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Some classes of natural almost Hermitian structures on the tangent bundles

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To Professor L. Tamássy on the occasion of his 80th birthday

Abstract. We construct a family of almost Hermitian structures on the tangent bundle of a Riemannian manifold by using another parametrization for the natural lifts of the Riemannian metric (see [6], [7], [8], [13]). Then we study the conditions under which we can get some classes in the classification obtained in [1].

Introduction

Consider an n-dimensional Riemannian manifold (M, g) and denote by $\tau : TM \longrightarrow M$ its tangent bundle. In the differential geometry of the tangent bundle TM of (M, g) one uses several Riemannian and semi-Riemannian metrics, induced by the Riemannian metric g on M. Among them, we may quote the Sasaki metric and the complete lift of the metric g. On the other hand, the natural lifts of g to TM induce some new Riemannian and pseudo-Riemannian geometric structures with many nice geometric properties (see [3], [2]). By similar methods one can get from g

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some natural almost complex structures on TM. The study of the almost Hermitian structures induced by g on TM is an interesting problem in the differential geometry of TM.

In the present paper we study some classes of natural almost Hermitian structures (G, J), induced on TM by the Riemannian metric g. These classes are of diagonal type and are obtained from the well known classification of the almost Hermitian structures in sixteen classes (see [1]).

Now we shall recall shortly the conditions characterizing the sixteen classes of almost Hermitian manifolds, obtained in [1]. Introduce the fundamental 2-form ϕ of the considered almost Hermitian structure on TM, defined by

$$\phi(X, Y) = G(JX, Y),$$

where X, Y are vector fields on TM.

Denote by ∇ the Levi–Civita connection of the metric G and consider the tensor field F of type $(0, 3)$, defined by

$$F(X, Y, Z) = (\nabla_X \phi)(Y, Z) = G((\nabla_X J)Y, Z)$$

for all vector fields X, Y, Z on the almost Hermitian manifold considered. The tensor field F has the following invariance and skew symmetry properties:

$$F(X, JY, JZ) = F(X, Z, Y) = -F(X, Y, Z).$$

We shall consider also the codifferential $\delta\phi$ of the 2-form ϕ and we shall use the notation

$$\theta = -\frac{1}{2(n-1)}\delta\phi.$$

The sixteen classes of almost Hermitian manifolds, obtained in [1] with the corresponding conditions, are

1. *the Kählerian manifolds, or the manifolds in the class $\mathcal{K}$, characterized by the property*

$$F = 0,$$

2. *the nearly Kählerian manifolds, or the manifolds in the class $\mathcal{NK} =$*

$\mathcal{W}_1$, characterized by

$$F(X,Y,Z) + F(Y,X,Z) = 0,$$

3. *the manifolds in the class $\mathcal{W}_3 = \mathcal{SK} \cap \mathcal{H}$ characterized by*

$$F(X,Y,Z) - F(JX,JY,Z) = 0, \quad \delta\phi = 0,$$

4. *the manifolds in the class $\mathcal{W}_1 \oplus \mathcal{W}_3$ characterized by*

$$F(X,Y,Z) + F(Y,X,Z) - F(JX,JY,Z) - F(JY,JX,Z) = 0, \quad \delta\phi = 0,$$

5. *the manifolds in the class $\mathcal{W}_4$ characterized by*

$$\begin{aligned} F(X,Y,Z) = {} & G(X,Y)\theta(Z) - G(X,Z)\theta(Y) \\ & + \phi(X,Y)\theta(JZ) - \phi(X,Z)\theta(JY), \end{aligned}$$

6. *the Hermitian manifolds, or manifolds in the class $\mathcal{H} = \mathcal{W}_3 \oplus \mathcal{W}_4$, characterized by*

$$F(X,Y,Z) - F(JX,JY,Z) = 0,$$

7. *the manifolds in the class $\mathcal{W}_1 \oplus \mathcal{W}_4$ characterized by*

$$\begin{aligned} F(X,Y,Z) + F(Y,X,Z) = {} & 2G(X,Y)\theta(Z) - G(X,Z)\theta(Y) \\ & - G(Y,Z)\theta(X) - \phi(X,Z)\theta(JY) \\ & - \phi(Y,Z)\theta(JX), \end{aligned}$$

