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Where do homogeneous polynomials attain their norm?

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Abstract. Given a fixed subset A of the unit sphere of a real finite-dimensional Banach space, how probable is it for a norm-one homogeneous polynomial to attain its norm on A? We study the linear case in Section 1, and in Section 2 consider the case of k-homogeneous polynomials on ℓ_∞^n.

If E is a real Banach space, a k-homogeneous polynomial $P : E \longrightarrow \mathbb{R}$ is a function that can be written as $P(x) = B(x, \dots, x)$, where B is a continuous k-linear form over E. The space $\mathcal{P}(^kE)$ of all such polynomials can be normed by considering the largest value of $|P|$ over the unit sphere S_E of E:

$$\|P\| = \sup\{|P(x)| : \|x\| = 1\}.$$

In recent years there has been considerable interest in the study of geometric properties of the space $\mathcal{P}(^kE)$. It has been shown not to be strictly convex [13], extremal and smooth points of the unit ball have been characterised for a number of spaces ([3], [4], [5], [6], [7], [8], [9], [10]), and related problems have been considered, often in the finite-dimensional setting ([1], [2], [11], [12]).

The question which gave rise to the material presented here can be loosely posed as: how probable is it for a polynomial to attain its norm at a given point or in a given subset A of the unit sphere of E? What we want is to measure – or otherwise give an indication of the size of – the

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set of norm-one polynomials for which

$$\|P\| = \sup\{|P(x)| : x \in A\}.$$

We restrict ourselves to finite dimensional real Banach spaces. Since here the unit sphere is compact, all norms are attained. Also, to simplify computations we will be looking at norm-one polynomials with $P(x) = 1$ instead of $|P(x)| = 1$, so we are considering half the set of polynomials P.

In Section 1 we study the case $k = 1$ of linear forms over E with a C^2 norm. It is perhaps noteworthy that the point of view of the geometer [14] and that of the functional analyst do not coincide. The geometer considers tangents to a convex body as points in a euclidean sphere; for the functional analyst – duality being what it is to functional analysts – it seems more natural to consider tangents as points in the dual (non-euclidean) sphere. The resulting measures are different. We adopt the functional analytic point of view here.

In Section 2 we consider the case of general k-homogeneous polynomials on E, with emphasis on $E = \ell_\infty^n$. It is important to note that the space of polynomials $\mathcal{P}(^k\mathbb{R}^n)$ has no preferred or canonical basis, so Lebesgue measure of subsets depends on an inevitably arbitrary choice. We limit ourselves to indicating zero measure and positive measure subsets.

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1. Linear forms

Given a subset $A \subset S_E$ of the unit sphere of E, we want to measure the set $A' = \{\gamma \in S_{E'} : \sup_{x \in A} \gamma(x) = 1\}$ considered as an $(n-1)$-dimensional surface in $\mathbb{R}^n$. It is clear that if $A = \{x_0\}$, then A' (which is a convex subset of $S_{E'}$) has positive measure if and only if it contains n linearly independent functionals (we will see later that this can be generalized for homogeneous polynomials of any degree).

Let us assume that we have a norm N on $\mathbb{R}^n$ which is twice continuously differentiable on $A \subset S_E$. Then for each $x \in A$ the differential $DN(x)$ is the unique norm-one linear functional that takes the value 1 at x. Therefore, $A' = DN(A)$. Moreover, if $U \subset \mathbb{R}^{n-1}$ is an open set and $\sigma : U \to A$ is a parametrization of A, the composition $DN \circ \sigma$ will be a parametrization of A'. We obtain the following result regarding the measure of A'.

Theorem 1. *With the above notation, the Lebesgue measure of A' can be calculated as*

$$\mu(A') = \int_A \frac{|\lambda_1(x) \cdots \lambda_{n-1}(x)|}{\|DN(x)\|_2 \, \|x\|_2} \, dx.$$

where $\lambda_1(x), \ldots, \lambda_{n-1}(x)$ are the non-zero eigenvalues of the hessian $HN(x)$, and $\| \, . \, \|_2$ denotes the euclidean norm.

