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Two bounded solutions of opposite sign for nonlinear hemivariational inequalities at resonance

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Abstract. In this paper we study quasilinear hemivariational inequalities at resonance at the first eigenvalue of the p-Laplacian. For such problems we establish the existence of at least two bounded solutions: one positive and the other negative. Our approach is based on the method of upper-lower solutions and on techniques from the theory of nonlinear operator of monotone type.

1. Introduction

In this paper we consider the following nonlinear hemivariational inequality at resonance:

$$\begin{cases} -\operatorname{div}(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)) - \lambda_1|x(z)|^{p-2}x(z) \in \partial j(z, x(z)) \\ \qquad \text{almost everywhere on } Z \\ x|_\Gamma = 0. \end{cases} \tag{HVI}$$

Here $2 \leq p < +\infty$, $Z \subseteq \mathbb{R}^N$ is a bounded domain with a $C^{1,\alpha}$-boundary Γ (where $0 < \alpha < 1$). By λ_1 we denote the first eigenvalue of the negative

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p-Laplacian $-\Delta_p x = -\mathrm{div}(\|\nabla x\|_{\mathbb{R}^N}^{p-2} \nabla x)$ with the Dirichlet boundary condition (i.e. of $(-\Delta_p, W_0^{1,p}(Z))$). By $j : Z \times \mathbb{R} \longmapsto \mathbb{R}$ we mean a function, which is measurable in the first variable and locally Lipschitz in the second variable. By $\partial j(z, \zeta)$ we denote the subdifferential of $j(z, \cdot)$ in the sense of Clarke [3] (see Section 2). Exploiting the methods of upper-lower solutions together with techniques from the theory of nonlinear operators of monotone type, we prove the existence of at least two bounded solutions for problem (HVI), of which one is strictly positive and the other strictly negative.

Hemivariational inequalities are a new type of variational expressions, which arise in physics and engineering problems, when we deal with nonsmooth energy functionals. Such functionals appear if one wants to consider more realistic mechanical laws of nonmonotone and multivalued nature. For concrete applications of hemivariational inequalities in mechanics and engineering we refer to Naniewicz–Panagiotopoulos [16] and Panagiotopoulos [17]. Resonant hemivariational inequalities were studied recently by Goeleven–Motreanu–Panagiotopoulos [10] (semilinear problem, i.e. $p = 2$) and by Gasiński–Papageorgiou [4], [5] (quasilinear problem). In these works the approach is variational and it is based on the nonsmooth critical point theory of Chang [2]. Nonresonant eigenvalue problems for hemivariational inequalities were investigated by Goeleven–Motreanu–Panagiotopoulos [9] (semilinear problem, i.e. $p = 2$) and by Gasiński–Papageorgiou [6], [7] (quasilinear problem). However, none of the aforementioned work addresses the problem of existence of positive and negative solutions.

2. Preliminaries

Problem (HVI) involves the first eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(Z))$. This is defined as follows. Consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\mathrm{div}(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2} \nabla x(z)) = \lambda |x(z)|^{p-2} x(z) \\ \qquad \text{almost everywhere on } Z \\ x|_\Gamma = 0. \end{cases} \tag{EP}$$

The least real number λ for which (EP) has a nontrivial solution is called the first eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(Z))$. It is known (see ANANE [1] and LINDQVIST [15]) that λ_1 is positive, isolated and simple (i.e. the associated eigenfunctions are constant multiple of each other). Moreover, there is a variational characterization of $\lambda_1 > 0$ by means of the Rayleigh quotient, namely we have

$$\lambda_1 = \min\left\{ \frac{\|\nabla x\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z),\ x \neq 0 \right\}. \tag{RQ}$$

The above minimum is attained at the normalized principal eigenfunction u_1. Note that, if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so we infer that the first eigenfunction u_1 does not change its sign on Z. In fact we can show that $u_1(z) \neq 0$ almost everywhere on Z and so we can assume that $u_1 > 0$ almost everywhere on Z. Moreover, since Γ is $C^{1,\alpha}$, from nonlinear elliptic regularity theory (see LIEBERMAN [14]), we know that the solution of (EP) is in $C^{1,\alpha'}(\overline{Z})$, with some $0 < \alpha' < 1$. For details on the first eigenvalue we refer to ANANE [1] and LINDQVIST [15].

