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Some remarks on the Lorenz five component model

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Dedicated to Professor L. Tamássy on his 80th birthday

Abstract. Some old aspects from the theory of the Lorenz five component model are discussed and some of its new properties are pointed out.

1. Introduction

The study of balanced dynamics is a central subject in geophysical fluid dynamics. It can be realized using three different approaches. The most recent one is due to Lorenz [6] and is called the five component model. Its dynamics is usually described by the following set of differential equations:

$$\begin{cases} \dot{x}_1 = -x_2x_3 + bx_2x_5 \\ \dot{x}_2 = x_1x_3 - bx_1x_5 \\ \dot{x}_3 = -x_1x_2 \\ \dot{x}_4 = -\dfrac{x_5}{\varepsilon} \\ \dot{x}_5 = \dfrac{x_4}{\varepsilon} + bx_1x_2, \end{cases} \tag{1.1}$$

where $b, \varepsilon \in \mathbb{R}$.

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The goal of our paper is to present some new properties of this dynamics from the Poisson geometry point of view.

2. Stability problem – Periodic solutions

Let us start with the following result dues to Bokhove [2]. More exactly we have:

Proposition 2.1 ([2])**.** *The dynamics* (1.1) *has the following Hamilton–Poisson realization:*

$$(\mathbb{R}^5, \Pi_{\varepsilon,b}, H),$$

where

$$\Pi_{\varepsilon,b} = \begin{bmatrix} 0 & 0 & -x_2 & 0 & bx_2 \\ 0 & 0 & x_1 & 0 & -bx_1 \\ x_2 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{1}{\varepsilon} \\ -bx_2 & bx_1 & 0 & \dfrac{1}{\varepsilon} & 0 \end{bmatrix}$$

and

$$H(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2).$$

Proposition 2.2. *There exists only one functionally independent Casimir of our configuration* $(\mathbb{R}^5, \Pi_{\varepsilon,b})$.

Proof. For the proof we shall use the technique of Bermejo and Fairen [1].

Let

$$(\Pi_{\varepsilon,b})_{2m} = \begin{bmatrix} 0 & x_1 & 0 & -bx_1 \\ -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\varepsilon} \\ bx_1 & 0 & \dfrac{1}{\varepsilon} & 0 \end{bmatrix}.$$

Then

$$\det(\Pi_{\varepsilon,b})_{2m} = \frac{x_1^2}{\varepsilon} \neq 0$$

iff

$$x_1 \neq 0.$$

It follows that

$$(\Pi_{\varepsilon,b})^{-1}_{2m} = \begin{bmatrix} 0 & -\dfrac{1}{x_1} & 0 & 0 \\ \dfrac{1}{x_1} & 0 & -b\varepsilon & 0 \\ 0 & b\varepsilon & 0 & \varepsilon \\ 0 & 0 & -\varepsilon & 0 \end{bmatrix}.$$

Hence

$$(\Pi_{\varepsilon,b})_{n-2m} = [0,\ -x_2,\ 0,\ bx_2];$$

$$\begin{aligned} \Gamma &= [(\Pi_{\varepsilon,b})_{n-2m} \cdot (\Pi_{\varepsilon,b})^{-1}_{2m}]^t \\ &= \left[-\frac{x_2}{x_1}, 0, 0, 0\right]^t. \end{aligned}$$

Therefore

$$dx_1 = -\frac{x_2}{x_1} dx_2$$

or equivalent

$$C(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{2}(x_1^2 + x_2^2)$$

is the Casimir of our configuration $(\mathbb{R}^5, \Pi_{\varepsilon,b})$. □

It is not hard to see that the equilibrium states of our system are:

$$e_1^M = (M, 0, 0, 0, 0), \quad M \in \mathbb{R};$$

$$e_2^M = (0, M, 0, 0, 0), \quad M \in \mathbb{R};$$

$$e_3^M = (0, 0, M, 0, 0), \quad M \in \mathbb{R}.$$

We shall now discuss their nonlinear stability. Recall that an equilibrium state x_e is nonlinear stable if the trajectories close to x_e stay close to x_e for each $t \in \mathbb{R}$. In other words, at least one neighborhood of x_e must be flow invariant.

Proposition 2.3. *The equilibrium state e_1^M, $M \in \mathbb{R}$, $M \neq 0$ is nonlinear stable.*

Proof. We shall make the proof using the energy-Casimir method, see [5], [7] or [9].