Proof. We have

$$\begin{aligned} \mu(A') &= \mu((DN \circ \sigma)(U)) \\ &= \int_U \left\|(DN \circ \sigma)_{u_1}(u) \wedge \cdots \wedge (DN \circ \sigma)_{u_{n-1}}(u)\right\| du \\ &= \int_U \left\|HN(\sigma(u)) \, (\sigma_{u_1}(u)) \wedge \cdots \wedge HN(\sigma(u)) \, (\sigma_{u_{n-1}}(u))\right\| \, du \quad (1) \end{aligned}$$

where HN is the hessian of N. It is easily verified that this hessian has the following properties:

i) $HN(x)(x) = 0$.

ii) $HN(\sigma(u)) \, (\sigma_{u_i}(u)) = (DN \circ \sigma)_{u_i}(u)$ for each $u \in U$. Consequently, $HN(x)(T_{S_E}(x)) = T_{S_{E'}}(DN(x))$, where $T_{S_E}(x)$ and $T_{S_{E'}}(DN(x))$ are the tangent hyperplanes to S_E and $S_{E'}$ at x and $DN(x)$ respectively.

iii) $HN(x)$ is a symmetric matrix, so we can find an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $HN(x)$, $\{v_1, \ldots, v_{n-1}, \frac{x}{\|x\|_2}\}$, with eigenvalues $\lambda_1(x), \ldots, \lambda_{n-1}(x), 0$.

The norm $\left\|HN(\sigma(u)) \, (\sigma_{u_1}(u)) \wedge \cdots \wedge HN(\sigma(u)) \, (\sigma_{u_{n-1}}(u))\right\|$ can be computed as

$$\left|\det\left(\widehat{HN(\sigma(u))}\right)\right| \sigma_{u_1}(u) \wedge \cdots \wedge \sigma_{u_{n-1}}(u)$$

where $\widehat{HN(\sigma(u))}$ denotes the operator $HN(\sigma(u))$ considered from $T_{S_E}(x)$ to $T_{S_{E'}}(DN(x))$. Since the vectors $v_1, \ldots, v_{n-1}$ span the image of $HN(x)$

and $HN(x)(T_{S_E}(x)) = T_{S_{E'}}(DN(x))$, we have that $[x]^\perp = [v_1, \dots, v_{n-1}] = T_{S_{E'}}(DN(x))$. So, if $P : \mathbb{R}^n \to [v_1, \dots, v_{n-1}]$ is the orthogonal projection, the operator $HN(x) : T_{S_E}(x) \to T_{S_{E'}}(DN(x))$ can be seen as the composition of

$$P|_{T_{S_E}(x)} : T_{S_E}(x) \to [v_1, \dots, v_{n-1}] = T_{S_{E'}}(DN(x))$$

and the diagonal operator

$$T_{S_{E'}}(DN(x)) \to T_{S_{E'}}(DN(x))$$
$$v_i \mapsto \lambda_i(x) v_i.$$

The determinant of this last operator is the product $\lambda_1(x) \cdots \lambda_{n-1}(x)$. Since $T_{S_E}(x)$ coincides with $[DN(x)]^\perp$, the determinant of $P \; |_{T_{S_E}(x)}$ is $\frac{\langle DN(x), x\rangle}{\|DN(x)\|_2 \|x\|_2} = \frac{1}{\|DN(x)\|_2 \|x\|_2}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product. Therefore, the determinant of the operator $\widehat{HN(x)} : T_{S_E}(x) \to T_{S_{E'}}(DN(x))$ is

$$\frac{\lambda_1(x) \cdots \lambda_{n-1}(x)}{\|DN(x)\|_2 \|x\|_2}$$

and from (1) we have

$$\begin{aligned} \mu(A') &= \int_U \frac{|\lambda_1(\sigma(u)) \cdots \lambda_{n-1}(\sigma(u))|}{\|DN(\sigma(u))\|_2 \, \|\sigma(u)\|_2} \, \left\| \sigma_{u_1}(u) \wedge \cdots \wedge \sigma_{u_{n-1}}(u) \right\| \, du \\ &= \int_A \frac{|\lambda_1(x) \cdots \lambda_{n-1}(x)|}{\|DN(x)\|_2 \, \|x\|_2} \, dx. \end{aligned}$$

□

Note that the function

$$x \mapsto \frac{|\lambda_1(x) \cdots \lambda_{n-1}(x)|}{\|DN(x)\|_2 \, \|x\|_2}$$

is therefore the (non-normalized) density of linear functionals attaining their norms at subsets of S_E.