As we already mentioned, our approach will use the theory of nonlinear operators of monotone type. For the convenience of the reader we recall some basic definitions and facts from this theory, which we will need in the sequel. Details can be found in the books of HU–PAPAGEORGIOU [11] and SHOWALTER [18].

Let X be a reflexive Banach space and X^* its topological dual. A map $A : X \supseteq D \longmapsto 2^{X^*}$ is said to be *monotone*, if for all $x, y \in D$ and all $x^* \in Ax$, $y^* \in Ay$, we have $\langle x^* - y^*, x - y\rangle \geq 0$ (here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*)). If in addition $\langle x^* - y^*, x - y\rangle = 0$ implies that $x = y$, then we say that A is strictly monotone. The map A is said to be *maximal monotone*, if it is monotone and the fact that $\langle x^* - y^*, x - y\rangle \geq 0$ for all $x \in D$ and all $x^* \in Ax$, implies that $y \in D$ and $y^* \in Ay$. This means that the graph of A is maximal with respect to inclusion among the graphs of all monotone maps. A map $A : X \longmapsto X^*$ which is single-valued and everywhere defined (i.e. $D = X$), is said to be *demicontinuous*, if the convergence $x_n \longrightarrow x$ in X, implies that $Ax_n \longrightarrow Ax$ weakly in X^*. A map $A : X \supseteq D \longmapsto X^*$ is said to be *coercive*, if D is bounded or D is unbounded and $\inf\{\|x^*\|_{X^*} : x^* \in Ax\} \longrightarrow +\infty$ as $\|x\|_X \to +\infty$. A maximal monotone, coercive operator is surjective.

An operator $A : X \longmapsto 2^{X^*}$ is said to be *pseudomonotone*, if

a) for all $x \in X$, set Ax is nonempty, compact and weakly compact in X^*;

b) for every Y, finite dimensional subspace of X, operator $A|_Y$ is upper semicontinuous into X^* furnished with the weak topology (i.e. if $U \subseteq X^*$ is weakly open, then the set $\{x \in Y : Ax \subseteq U\}$ is open in Y);

c) if $x_n \longrightarrow x$ weakly in X, $x_n^* \in Ax_n$ for $n \geq 1$ and $\limsup \langle x_n^*, x_n - x\rangle \leq 0$, then for every $y \in X$, there exists $x^*(y) \in Ax$ such that $\langle x^*(y), x - y\rangle \leq \liminf \langle x_n^*, x_n - y\rangle$.

If A is bounded (i.e. maps bounded sets into bounded sets) and satisfies condition (c), then satisfies condition (b) too. An operator $A : X \longmapsto 2^{X^*}$ is said to be *generalized pseudomonotone*, if $x_n \longrightarrow x$ weakly in X, $x_n^* \longrightarrow x^*$ weakly in X^*, $x_n^* \in Ax_n$ for $n \geq 1$ and $\limsup\langle x_n^*, x_n - x\rangle \leq 0$, imply that $x^* \in Ax$ and $\langle x_n^*, x_n\rangle \longrightarrow \langle x^*, x\rangle$. Every maximal monotone operator is generalized pseudomonotone. Also a pseudomonotone operator is generalized pseudomonotone, while the converse is true if the operator is bounded and has nonempty, convex and weakly compact values. A pseudomonotone, coercive operator is surjective.

Finally, let X be a Banach space. A function $\phi : X \longmapsto \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$, there exists a neighbourhood U of x and a constant $k > 0$ depending on U, such that $|\phi(z) - \phi(y)| \leq k\|z - y\|_X$ for all $z, y \in U$. From convex analysis we know that a proper, convex and lower semicontinuous function $g : X \longmapsto \overline{\mathbb{R}} \stackrel{\mathrm{df}}{=} \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom} g \stackrel{\mathrm{df}}{=} \{x \in X : g(x) < +\infty\}$. In analogy with the directional derivative of a convex function, we define the *generalized directional derivative* of a locally Lipschitz function ϕ at $x \in X$ in the direction $h \in X$, by

$$\phi^0(x;h) \stackrel{\mathrm{df}}{=} \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{\phi(x' + th) - \phi(x')}{t}.$$