8. *the manifolds in the class $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ characterized by*

$$F(X,Y,Z) + F(Y,X,Z) - F(JX,JY,Z) - F(JY,JX,Z) = 0,$$

9. *the almost Kählerian manifolds, or those in the class $\mathcal{AK} = \mathcal{W}_2$, characterized by*

$$d\phi = 0 \Longleftrightarrow F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = 0,$$

10. *the quasi-Kählerian manifolds, or those in the class $\mathcal{QK} = \mathcal{W}_1 \oplus \mathcal{W}_2$, characterized by*

$$F(X,Y,Z) + F(JX,JY,Z) = 0,$$

11. *the manifolds in the class* $\mathcal{W}_2 \oplus \mathcal{W}_3$ *characterized by*

$$\sum_{\mathrm{cycl}(X,Y,Z)} [F(X,Y,Z) - F(JX,JY,Z)] = 0, \quad \delta\phi = 0$$

12. *the semi-Kähler manifolds, or those in the class* $\mathcal{SK} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$*, characterized by*

$$\delta\phi = 0,$$

13. *the manifolds in the class* $\mathcal{W}_2 \oplus \mathcal{W}_4$ *characterized by*

$$\sum_{\mathrm{cycl}(X,Y,Z)} [F(X,Y,Z) + 2\phi(X,Y)\theta(Z)] = 0,$$

14. *the manifolds in the class* $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ *characterized by*

$$\begin{aligned} F(X,Y,Z) + F(JX,JY,Z) &= 2G(X,Y)\theta(Z) - 2G(X,Z)\theta(Y) \\ &\quad + \phi(X,Y)\theta(JZ) - 2\phi(X,Z)\theta(JY), \end{aligned}$$

15. *the manifolds in the class* $\mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *characterized by*

$$\sum_{\mathrm{cycl}(X,Y,Z)} [F(X,Y,Z) - F(JX,JY,Z)] = 0,$$

16. *the almost Hermitian manifolds, or the manifolds in the class* $\mathcal{W}$*, for which there is no condition.*

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). We use computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. Some quite complicate computations have been made by using the Ricci package under Mathematica for doing tensor computations. The well known summation convention is used throughout this paper, the range of the indices h, i, j, k, l being always $\{1, \ldots, n\}$.

1. Natural almost complex structures of diagonal type on TM

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that there is a structure of $2n$-dimensional smooth manifold on TM, induced by the structure of smooth n-dimensional manifold of M. Every local chart $(U, \varphi) = (U, x^1, \dots, x^n)$ on M induces a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$ on TM, as follows. For a tangent vector $y \in \tau^{-1}(U) \subset TM$, the first n local coordinates $x^1, \dots, x^n$ are the local coordinates $x^1, \dots, x^n$ of its base point $x = \tau(y)$ in the local chart (U, φ) (in fact we made an abuse of notation, identifying x^i with $\tau^* x^i = x^i \circ \tau$, $i = 1, \dots, n$). The last n local coordinates $y^1, \dots, y^n$ of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis $\Big(\big(\frac{\partial}{\partial x^1}\big)_{\tau(y)}, \dots, \big(\frac{\partial}{\partial x^n}\big)_{\tau(y)} \Big)$, defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it. An M-tensor field of type (p, q) on TM is defined by sets of n^{p+q} components (functions depending on x^i and y^i), with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type (p, q) on the base manifold M (see [5] for further details); e.g. the components y^i, $i = 1, \dots, n$, corresponding to the last n local coordinates of a tangent vector y assigned to the induced local chart $(\tau^{-1}(U), \Phi)$ define an M-tensor field of type $(1, 0)$ on TM. A usual tensor field of type (p, q) on M may be thought of as an M-tensor field of type (p, q) on TM. If the considered tensor field on M is only covariant, the corresponding M-tensor field on TM may be identified with the induced (pullback by τ) tensor field on TM. Some useful M-tensor fields on TM may be obtained as follows. Let $u : [0, \infty) \longrightarrow \mathbb{R}$ be a smooth function and let $\|y\|^2 = g_{\tau(y)}(y, y)$ be the square of the norm of the tangent vector $y \in \tau^{-1}(U)$. If δ^i_j are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field I on M), then the components $u(\|y\|^2)\delta^i_j$ define an M-tensor field of type $(1, 1)$ on TM. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field g on M in the local chart (U, φ), then the components