We may compare this with the density of linear functionals from the geometer's point of view [14]. Instead of considering $A' = DN(A)$, consider $A' = (s \circ DN)(A)$, where s is the euclidean normalization $s(y) =$

$\frac{y}{\|y\|_2}$. Since $Ds(DN(x))$ is $\frac{1}{\|DN(x)\|_2}$ times the orthogonal projection onto $[DN(x)]^\perp$ and the image of $HN(x)$ is $[x]^\perp$, we obtain the geometer's density

$$x \mapsto \frac{|\lambda_1(x) \cdots \lambda_{n-1}(x)|}{\|DN(x)\|_2^3 \, \|x\|_2^2}.$$

2. k-homogeneous polynomials

The space $\mathcal{P}(^k\mathbb{R}^n)$ of k-homogeneous polynomials on $\mathbb{R}^n$ is a vector space of dimension $d_{n,k} = \frac{(n+k-1)!}{(n-1)!k!}$. Therefore, we can identify it with $\mathbb{R}^{d_{n,k}}$ (in a non-unique way). With this identification, $S_{\mathcal{P}(^k\mathbb{R}^n)}$ is a $(d_{n,k}-1)$-dimensional surface of $\mathbb{R}^{d_{n,k}}$ and we will consider a measure μ on $S_{\mathcal{P}(^k\mathbb{R}^n)}$ which is non-zero on relatively open subsets of $S_{\mathcal{P}(^k\mathbb{R}^n)}$ but is zero on subsets of lower dimension than $d_{n,k}-1$. For example, the Lebesgue measure of hypersurfaces on $\mathbb{R}^{d_{n,k}}$ is such a measure.

We will say that a point x_0 in the unit ball B_E of E is a *vertex* if there are n linearly independent functionals that attain their norms at x_0. This is equivalent to saying that the set $\{\gamma \in S_{E'} : \gamma(x_0) = 1\}$ has positive measure. We want to show that this is equivalent to the fact that the set $\{P \in S_{\mathcal{P}(^k\mathbb{R}^n)} : P(x_0) = 1\}$ has positive measure. First we need the following:

Lemma 2. *If P attains its norm at $x_0 \in S_E$, then $DP(x_0)$ attains its norm at x_0 and therefore $\|DP(x_0)\| = \|P\|$.*

Proof. For $y \in S_E$ define $\alpha : [0,1] \to \mathbb{R}$ as

$$\alpha(t) = P(x_0 + t(y - x_0)).$$

Since P attains its norm at x_0, $t = 0$ is a maximum of α and therefore, $\alpha'(0) \le 0$. Then, we have

$$0 \ge \alpha'(0) = DP(x_0)(y - x_0) = DP(x_0)(y) - DP(x_0)(x_0)$$

which means that x_0 is a maximum of $DP(x_0)$ on S_E. $\square$

Theorem 3. *The point x_0 is a vertex of S_E if and only if the set $\{P \in S_{\mathcal{P}(^k\mathbb{R}^n)} : P(x_0) = 1\}$ has positive measure in $S_{\mathcal{P}(^k\mathbb{R}^n)}$.*

Proof. If x_0 is a vertex of S_E, there are norm-one linearly independent functionals $\gamma_1, \dots, \gamma_n$ such that $\gamma_1(x_0) = \cdots = \gamma_n(x_0) = 1$. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ such that $\alpha_1 + \cdots + \alpha_n = k$ the polynomials $P_\alpha = \gamma_1^{\alpha_1} + \cdots + \gamma_n^{\alpha_n}$ are $d_{n,k}$ linearly independent polynomials of norm one with $P_\alpha(x_0) = 1$. Therefore, $\mathrm{co}(\{P_\alpha\}_\alpha)$ contains a relatively open subset of the $(d_{n,k} - 1)$-dimensional surface S_E. Consequently, $\mathrm{co}(\{P_\alpha\}_\alpha)$ has positive measure and so does $\{P \in S_{\mathcal{P}(^k\mathbb{R}^n)} : P(x_0) = 1\}$.