The function $X \ni h \longmapsto \phi^0(x;h) \in \mathbb{R}$ is sublinear, continuous and by the Hahn–Banach theorem it is the support function of a nonempty, convex and w^*-compact subset of X^*, defined by

$$\partial\phi(x) \stackrel{\mathrm{df}}{=} \{x^* \in X^* : \ \langle x^*, h\rangle \leq \phi^0(x;h) \text{ for all } h \in X\}.$$

The multifunction $X \ni x \longmapsto \partial\phi(x) \in 2^{X^*}$ is called *generalized* or *Clarke's subdifferential* of ϕ at x. If $\phi, \psi : X \longmapsto \mathbb{R}$ are locally Lipschitz functions, then $\partial(\phi+\psi)(x) \subseteq \partial\phi(x)+\partial\psi(x)$ and $\partial(t\phi)(x) = t\partial\phi(x)$ for all $t \in \mathbb{R}$ and all $x \in X$. Moreover, if $\phi : X \longmapsto \mathbb{R}$ is also convex, then the subdifferential of ϕ in the sense of convex analysis coincides with the generalized subdifferential introduced above. If ϕ is strictly differentiable at x (in particular if ϕ is continuously Gateaux differentiable at x), then $\partial\phi(x) = \{\phi'(x)\}$.

3. Auxiliary results

In the sequel, we will assume that $p \geq 2$ and that p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. By p^* we will denote the Sobolev critical exponent, defined by

$$p^* \stackrel{\mathrm{df}}{=} \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}$$

and by $p^{*\prime}$ the number such that $\frac{1}{p^*} + \frac{1}{p^{*\prime}} = 1$. Note that $1 \leq p^{*\prime} < p' \leq 2 \leq p < p^* \leq +\infty$.

Our hypotheses on the nonsmooth potential function $j(z,\zeta)$ are the following:

$\underline{H(j)}$: j: $Z \times \mathbb{R} \longmapsto \mathbb{R}$ is a function such that:

(i) for all $\zeta \in \mathbb{R}$, the function $Z \ni z \longmapsto j(z,\zeta) \in \mathbb{R}$ is measurable and $j(\cdot,0) \in L^{p^{*\prime}}(Z)$;

(ii) for almost all $z \in Z$, the function $\mathbb{R} \ni \zeta \longmapsto j(z,\zeta) \in \mathbb{R}$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $\eta \in \partial j(z,\zeta)$, we have $|\eta| \leq a(z)+c|\zeta|^{r-1}$ with some $a \in L^\infty(Z)_+$, $c > 0$ and $1 \leq r < p^*$;

(iv) there exists a function $\vartheta \in L^\infty(Z)$ such that for almost all $z \in Z$, we have $\vartheta(z) \leq 0$, with strict inequality on a set of positive Lebesgue measure, such that

$$\limsup_{|\zeta|\to+\infty} \frac{u(z,\zeta)}{|\zeta|^{p-2}\zeta} = \vartheta(z),$$

uniformly for almost all $z \in Z$ and all $u(z,\zeta) \in \partial j(z,\zeta)$;

(v) $\limsup_{\zeta \to 0} \frac{u(z,\zeta)}{|\zeta|^{p-2}\zeta} > 0$ uniformly for almost all $z \in Z$ and all $u(z,\zeta) \in \partial j(z,\zeta)$.

Proposition 3.1. *If hypotheses* H(j) *hold, then there exists* $\beta > 0$ *such that for all* $x \in W_0^{1,p}(Z)$*, we have*

$$\eta(x) \stackrel{\mathrm{df}}{=} \|\nabla x\|_p^p - \int_Z \big(\lambda_1 + \vartheta(z)\big)|x(z)|^p \, dz \geq \beta \|\nabla x\|_p^p.$$