Let H_φ be the energy-Casimir function given by

$$H_\varphi(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2) + \varphi\left(\frac{1}{2}(x_1^2 + x_2^2)\right),$$

where $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$.

Then we have

$$\begin{aligned}\delta H_\varphi &= x_1\delta x_1 + 2x_2\delta x_2 + x_3\delta x_3 + x_4\delta x_4 + x_5\delta x_5 \\ &\quad + \varphi'(x_1\delta x_1 + x_2\delta x_2),\end{aligned}$$

where

$$\varphi' = \frac{\partial\varphi}{\partial\left(\frac{1}{2}(x_1^2 + x_2^2)\right)}.$$

Hence

$$\delta H_\varphi(M, 0, 0, 0, 0) = 0$$

if and only if

$$\varphi'\left(\frac{1}{2}M^2\right) = -1. \tag{2.1}$$

The second variation of H_φ is given by

$$\begin{aligned}\delta^2 H_\varphi &= (\delta x_1)^2 + 2(\delta x_2)^2 + (\delta x_3)^2 + (\delta x_4)^2 + (\delta x_5)^2 \\ &\quad + \varphi''(x_1\delta x_1 + x_2\delta x_2)^2 + \varphi'[(\delta x_1)^2 + (\delta x_2)^2].\end{aligned}$$

At the equilibrium of interest we have via (2.1)

$$\begin{aligned}\delta^2 H_\varphi(M, 0, 0, 0, 0) &= (\delta x_2)^2 + (\delta x_3)^2 + (\delta x_4)^2 + (\delta x_5)^2 \\ &\quad + \varphi''\left(\frac{1}{2}M^2\right) M^2(\delta x_1)^2.\end{aligned}$$

If we can choose φ such that

$$\varphi''\left(\frac{1}{2}M^2\right) > 0 \tag{2.2}$$

then the second variation of H_φ at the equilibrium of interest is positive definite and so we can conclude that the equilibrium state $(M, 0, 0, 0, 0)$, $M \in \mathbb{R}$, $M \neq 0$ is nonlinear stable.

For instance such a φ is given by

$$\varphi(x) = \left(x - \frac{1}{2}M^2\right)^2 - x. \qquad \square$$

Using now the linear part of our system (1.1) at the equilibrium of interest e_2^M [resp. e_3^M] we have immediately:

Proposition 2.4. *The equilibrium states* e_2^M, e_3^M, $M \in \mathbb{R}$, $M \neq 0$, *have the following behaviour:*

i) $e_2^M = (0, M, 0, 0, 0)$, $M \in \mathbb{R}$, $M \neq 0$ *is unstable.*

ii) $e_3^M = (0, 0, M, 0, 0)$, $M \in \mathbb{R}$, $M \neq 0$ *is spectrally stable.*

If we take now the function $H \in C^\infty(\mathbb{R}^5, \mathbb{R})$ given by

$$H(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2)$$

as a Lyapunov function, then via Lyapunov theorem we have

Proposition 2.5. *The equilibrium state* $e^0 = (0, 0, 0, 0, 0)$ *is nonlinear stable.*

Remark 2.1. It is an open problem to decide the nonlinear stability or instability of the equilibrium states $e_3^M = (0, 0, M, 0, 0)$, $M \in \mathbb{R}$, $M \neq 0$.

We shall discuss now the existence of the periodic solutions for the dynamics (1.1).

Let K_φ be a first integral of the dynamics (1.1) given by

$$\begin{aligned} K_\varphi(x_1, x_2, x_3, x_4, x_5) &= \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2) \\ &\quad + \varphi\left(\frac{1}{2}(x_1^2 + x_2^2)\right) - \frac{1}{2}M^2 - \varphi\left(\frac{1}{2}M^2\right) \end{aligned}$$

where $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies to the conditions (2.1), (2.2). Then we have

i) $K_\varphi \in C^\infty(\mathbb{R}^5, \mathbb{R})$;

ii) $K_\varphi(M, 0, 0, 0, 0) = 0$;

iii) $dK_\varphi(M, 0, 0, 0, 0) = 0$;

iv) $d^2K_\varphi(M, 0, 0, 0, 0)$ is positive definite.