Conversely, suppose $\{P \in S_{\mathcal{P}(^k\mathbb{R}^n)} : P(x_0) = 1\}$ has positive measure. Being convex, it contains linearly independent polynomials $P_1, \dots, P_{d_{n,k}}$. Consider the linear functionals $\phi_i = DP_i(x_0)$ for $i = 1, \dots, d_{n,k}$. The previous lemma affirms that $\|\phi_i\| = 1$ for all i, so to see that x_0 is a vertex it is enough to show that $\phi_1, \dots, \phi_{d_{n,k}}$ span E'. If $\gamma \in E'$, the polynomial γ^k is a linear combination of the polynomials $P_1, \dots, P_{d_{n,k}}$:

$$\gamma^k = \sum_i \lambda_i P_i$$

and taking differentials on x_0 we have

$$k\gamma(x_0)^{k-1}\gamma = \sum_i \lambda_i \; DP_i(x_0) = \sum_i \lambda_i \; \phi_i$$

which shows that every γ which is non-zero at x_0 is spanned by the ϕ_i's. Since the set of linear functionals that are non-zero at x_0 is dense in E' we conclude that the span of $\phi_1, \dots, \phi_{d_{n,k}}$ is E' and consequently x_0 is a vertex. $\square$

We have seen that vertices behave similarly for linear functionals and for polynomials. However, not every subset of the unit sphere has the same behavior in both cases. For simplicity, we start looking at ℓ_∞^n, that is, $\mathbb{R}^n$ with the supremum norm. For $1 \le r \le n-1$, consider in the unit cube of ℓ_∞^n the r-*face* given by

$$A_r = \{(\underbrace{1, \dots, 1}_{n-r}, u_1, \dots, u_r) : -1 < u_l < 1 \text{ for } l = 1, \dots, r\}.$$

Note that for $r = 1$, A_r is an edge, while for $r = n-1$, A_r is a face of the unit cube of ℓ_∞^n. The set of linear functionals that attain their norm at A_r is

$$\{(\alpha_1, \dots, \alpha_{n-r}, 0, \dots, 0) : \alpha_1 + \cdots + \alpha_{n-r} = 1, \; \alpha_i \ge 0 \text{ for all } i\},$$

which has zero measure in $S_{(\ell_\infty^n)'} = S_{\ell_1^n}$. However, we will see that the set of norm-one 2-homogenous polynomials that attain their norm at A_r has positive measure in the unit sphere of $\mathcal{P}(^2\ell_\infty^n)$ for any $r = 1, \ldots, n-1$. The polynomials

$$\begin{array}{ll} x_i^2, \qquad x_i x_j & \text{for } 1 \leq i < j \leq n-r \\ x_i^2 - \left(\dfrac{x_{n-r+l} - u_l x_i}{2}\right)^2 & \text{for } 1 \leq i \leq n-r \\ & \text{and } 1 \leq l \leq r \\ x_1^2 - \left(\dfrac{x_{n-r+l} + x_{n-r+k} - (u_l + u_k)x_1}{4}\right)^2 & \text{for } 1 \leq i \leq n-r \\ & \text{and } 1 \leq l < k \leq r \end{array}$$

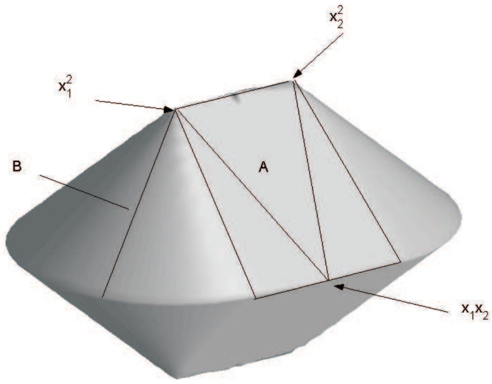
are linearly independent, have norm one and take the value 1 at the point $(1, \ldots, 1, u_1, \ldots, u_r)$ in A_r. There are $\frac{(n-r+1)(n-r)}{2}$ polynomials of the first class, $(n-r)r$ of the second, and $\frac{r(r-1)}{2}$ of the third, which gives a total number of $\frac{n(n+1)}{2} - r$. Taking the convex hull of these polynomials and moving the r free parameters $u_1, \ldots, u_r$ we get a $(d_{n,2} - 1)$-dimensional surface on the unit sphere of $\mathcal{P}(^2\ell_\infty^n)$ consisting of polynomials that attain their norm at A_r. Therefore, the set