Proof. Suppose that the above inequality is not true. Then we can find $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that $\|\nabla x_n\|_p = 1$ and $\eta(x_n) \searrow 0$. Note that from (RQ) and the properties of $\vartheta \in L^\infty(Z)$ (hypothesis H(j)(iv)), we have that $\eta(x) \geq 0$ for all $x \in W_0^{1,p}(Z)$. From Poincare's inequality, we have that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded and so by passing to a subsequence if necessary, we may assume that $x_n \longrightarrow x$ weakly in $W_0^{1,p}(Z)$, $x_n \longrightarrow x$ in $L^p(Z)$, $x_n(z) \longrightarrow x(z)$ almost everywhere on Z and $|x_n(z)| \leq \chi(z)$ almost everywhere on Z for all $n \geq 1$ with some $\chi \in L^p(Z)$. Exploiting the weak lower semicontinuity of the norm in a Banach space and the fact that $\eta(x) \geq 0$ for all $x \in W_0^{1,p}(Z)$, we have that

$$\begin{aligned} 0 = \lim_{n\to+\infty} & \Big\{ \|\nabla x_n\|_p^p - \int_Z \big(\lambda_1 + \vartheta(z)\big)|x_n(z)|^p \, dz \Big\} \\ & \geq \|\nabla x\|_p^p - \int_Z \big(\lambda_1 + \vartheta(z)\big)|x(z)|^p \, dz \geq 0, \end{aligned}$$

thus using also hypothesis H(j)(iv) and (RQ), we get

$$\|\nabla x\|_p^p = \int_Z \big(\lambda_1 + \vartheta(z)\big)|x(z)|^p \, dz = \lambda_1 \|x\|_p^p.$$

So $x = 0$ or $x = \pm u_1$. But if $x = \pm u_1$, then from the positivity of u_1 and the properties of ϑ, we would have that

$$\int_Z \big(\lambda_1 + \vartheta(z)\big)|x(z)|^p \, dz < \lambda_1 \|x\|_p^p,$$

a contradiction. Therefore $x = 0$ and we have that $\|\nabla x_n\|_p \longrightarrow 0$. As also $\nabla x_n \longrightarrow 0$ weakly in $L^p(Z;\mathbb{R}^N)$ and space $L^p(Z;\mathbb{R}^N)$ is uniformly convex, from the Kadec–Klee property (see Hu–Papageorgiou [11], Definition I.1.72(d) and Lemma I.1.74, p. 28), we get that

$$\nabla x_n \longrightarrow 0 \ \text{ in } \ L^p(Z;\mathbb{R}^N),$$

which is a contradiction to the fact that $\|\nabla x_n\|_p = 1$ for $n \geq 1$. This implies our proposition. $\square$

Proposition 3.2. *If hypotheses* $\mathrm{H}(j)$ *hold, then for every* $\varepsilon > 0$, *there exists* $\gamma \in L^\infty(Z)_+$, *such that for all* $u(z,\zeta) \in \partial j(z,\zeta)$, *we have*

$$u(z,\zeta) \leq (\vartheta(z)+\varepsilon)|\zeta|^{p-2}\zeta + \gamma(z) \quad \textit{for a.a. } z \in Z \quad \textit{and all } \ \zeta \geq 0,$$

$$u(z,\zeta) \geq (\vartheta(z)+\varepsilon)|\zeta|^{p-2}\zeta - \gamma(z) \quad \textit{for a.a. } z \in Z \quad \textit{and all } \ \zeta \leq 0.$$

Proof. By virtue of hypothesis $\mathrm{H}(j)$(iv), for a given $\varepsilon > 0$, we can find $M = M(\varepsilon) > 0$, such that for almost all $z \in Z$ and all $u(z,\zeta) \in \partial j(z,\zeta)$, we have

$$u(z,\zeta) \leq (\vartheta(z)+\varepsilon)|\zeta|^{p-2}\zeta \quad \text{for} \ \ \zeta \geq M,$$

$$u(z,\zeta) \geq (\vartheta(z)+\varepsilon)|\zeta|^{p-2}\zeta \quad \text{for} \ \ \zeta \leq -M.$$

On the other hand, from $\mathrm{H}(j)$(iii), for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, such that $|\zeta| < M$, we have that

$$|u(z,\zeta)| \leq a(z) + cM^{r-1}.$$

Combining all above estimates, we obtain our proposition with
$\gamma \overset{\mathrm{df}}{=} |\vartheta(z)+\varepsilon|M^{p-1} + a(z) + cM^{r-1}$. $\square$

Let $\beta > 0$ be as in Proposition 3.1 and $\gamma \in L^\infty(Z)_+$ as in Proposition 3.2, with $\varepsilon = \frac{\beta\lambda_1}{2}$. Let us consider the following auxiliary nonlinear problem:

$$\begin{cases} -\operatorname{div}(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)) - \lambda_1|x(z)|^{p-2}x(z) \\ \qquad = \left(\vartheta(z) + \dfrac{\beta\lambda_1}{2}\right)|x(z)|^{p-2}x(z) + \gamma(z) \quad \text{a.e. on } Z \\ x|_\Gamma = 0. \end{cases} \tag{$\mathrm{HVI}_{\beta\gamma}$}$$