$u(\|y\|^2)g_{ij}$ define a symmetric M-tensor field of type $(0,2)$ on TM. The components $g_{0i} = y^k g_{ki}$, as well as $u(\|y\|^2)g_{0i}$ define M-tensor fields of type $(0,1)$ on TM. Of course, all the components considered above are in the induced local chart $(\tau^{-1}(U), \Phi)$.

We shall use the horizontal distribution HTM, defined by the Levi Civita connection $\dot{\nabla}$ of g, in order to define some first order natural lifts to TM of the Riemannian metric g on M. Denote by $VTM = \operatorname{Ker}\tau_* \subset TTM$ the vertical distribution on TM. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM. \tag{1}$$

If $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots y^n)$ is a local chart on TM, induced by the local chart $(U, \varphi) = (U, x^1, \dots, x^n)$, the local vector fields $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}$ on $\tau^{-1}(U)$ define a local frame for VTM over $\tau^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}$ define a local frame for HTM over $\tau^{-1}(U)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k\Gamma^h_{ki}$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set of vector fields $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \dots \frac{\delta}{\delta x^n})$ defines a local frame on TM, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i}\right)^V, \quad \frac{\delta}{\delta x^i} = \left(\frac{\partial}{\partial x^i}\right)^H,$$

where X^V and X^H denote the vertical and horizontal lifts of the vector field X on M.

Lemma 1. *If $n > 1$ and u, v are smooth functions on TM such that*

$$ug_{ij} + vg_{0i}g_{0j} = 0, \ g_{0i} = y^h g_{hi}, \quad y \in \tau^{-1}(U)$$

on the domain of any induced local chart on TM, then $u = 0$, $v = 0$.

The proof is obtained easily by transvecting the given relation with g^{ij} and y^j (recall that the functions $g^{ij}(x)$ are the components of the inverse of the matrix $(g_{ij}(x))$, associated with g in the local chart (U, φ) on M; moreover, the components $g^{ij}(x)$ define a tensor field of type $(2,0)$ on M).

Remark. From a relation of the type

$$u\delta^i_j + vg_{0j}y^i = 0, \quad y \in \tau^{-1}(U),$$

we obtain in a similar way $u = v = 0$.

Let $C = y^i \frac{\partial}{\partial y^i}$ be the Liouville vector field on TM and consider the corresponding horizontal vector field $\widetilde{C} = y^i \frac{\delta}{\delta x^i}$ on TM obtained in a similar way.

Since we work in a fixed local chart (U, φ) on M and in the corresponding induced local chart $(\tau^{-1}(U), \Phi)$ on TM, we shall use the following simpler (but less clear) notations:

$$\frac{\partial}{\partial y^i} = \partial_i, \quad \frac{\delta}{\delta x^i} = \delta_i.$$

Denote by

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^iy^k, \quad y \in \tau^{-1}(U) \tag{2}$$

the energy density defined by g in the tangent vector y. We have $t \in [0, \infty)$ for all $y \in TM$. Consider the real valued smooth functions a_1, a_2, b_1, b_2 defined on $[0, \infty) \subset \mathbb{R}$ and define a diagonal natural 1st order almost complex structure on TM, by using these coefficients and the Riemannian metric g, just as the natural 1st order lifts of g to TM are obtained in [3]. Then the expression of J is given by

$$\begin{cases} JX^H_y = a_1(t)X^V_y + b_1(t)g_{\tau(y)}(y, X)C_y, \\ JX^V_y = -a_2(t)X^H_y - b_2(t)g_{\tau(y)}(y, X)\widetilde{C}_y. \end{cases} \tag{3}$$