$$\{P \in \mathcal{P}(^2\ell_\infty^n) : P \text{ attains its norm at } A_r\}$$

has positive measure on $S_{\mathcal{P}(^2\ell_\infty^n)}$. Clearly the same holds for all other r-faces such as

$$\{(1, u_1, -1, \ldots, 1, \ldots, -1, \ldots, u_r) : -1 < u_l < 1 \text{ for } l = 1, \ldots, r\}.$$

As an example, consider the case $n = 2$. The following figure represents the unit sphere of the space of 2-homogeneous polynomials on ℓ_∞^2. The triangle A is the convex hull of x_1^2, x_2^2 and x_1x_2, all of which take the value 1 at the vertex $(1, 1)$. The rectangle that contains the triangle is the set of all polynomials that attain their norms at $(1, 1)$. The segment B is the set of polynomials attaining their norm at the point $(1, t)$ for some fixed t with $-1 < t < 1$. By itself, it has zero measure but the union of these segments over t is a semiconic surface with positive measure.



If the unit sphere of our space is not a cube but has curved edges or faces (like a cylinder, for example), something similar can be done. If E denotes $\mathbb{R}^n$ with an arbitrary norm, we have the following.

Proposition 4. *Let $A \subset S_E$ be such that there exists an open subset $U \subset \mathbb{R}^r$ and a differentiable (non-degenerate) parametrization of A*

$$U \longrightarrow A \qquad u \longmapsto x_u.$$

Assume that for $u \in U$ there are norm-one linearly independent functionals $\gamma_1^u, \dots, \gamma_{n-r}^u$ such that $\gamma_i^u(x_u) = 1$ and that the mapping $u \mapsto \gamma_i^u$ is differentiable for each i. Then the subset of norm-one 2-homogeneous polynomials that attain their norm at A has positive measure.

Proof. Taking a smaller U if necessary, we can find norm one linear functionals $\varphi_1^u, \dots, \varphi_r^u$ such that $\{\gamma_1^u, \dots, \gamma_{n-r}^u, \varphi_1^u, \dots, \varphi_r^u\}$ is a basis of $\mathbb{R}^n$ and the mapping $u \mapsto \varphi_l^u$ is differentiable for each l. Now, for each u, the polynomials

$$(\gamma_i^u)^2, \qquad \gamma_i^u\gamma_j^u \qquad \text{for } 1 \le i < j \le n-r$$

$$(\gamma_i^u)^2 - \left(\frac{\varphi_l^u - \varphi_l^u(x_u)\gamma_i^u}{2}\right)^2 \qquad \begin{array}{l}\text{for } 1 \le i \le n-r \\ \text{and } 1 \le l \le r\end{array}$$

$$(\gamma_1^u)^2 - \left(\frac{\varphi_l^u + \varphi_k^u - (\varphi_l^u(x_u) + \varphi_l^u(x_u))\gamma_1^u}{4}\right)^2 \qquad \begin{array}{l}\text{for } 1 \le i \le n-r \\ \text{and } 1 \le l < k \le r\end{array}$$

are $\frac{n(n+1)}{2} - r$ norm-one linearly independent polynomials that take the value 1 at x_u. If we enumerate them as $\{P_k^u : k = 1, \dots, s\}$ were $s =$

$\frac{n(n+1)}{2} - r$, the set

$$\left\{ \sum_{j=1}^{s} \alpha_j P_j^u : \alpha_1 + \cdots + \alpha_s = 1,\ \alpha_i \geq 0 \text{ for each } i \text{ and } u \in U \right\}$$

is a $(d_{n,k} - 1)$-dimensional surface in $S_{\mathcal{P}(^2\ell_\infty^n)}$. Since all these polynomials attain their norm at A, the subset of all polynomials attaining their norm at A has positive measure in $S_{\mathcal{P}(^2\ell_\infty^n)}$. □

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