Proposition 3.3. *If hypotheses* $\mathrm{H}(j)$ *hold, then problem* $(\mathrm{HVI}_{\beta\gamma})$ *has a solution* $\overline{\phi} \in C^{1,\alpha}(\overline{Z})$ *(with* $0 < \alpha < 1$*), such that* $\overline{\phi}(z) > 0$ *for all* $z \in Z$ *and* $\frac{\partial\overline{\phi}}{\partial n}(z') < 0$ *for all* $z' \in \Gamma$ *where* $\overline{\phi}(z') = 0$ *(here* $n(z')$ *denotes the outward normal to* Γ *at* $z' \in \Gamma$*).*

PROOF. Let $A : W_0^{1,p}(Z) \longmapsto W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$\langle Ax, y\rangle \stackrel{\mathrm{df}}{=} \int_Z \|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}(\nabla x(z), \nabla y(z))_{\mathbb{R}^N}\, dz \quad \forall x, y \in W_0^{1,p}(Z)$$

(by $\langle \cdot\, , \cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z))$). It is easy to check that A is demicontinuous and strongly monotone, hence maximal monotone (see HU–PAPAGEORGIOU [11], Corollary III.1.35, p. 309). Let $J : W_0^{1,p}(Z) \longmapsto L^{p'}(Z) \subseteq W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$J(x)(\cdot) \stackrel{\mathrm{df}}{=} \left(\lambda_1 + \vartheta(\cdot) + \frac{\beta\lambda_1}{2}\right)|x(\cdot)|^{p-2}x(\cdot).$$

By virtue of the compactness of the embedding $W_0^{1,p}(Z) \subseteq L^p(Z)$, operator J is completely continuous. Hence it follows easily that the operator

$$V \stackrel{\mathrm{df}}{=} A - J : W_0^{1,p}(Z) \longrightarrow W^{-1,p'}(Z)$$

is pseudomonotone (see ZEIDLER [20], Proposition 27.6(a), p. 586 and Proposition 27.7(d), p. 588).

Next using Proposition 3.1 and (RQ), for every $x \in W_0^{1,p}(Z)$, we have

$$\begin{aligned}\langle Vx, x\rangle &= \|\nabla x\|_p^p - \int_Z \left(\lambda_1 + \vartheta(z) + \frac{\beta\lambda_1}{2}\right)|x(z)|^p\, dz\\ &\geq \beta\|\nabla x\|_p^p - \frac{\beta\lambda_1}{2}\|x\|_p^p \geq \frac{\beta}{2}\|\nabla x\|_p^p.\end{aligned}$$

So V is coercive. But recall that a pseudomonotone, coercive operator is surjective. Thus we can find $\overline{\phi} \in W_0^{1,p}(Z)$, such that $V\overline{\phi} = \gamma$. Now, we have

$$\langle A\overline{\phi}, y\rangle - \big(J\overline{\phi}, y\big)_{pp'} = (\gamma, y)_{pp'} \quad \forall y \in W_0^{1,p}(Z) \tag{3.1}$$

(where by $(\cdot\, , \cdot)_{pp'}$, we denote the duality brackets for the pair $(L^p(Z), L^{p'}(Z))$). Integrating (3.1) by parts, we get

$$\big\langle -\operatorname{div}\big(\|\nabla\overline{\phi}\|_{\mathbb{R}^N}^{p-2}\nabla\overline{\phi}\big), y\big\rangle - \big(J\overline{\phi}, y\big)_{pp'} = (\gamma, y)_{pp'}.$$

Note that by virtue of representation theorem for the elements of $W^{-1,p'}(Z) = \big(W_0^{1,p}(Z)\big)^*$, we have that

$$\operatorname{div}\big(\|\nabla\overline{\phi}\|^{p-2}\nabla\overline{\phi}\big) \in W^{-1,p'}(Z)$$

(see e.g. Hu–Papageorgiou [11], p. 866 or Showalter [18], p. 54). From the last equality which is true for every $y \in W_0^{1,p}(Z)$, we obtain that $\overline{\phi}$ is a solution of $(\mathrm{HVI}_{\beta\gamma})$.