The expression of J in adapted local frames is given by

$$J\delta_i = a_1(t)\partial_i + b_1(t)g_{0i}C,$$

$$J\partial_i = -a_2(t)\delta_i - b_2(t)g_{0i}\widetilde{C}.$$

Remark that one can consider the case of the general natural tensor fields J on TM. This case has been dealt with in [9], [10]. In this case we have another four coefficients a_3, b_3, a_4, b_4 and the computations involved in the study of the corresponding almost complex structure J on TM become really complicate. In fact, the tensor fields of this type define the most general natural lift of type $(1,1)$ of the metric g.

Proposition 2. *The operator J defines an almost complex structure on TM if and only if*

$$a_1 a_2 = 1, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1. \tag{4}$$

Proof. The relations are easily obtained from the property $J^2 = -I$ of J and Lemma 1. □

The so obtained almost complex structures defined by the tensor field J on TM are called *natural almost complex structures of diagonal type,* obtained from the Riemannian metric g by using the parameters a_1, a_2, b_1, b_2. We use the word diagonal for these almost complex structures, since the $(2n\times 2n)$-matrix associated with J has with respect to the adapted local frame $(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$ two $n\times n$-blocks on the second diagonal

$$J = \begin{pmatrix} 0 & -a_2\delta^i_j - b_2 y^i g_{0j} \\ a_1\delta^i_j + b_1 y^i g_{0j} & 0 \end{pmatrix}.$$

Remark. From the conditions (4) we have that the coefficients a_1, a_2, a_1+2tb_1, a_2+2tb_2 cannot vanish and have the same sign. We assume that $a_1 > 0$, $a_2 > 0$, $a_1 + 2tb_1 > 0$, $a_2 + 2tb_2 > 0$ for all $t \geq 0$.

2. Natural diagonal almost Hermitian structures on TM

Consider a natural Riemannian metric of diagonal type G on TM (see [3], see also [4], [8]), induced by g and given by

$$\begin{cases} G_y(X^H, Y^H) = c_1(t) g_{\tau(y)}(X,Y) + d_1(t) g_{\tau(y)}(y,X) g_{\tau(y)}(y,Y) \\ G_y(X^V, Y^V) = c_2(t) g_{\tau(y)}(X,Y) + d_2(t) g_{\tau(y)}(y,X) g_{\tau(y)}(y,Y) \\ G_y(X^H, Y^V) = G_y(Y^V, X^H) = G_y(X^V, Y^H) = G_y(Y^H, X^V) = 0. \end{cases} \tag{5}$$

The expression of G in local adapted frames is defined by the M-tensor fields

$$G_{ij}^{(1)} = G(\delta_i, \delta_j) = c_1 g_{ij} + d_1 g_{0i} g_{0j},$$

$$G_{ij}^{(2)} = G(\partial_i, \partial_j) = c_2 g_{ij} + d_2 g_{0i} g_{0j},$$

and the associated $(2n \times 2n)$-matrix with respect to the adapted local frame $(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$ has two $(n \times n)$-blocks on the first diagonal:

$$G = \begin{pmatrix} c_1 g_{ij} + b_1 g_{0i} g_{0j} & 0 \\ 0 & a_2 g_{ij} + b_2 g_{0i} g_{0j} \end{pmatrix} = \begin{pmatrix} G_{ij}^{(1)} & 0 \\ 0 & G_{ij}^{(2)} \end{pmatrix}.$$

The coefficients c_1, c_2, d_1, d_2 are smooth functions depending on the energy density $t \in [0, \infty)$. The conditions for G to be positive definite are given by

$$\begin{cases} c_1 > 0, & c_2 > 0, \\ c_1 + 2td_1 > 0, & c_2 + 2td_2 > 0. \end{cases} \tag{6}$$

These relations are obtained by evaluating the condition
$G(v^i \partial_i + w^i \delta_i, v^j \partial_j + w^j \delta_j) \geq 0$, where v^i, w^i; $i, j = 1, \dots, n$, are arbitrary real numbers.