Then via the Moser theorem [8] we have

Proposition 2.6. *For each* $\varepsilon > 0$ *sufficiently small any integral surface*

$$K_\varphi(x_1, x_2, x_3, x_4, x_5) = \varepsilon^2$$

contains at least one periodic solution for the dynamics (1.1) *whose periods are close to those of the corresponding linear system.*

Let H be the Hamiltonian (or the energy of the dynamics (1.1)), i.e.

$$H(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2).$$

Then we have

i) $H \in C^\infty(\mathbb{R}^5, \mathbb{R})$;

ii) $H(0, 0, 0, 0, 0) = 0$;

iii) $DH(0, 0, 0, 0, 0) = 0$;

iv) $D^2 H(0, 0, 0, 0, 0)$ is positive definite.

Then via the Moser [8] we have

Proposition 2.7. *For each $\varepsilon > 0$ sufficiently small any integral surface*

$$H(x_1, x_2, x_3, x_4, x_5) = \varepsilon^2$$

contains at least one periodic solution for the dynamics (1.1) *whose periods are close to those of the corresponding linear system.*

3. The reduced phase space and the reduced dynamics

It is clear that the function $C \in C^\infty(\mathbb{R}^5, \mathbb{R})$ given by

$$C(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + x_2^2)$$

is constant of motion (1.1). Let us make the change of variables:

$$\begin{cases} x_1 = \sqrt{2C}\cos\varphi \\ x_2 = \sqrt{2C}\sin\varphi. \end{cases}$$

Then the dynamics (1.1) takes the following form:

$$\begin{cases} \dfrac{d\varphi}{dt} = x_3 - bx_5 \\ \dfrac{dx_3}{dt} = -C\sin 2\varphi \\ \dfrac{dx_4}{dt} = -\dfrac{x_5}{\varepsilon} \\ \dfrac{dx_5}{dt} = \dfrac{x_4}{\varepsilon} + bC\sin 2\varphi \end{cases} \tag{3.1}$$

and it is usually called the reduced dynamics.

Proposition 3.1 ([3])**.** *The reduced dynamics* (3.1) *has the following Hamilton–Poisson realization:*

$$(M_C, \Pi_{\text{red}}, H_{\text{red}}),$$

where

$$M_C = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 = 2C\};$$

$$\Pi_{\text{red}} = \begin{bmatrix} 0 & 1 & 0 & -b \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\varepsilon} \\ b & 0 & \dfrac{1}{\varepsilon} & 0 \end{bmatrix};$$

$$H_{\text{red}}(\varphi, x_3, x_4, x_5) = -\frac{C}{2}\cos 2\varphi + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2);$$

$$C \in \mathbb{R}, \quad C > 0.$$

Since

$$\det(\Pi_{\text{red}}) = -\frac{1}{\varepsilon^2} \neq 0,$$

we have immediately:

Proposition 3.2. *The reduced dynamics* (3.1) *has the following Hamiltonian realization:*

$$(M_C, \omega_C, H_{\text{red}}),$$

where

$$\omega_C = -d\varphi \wedge dx_3 - \varepsilon b dx_3 \wedge dx_4 + \varepsilon dx_4 \wedge dx_5.$$

Remark 3.1. It is clear from the above considerations that (M_C, ω_C), $C \in \mathbb{R}$, $C > 0$, is in fact the symplectic foliation of the Poisson manifold $(\mathbb{R}^5, \Pi)$.

It is easy to see that the equilibrium states of our system (3.1) are

$$e_k = \left(\frac{k\pi}{2}, 0, 0, 0\right), \quad k \in \mathbb{Z}.$$

Proposition 3.3. *The equilibrium states*

$$e_{2l} = (l\pi, 0, 0, 0), \quad l \in \mathbb{Z}$$

are nonlinear stable.

Proof. Let $L \in C^\infty(M_C, \mathbb{R})$ be a smooth function given by

$$L(\varphi, x_3, x_4, x_5) = -\frac{C}{2}\cos 2\varphi + \frac{C}{2} + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2).$$

Then we have

i) $L(l\pi, 0, 0, 0) = 0$;

ii) $L(\varphi, x_3, x_4, x_5) > 0$, $(\forall)$ $(\varphi, x_3, x_4, x_5) \in M_C$, $(\varphi, x_3, x_4, x_5) \neq (l\pi, 0, 0, 0)$;

iii) $\dot{L} = 0$, along the trajectories of the system (3.1).