Let $\overline{\phi}^- \overset{\mathrm{df}}{=} \max\{-\overline{\phi}, 0\}$ (the negative part of $\overline{\phi}$). From Gilbarg–Trudinger [8], p. 145 (see also Hu–Papageorgiou [11], p. 866), we know that $\overline{\phi}^- \in W_0^{1,p}(Z)$ and

$$\nabla\overline{\phi}^-(z) = \begin{cases} -\nabla\overline{\phi}(z) & \text{a.e. on } \{z \in Z : \overline{\phi}(z) < 0\} \\ 0 & \text{a.e. on } \{z \in Z : \overline{\phi}(z) \geq 0\}. \end{cases}$$

If in (3.1), we put $y = -\overline{\phi}^-$ as our test function, we obtain

$$\|\nabla\overline{\phi}^-\|_p^p - \int_Z \left(\lambda_1 + \vartheta(z) + \frac{\beta\lambda_1}{2}\right) |\overline{\phi}^-(z)|^p \, dz = \left(\gamma, -\overline{\phi}^-\right)_{pp'}.$$

But $\left(\gamma, -\overline{\phi}^-\right)_{pp'} \leq 0$ (as $\gamma \geq 0$), so using also Proposition 3.1 and (RQ), we have

$$\frac{\beta}{2}\|\nabla\overline{\phi}^-\|_p^p \leq 0.$$

Thus, we have that $\|\nabla\overline{\phi}^-\|_p = 0$, hence $\overline{\phi}^- \equiv 0$ (since $\overline{\phi}^- \in W_0^{1,p}(Z)$). Therefore $\overline{\phi} \geq 0$. Since $\overline{\phi} \in W_0^{1,p}(Z)_+$ is a solution of $(\mathrm{HVI}_{\beta\gamma})$, we have

$$-\mathrm{div}\big(\|\nabla\overline{\phi}(z)\|_{\mathbb{R}^N}^{p-2}\nabla\overline{\phi}(z)\big) - \left(\lambda_1 + \vartheta(z) + \frac{\beta\lambda_1}{2}\right) |\overline{\phi}(z)|^{p-2}\overline{\phi}(z) \geq 0$$

almost everywhere on Z, and so

$$\mathrm{div}\big(\|\nabla\overline{\phi}(z)\|_{\mathbb{R}^N}^{p-2}\nabla\overline{\phi}(z)\big) \leq \overline{M}|\overline{\phi}(z)|^{p-2}\overline{\phi}(z)$$

almost everywhere on Z, with $\overline{M} \overset{\mathrm{df}}{=} \|\lambda_1 + \vartheta(\cdot) + \frac{\beta\lambda_1}{2}\|_\infty$.

Because $\overline{\phi} \in W_0^{1,p}(Z)$ is a solution of $(\mathrm{HVI}_{\beta\gamma})$, from Theorem 7.1, p. 286 of Ladyzhenskaya–Uraltseva [13], we have that $\overline{\phi} \in L^\infty(Z)$ and then using Theorem 1 of Lieberman [14], we deduce that $\overline{\phi} \in C^{1,\alpha}(\overline{Z})$ for some $0 < \alpha < 1$. This fact permits the use of Theorem 5 of Vazguez [19] to obtain that $\overline{\phi}(z) > 0$ for all $z \in Z$ and if for some $z' \in \Gamma$, $\overline{\phi}(z') = 0$, then for the outward normal derivative $\frac{\partial\overline{\phi}}{\partial n}$, we have $\frac{\partial\overline{\phi}}{\partial n}(z') < 0$. Therefore the function $\overline{\phi} \in C^{1,\alpha}(\overline{Z})$ is the desired solution of $(\mathrm{HVI}_{\beta\gamma})$. $\square$

Analogously, we consider the following auxiliary problem:

$$\begin{cases} -\operatorname{div}(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)) - \lambda_1|x(z)|^{p-2}x(z) \\ \qquad = \left(\vartheta(z) + \dfrac{\beta\lambda_1}{2}\right)|x(z)|^{p-2}x(z) - \gamma(z) \quad \text{a.e. on } Z \\ x|_\Gamma = 0. \end{cases} \tag{$\mathrm{HVI}'_{\beta\gamma}$}$$

In a similar fashion, we can prove the following existence result for problem $(\mathrm{HVI}'_{\beta\gamma})$.