Of course, we shall be interested in the conditions under which the metric G is almost Hermitian with respect to the almost complex structure J considered in the previous section, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields X, Y on TM.

Considering the coefficients of g_{ij} under the conditions

$$\begin{cases} G(J\delta_i, J\delta_j) = G(\delta_i, \delta_j), & G(J\partial_i, J\partial_j) = G(\partial_i, \partial_j), \\ G(J\partial_i, J\delta_j) = G(\partial_i, \delta_j), \end{cases} \tag{7}$$

we obtain the proportionalities

$$c_1 = \lambda a_1, \quad c_2 = \lambda a_2, \tag{8}$$

where $\lambda = \lambda(t)$ is a positive smooth function on $[0, \infty)$. (Recall the assumptions $a_1 > 0$, $a_2 > 0$.)

Similarly, considering the coefficients of $g_{0i}g_{0j}$ in the relations (7), we obtain the proportionalities

$$\begin{cases} c_1 + 2td_1 = (\lambda + 2t\mu)(a_1 + 2tb_1), \\ c_2 + 2td_2 = (\lambda + 2t\mu)(a_2 + 2tb_2), \end{cases} \tag{9}$$

where $\lambda + 2t\mu = \lambda(t) + 2t\mu(t)$ is a positive smooth function of $t \in [0, \infty)$. It was much more convenient to consider the proportionality factor in such a form in the expression of the general solution of the system obtained from the conditions (7). The conditions (6) are automatically fulfilled, due to the properties (4) of the coefficients a_1, a_2, b_1, b_2. Of course, we can easily obtain from (9) the explicit expressions of the coefficients d_1 and d_2:

$$\begin{cases} d_1 = \lambda b_1 + \mu(a_1 + 2tb_1), \\ d_2 = \lambda b_2 + \mu(a_2 + 2tb_2). \end{cases} \tag{10}$$

Hence we may state

Theorem 3. *Let J be a diagonal natural almost complex structure on TM defined by g and the parameters a_1, a_2, b_1, b_2. The family of diagonal natural, Riemannian metrics G on TM defined by g and the parameters c_1, c_2, d_1, d_2, such that (TM, G, J) is an almost Hermitian manifold, is given by* (5) *where*

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu,$$

where λ, μ are smooth functions depending on t and $\lambda > 0$, $\lambda + 2t\mu > 0$.

Remark. In the general case there are involved other coefficients a_3, b_3 for the expression of J and c_3, d_3 for the expression of G and the condition for G to be Hermitian with respect to J contains two other extra proportionality relations $c_3 = \lambda a_3$, $c_3 + 2td_3 = (\lambda + 2t\mu)(a_3 + 2tb_3)$, see [9], [10].

3. The fundamental formulae for the natural almost Hermitian structures on TM

Consider the tangent bundle of the n-dimensional Riemannian manifold (M, g), endowed with the diagonal natural almost Hermitian structure (G, J) defined in the first section. Denote by ∇ the Levi–Civita connection of G. Using the adapted local frame $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}) = (\partial_1, \dots, \partial_n, \delta_1, \dots, \delta_n)$, we get the following expressions for ∇:

$$\begin{cases} \nabla_{\partial_i}\partial_j = Q^h_{ij}\partial_h, & \nabla_{\delta_i}\partial_j = \Gamma^h_{ij}\partial_h + P^h_{ji}\delta_h, \\ \nabla_{\partial_i}\delta_j = P^h_{ij}\delta_h, & \nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h + S^h_{ij}\partial_h. \end{cases} \tag{11}$$