Hence L is a Lyapunov function and then via the Lyapunov theorem the equilibrium states

$$e_{2l} = (l\pi, 0, 0, 0), \quad l \in \mathbb{Z}$$

are nonlinear stable. □

Proposition 3.4. *The equilibrium states*

$$e_{2l+1} = \left(l\pi + \frac{\pi}{2}, 0, 0, 0\right), \quad l \in \mathbb{Z}$$

are unstable.

Proof. Indeed, let A be the matrix of the linear part of our dynamics (3.1),

$$A = \begin{bmatrix} 0 & 1 & 0 & -b \\ -2C\cos 2\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\varepsilon} \\ 2bC\cos 2\varphi & 0 & \dfrac{1}{\varepsilon} & 0 \end{bmatrix}.$$

Then we have

$$A(e_{2l+1}) = \begin{bmatrix} 0 & 1 & 0 & -b \\ 2C & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\varepsilon} \\ 2bC & 0 & \dfrac{1}{\varepsilon} & 0 \end{bmatrix};$$

$$p_{A(e_{2l+1})}(x) = x^4 + x^2\left(\frac{1}{\varepsilon^2} - 2C + 2b^2C\right) - 2C\frac{1}{\varepsilon^2}.$$

It is easy to see now that the characteristic equation

$$p_{A(e_{2l+1})}(x) = 0$$

has a positive root and so the equilibrium states

$$e_{2l+1} = \left(l\pi + \frac{\pi}{2}, 0, 0, 0\right), \quad l \in \mathbb{Z}$$

are unstable. □

Let L be a real valued function given by

$$L(\varphi, x_3, x_4, x_5) = -\frac{C}{2}\cos 2\varphi + \frac{C}{2} + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2).$$

Then we have

i) $L \in C^\infty(M_C, \mathbb{R})$;

ii) $L(\pi l, 0, 0, 0) = 0$;

iii) $dL(\pi l, 0, 0, 0) = 0$;

iv) $d^2L(\pi l, 0, 0, 0)$ is positive definite.

Then via Weinstein theorem [10] we have:

Proposition 3.5. *For each sufficiently small ε any integral surface*

$$L(\varphi, x_3, x_4, x_5) = \varepsilon^2$$

contains at least two periodic solutions of the dynamics (3.1) *whose periods are close to those of the linearized system.*

4. Geometric prequantization of the reduced dynamics (3.1)

We have seen in the previous section that the dynamics (3.1) has the following Hamiltonian realization:

$$(M_C, \omega_C, H_{\text{red}})$$

where

$$M_C = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 = 2C\};$$

$$\omega_C = -d\varphi \wedge dx_3 - \varepsilon b dx_3 \wedge dx_4 + \varepsilon dx_4 \wedge dx_5;$$

$$H_{\mathrm{red}}(\varphi, x_3, x_4, x_5) = -\frac{C}{2}\cos 2\varphi + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2);$$

$$C \in \mathbb{R}, \quad C > 0.$$

It is easy to see that

$$\omega_C = d\theta_C,$$

where

$$\theta_C = -\varphi dx_3 - \varepsilon b x_3 dx_4 + \varepsilon x_4 dx_5$$

and so (M_C, ω_C) is a quantizable manifold. Moreover, the Hilbert reprezentation space is given by

$$\mathcal{H} = L^2(M_C, \mathbb{C}),$$

and the prequantum operator δ_f has the following expression for each $f \in C^\infty(M_C, \mathbb{R})$:

$$\delta_f = -i\hbar\left[X_f - \frac{i}{\hbar}\theta_C(X_f)\right] + f,$$

where $\hbar$ is the Planck constant divided by 2π.

Therefore we have

Proposition 4.1. *The pair $(\mathcal{H}, \delta)$ gives rise to a prequantization of the reduced dynamics* (3.1).

Using now the same arguments as in [4] with obvious modifications we can prove:

Proposition 4.2. *Let $O(L^2(M_C, \mathbb{C}))$ be the space of self adjoint operators on the Hilbert space $L^2(M_C, \mathbb{C})$. Then the map*

$$f \in C^\infty(M_C, \mathbb{R}) \mapsto \delta_f \in O(L^2(M_C, \mathbb{C}))$$

gives rise to an irreducible representation of $C^\infty(M_C, \mathbb{R})$ on $O(L^2(M_C, \mathbb{C}))$.

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