - \frac{A_2 - a_2}{2}\left[(A_2 + a_2)\int_0^t (\eta(s))^2 ds - \int_0^t (\eta'(s))^2 ds + \eta(t)\eta'(t) - \right.\\ &\quad \left.\left. - \eta(0)\eta'(0) + A_1\eta^2(t) + A_1\eta^2(0)\right]\right\} =\\ &= \sqrt{\frac{A_2}{a_2}}\mathbb{E}\exp\left\{-\frac{A_2 - a_2}{2}\left[\eta(t)\eta'(t) - \eta(0)\eta'(0) + A_1\left(\eta^2(t) + \eta^2(0)\right)\right]\right\}.\end{aligned}$$

In this expectation we have again a quadratic form of normally distributed random variables, which can be calculated. This form does not give that the estimator of ω is normal.

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