The components Q^h_{ij}, P^h_{ij}, S^h_{ij} define M-tensor fields on TM and they can be obtained from the formulas which give the expressions of the classical Christoffel symbols. They can be expressed as functions depending on the essential parameters a_1, b_1, λ, μ involved in the definition of the diagonal natural almost Hermitian structure (G, J) on TM:

$$Q^h_{ij} = \frac{1}{2\lambda}\left(a_1 g^{hl} + \frac{b_1\lambda - a_1\mu}{\lambda + 2t\mu} y^h y^l\right)\left(\partial_i G^{(2)}_{jl} + \partial_j G^{(2)}_{il} - \partial_l G^{(2)}_{ij}\right),$$

$$P^h_{ij} = \frac{1}{2\lambda}\left(a_2 g^{hl} + \frac{b_2\lambda - a_2\mu}{\lambda + 2t\mu} y^h y^l\right)\left(\partial_i G^{(1)}_{jl} + G^{(2)}_{ik} R^k_{0jl}\right),$$

$$S^h_{ij} = -\frac{1}{2\lambda}\left(a_1 g^{hl} + \frac{b_1\lambda - a_1\mu}{\lambda + 2t\mu} y^h y^l\right)\left(\partial_l G^{(1)}_{ij} + G^{(2)}_{lk} R^k_{0ij}\right),$$

where we have used the notation $R^k_{0ij} = y^l R^k_{lij}$. Of course, we have to do the necessary computations, obtaining some quite complicated expressions.

Next, by using the expressions in the local adapted frame of the fundamental 2-form ϕ

$$\begin{aligned} &\phi(\partial_i, \partial_j) = 0, \quad \phi(\delta_i, \delta_j) = 0, \\ &\phi(\delta_i, \partial_j) = -\phi(\partial_i, \delta_j) = \lambda g_{ij} + \mu g_{0i} g_{0j}, \end{aligned} \tag{12}$$

we get he expressions of the tensor field F, given by the following M-tensor fields:

$$
\begin{aligned}
FYYY_{ijk} &= F(\partial_i, \partial_j, \partial_k) = (\nabla_{\partial_i}\phi)(\partial_j, \partial_k) = 0,\\
FYXX_{ijk} &= (\nabla_{\partial_i}\phi)(\delta_j, \delta_k) = 0,\\
FXYX_{ijk} &= (\nabla_{\delta_i}\phi)(\partial_j, \delta_k) = -FXXY_{ikj} = 0,\\
FYYX_{ijk} &= (\nabla_{\partial_i}\phi)(\partial_j, \delta_k) = -FYXY_{ikj}\\
&= \frac{1}{2a_1(a_1+2b_1t)}(a_1^2\lambda' - a_1^2\mu + a_1b_1\lambda + 2a_1b_1\lambda't - 2a_1b_1\mu t\\
&\quad - a_1a_1'\lambda - 2a_1'b_1\lambda t)(g_{ik}g_{0j} - g_{ij}g_{0k})\\
&\quad + \frac{1}{2a_1^2(a_1+2b_1t)}(a_1^2b_1\lambda' - a_1^2b_1\mu + a_1b_1^2\lambda + 2a_1b_1^2\lambda't\\
&\quad - 2a_1b_1^2\mu t - a_1a_1'b_1\lambda - 2a_1'b_1^2\lambda t)g_{0i}g_{0j}g_{0k}\\
&\quad - \frac{\lambda}{2a_1^2}R_{iljk}y^l - \frac{b_1\lambda}{2a_1^2(a_1+2b_1t)}R_{ilkh}g_{0j}y^ly^h,\\
FXYY_{ijk} &= (\nabla_{\delta_i}\phi)(\partial_j, \partial_k)\\
&= \frac{1}{2a_1(a_1+2b_1t)}(a_1a_1'\lambda - a_1b_1\lambda + a_1^2\lambda' - a_1^2\mu + 2a_1'b_1\lambda t\\
&\quad + 2a_1b_1\lambda't - 2a_1b_1\mu t)(g_{ij}g_{0k} - g_{ik}g_{0j}) + \frac{\lambda}{2a_1^2}R_{hijk}y^h\\
&\quad + \frac{b_1\lambda}{2a_1^2(a_1+2b_1t)}(R_{hikl}g_{0j} - R_{hijl}g_{0k})y^hy^l,\\
FXXX_{ijk} &= (\nabla_{\delta_i}\phi)(\delta_j, \delta_k)\\
&= \frac{1}{2}(a_1a_1'\lambda - a_1b_1\lambda + a_1^2\lambda' - a_1^2\mu + 2a_1'b_1\lambda t + 2a_1b_1\lambda't\\
&\quad - 2a_1b_1\mu t)(g_{ik}g_{0j} - g_{ij}g_{0k}) - \frac{\lambda}{2}R_{hijk}y^h.
\end{aligned}
$$

Now, the local expression of the 1-form $\theta = -\frac{1}{2(n-1)}\delta\phi$ can be obtained easily:

$$
\begin{aligned}
\theta Y_k &= \theta(\partial_k) = 0,\\
\theta X_k &= \theta(\delta_k) = \frac{(-\lambda' + \mu)(a_1+2b_1t)}{2\lambda}g_{0k}.
\end{aligned}
$$

Similar expressions are obtained for the tensor field defined by $F(JX,JY,Z)$, where X, Y, Z are vector fields on TM. We get

$$
\begin{aligned}
FJYJYY_{ijk} &= F(J\partial_i, J\partial_j, \partial_k) = (\nabla_{J\partial_i}\phi)(J\partial_j, \partial_k) = 0,\\
FJYJXX_{ijk} &= (\nabla_{J\partial_i}\phi)(J\delta_j, \delta_k) = 0,\\
FJXJYX_{ijk} &= (\nabla_{J\delta_i}\phi)(J\partial_j, \delta_k) = -FJXJXY_{ikj} = 0,\\
FJYJYX_{ijk} &= (\nabla_{J\partial_i}\phi)(J\partial_j, \delta_k)\\
&= \frac{1}{2a_1(a_1+2b_1t)}(a_1^2\lambda' - a_1^2\mu - a_1b_1\lambda + 2a_1b_1\lambda' t - 2a_1b_1\mu t\\
&\quad + a_1a_1'\lambda + 2a_1'b_1\lambda t)(g_{ik}g_{0j} - g_{ij}g_{0k})\\
&\quad + \frac{1}{2a_1^2(a_1+2b_1t)}(a_1^2b_1\lambda' - a_1^2b_1\mu - a_1b_1^2\lambda + 2a_1b_1^2\lambda' t\\
&\quad - 2a_1b_1^2\mu t + a_1a_1'b_1\lambda + 2a_1'b_1^2\lambda t)g_{0i}g_{0j}g_{0k} - \frac{\lambda}{2a_1^2}R_{hijk}y^h\\
&\quad - \frac{b_1\lambda}{2a_1^2(a_1+2b_1t)}R_{hikl}g_{0j}y^hy^l,\\
FJXJYY_{ijk} &= (\nabla_{\delta_i}\phi)(\partial_j, \partial_k)\\
&= \frac{1}{2a_1(a_1+2b_1t)}(a_1a_1'\lambda - a_1b_1\lambda - a_1^2\lambda' + a_1^2\mu + 2a_1'b_1\lambda t\\
&\quad - 2a_1b_1\lambda' t + 2a_1b_1\mu t)(g_{ik}g_{0j} - g_{ij}g_{0k}) - \frac{\lambda}{2a_1^2}R_{hijk}y^h\\
&\quad + \frac{b_1\lambda}{2a_1^2(a_1+2b_1t)}(R_{hijl}g_{0k} - R_{hikl}g_{0j})y^hy^l,\\
FJXJXX_{ijk} &= (\nabla_{\delta_i}\phi)(\delta_j, \delta_k)\\
&= \frac{1}{2}(a_1a_1'\lambda - a_1b_1\lambda - a_1^2\lambda' + a_1^2\mu + 2a_1'b_1\lambda t - 2a_1b_1\lambda' t\\
&\quad + 2a_1b_1\mu t)(g_{ij}g_{0k} - g_{ik}g_{0j}) + \frac{\lambda}{2}R_{hijk}y^h.
\end{aligned}
$$

4. The classification

In this section we use the expressions obtained in Section 3 in order to find the classes of diagonal natural almost Hermitian structures defined on TM. Our task is to find the conditions under which the almost Hermitian structure (G, J) belongs to one of the sixteen classes quoted in the Introduction. The computations done in order to check various properties of the almost Hermitian structure (G, J) are quite long and hard. By using systematically the Ricci package for doing tensor computations we succeeded in getting our main results. We shall not present all the cases but only some hints. E.g., if we want to check the properties of the almost Hermitian structures (G, J) in the class $\mathcal{G}_2$, we have to consider the expression

$$\sum_{\mathrm{cycl}(i,j,k)} [FXXX_{ijk} - FJXJXX_{ijk}],$$

and other 3 expressions involving $FYYY$, $FYYX$, $FYXX$. In fact, one obtains that these expressions are automatically zero. If we want to check the properties of the almost Hermitian structures (G, J) in the class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, we have to consider the expression

$$FXXX_{ijk} + FJXJXX_{ijk} - 2\theta X_k G^{(1)}_{ij} + 2\theta X_j G^{(1)}_{ik}$$

and five other more complicated expressions involving the other components of F. These expressions too do automatically vanish. In the case of the semi-Kählerian manifolds (almost Hermitian structures in the class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$), we have to study the vanishing of the components θX_i, θY_i, obtaining $\mu = \lambda'$. Now we state the results:

Theorem 4. *The diagonal natural almost Hermitian structures* (G, J) *on* TM *are automatically in each of the classes* $\mathcal{W}$, $\mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \oplus \mathcal{W}_4$.

Theorem 5. *The diagonal natural almost Hermitian structure* (G, J) *on* TM *is in one of the classes* $\mathcal{SK} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$, $\mathcal{QK} = \mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{AK} = \mathcal{W}_2$, *if and only if*

$$\mu = \lambda'.$$

Remark. The property that the almost Hermitian structure (G, J) is almost Kählerian (in the class $\mathcal{AK} = \mathcal{W}_2$) can be checked directly by computing the exterior differential $d\phi$ of its fundamental 2-form.

Theorem 6. *Assume that $a_1 \neq \sqrt{2ct}$. The diagonal natural almost Hermitian structure (G, J) on TM is in one of the classes $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_4$, $\mathcal{H} = \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_4$, if and only if the base manifold (M, g) has constant sectional curvature c and the coefficient b_1 is expressed as*

$$b_1 = \frac{a_1 a_1' - c}{a_1 - 2a_1' t}.$$

Remark. The property that the almost Hermitian structure (G, J) is Hermitian (in the class $\mathcal{H} = \mathcal{W}_3 \oplus \mathcal{W}_4$) can be checked directly by computing the Nijenhuis tensor field of J.

Finally, we get

Theorem 7. *Assume that $a_1 \neq \sqrt{2ct}$. The diagonal natural almost Hermitian structure (G, J) on TM is in one of the classes $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_3 = \mathcal{SK} \cap \mathcal{H}$, $\mathcal{W}_1 = \mathcal{NK}$, $\mathcal{K}$ if and only if $\mu = \lambda'$ and b_1 is expressed as in the Theorem 6.*

Remark. The property that the almost Hermitian structure (G, J) is Kählerian (in the class $\mathcal{K}$) can be checked directly by computing the Nijenhuis tensor field of J and the exterior differential of the 2-form ϕ.

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