Proposition 3.4. *If hypotheses* $\mathrm{H}(j)$ *hold, then problem* $(\mathrm{HVI}'_{\beta\gamma})$ *has a solution* $\underline{\phi} \in C^{1,\alpha}(\overline{Z})$ *(with* $0 < \alpha < 1$*), such that* $\underline{\phi}(z) < 0$ *for all* $z \in Z$ *and* $\frac{\partial\underline{\phi}}{\partial n}(z') > 0$ *for all* $z' \in \Gamma$ *where* $\underline{\phi}(z') = 0$ *(again* $n(z')$ *is the outward normal to* Γ *at* $z' \in \Gamma$*).*

Now, we introduce the notions of upper and lower solutions which will be our basic tools in the existence theorems in Section 4. To this end, we define two functions $\underline{g}, \overline{g} : Z \times \mathbb{R} \longmapsto \mathbb{R}$, as follows

$$\underline{g}(z,\zeta) \stackrel{\mathrm{df}}{=} \inf\{\eta : \ \eta \in \partial j(z,\zeta)\}$$

$$\overline{g}(z,\zeta) \stackrel{\mathrm{df}}{=} \sup\{\eta : \eta \in \partial j(z,\zeta)\}.$$

By redefining j on a Lebesgue-null subset of Z, without any loss of generality, we may assume that j is Borel measurable and for all $z \in Z$, function $j(z,\cdot)$ is locally Lipschitz. From Section 2, we know that

$$\begin{aligned} j^0(z,\zeta;\xi) &= \limsup_{\substack{\zeta' \to \zeta \\ t \searrow 0}} \frac{j(z,\zeta' + t\xi) - j(z,\zeta')}{t} \\ &= \inf_{\varepsilon > 0} \sup_{\substack{|\zeta' - \zeta| < \varepsilon \\ 0 < t < \varepsilon;\ \zeta', t \in \mathbb{Q}}} \frac{j(z,\zeta' + t\xi) - j(z,\zeta')}{t} \end{aligned}$$

and so the function $Z \times \mathbb{R} \times \mathbb{R} \ni (z,\zeta,\xi) \longmapsto j^0(z,\zeta;\xi) \in \mathbb{R}$ is Borel measurable. Since

$$\partial j(z,\zeta) = \{\eta \in \mathbb{R} : \eta\xi \le j^0(z,\zeta;\xi) \quad \forall \xi \in \mathbb{R}\},$$

we have that

$$\operatorname{Gr}\partial j = \{(z,\zeta,\eta) \in Z \times \mathbb{R} \times \mathbb{R} : \eta \in \partial j(z,\zeta)\} \in \mathcal{B}(Z) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}),$$

with $\mathcal{B}(Z)$ (resp. $\mathcal{B}(\mathbb{R})$) being the Borel σ-field of Z (resp. $\mathbb{R}$). For every $\mu \in \mathbb{R}$, we have that

$$\big\{(z,\zeta) \in Z \times \mathbb{R} : \underline{g}(z,\zeta) < \mu\big\} = \operatorname{proj}_{Z\times\mathbb{R}}\big(\operatorname{Gr}\partial j \cap (Z \times \mathbb{R} \times (-\infty,\mu))\big).$$

Since the subdifferential multifunction has compact values, from Theorem II.1.22, p. 146 of Hu–Papageorgiou [11], we infer that the above projection belongs to $\mathcal{B}(Z \times \mathbb{R}) = \mathcal{B}(Z) \times \mathcal{B}(\mathbb{R})$. Hence $\underline{g}$ is measurable. Similarly we obtain that $\overline{g}$ is measurable. Note that by virtue of hypotheses H(j)(iii) and (iv), for every $x \in W^{1,p}(Z)$, we have that $\underline{g}(\cdot,x(\cdot))$, $\overline{g}(\cdot,x(\cdot)) \in L^{p'}(Z)$.

Definition 3.5. (a) A function $\overline{w} \in W^{1,p}(Z)$ is an *"upper solution"* of (HVI), if $\overline{w}|_\Gamma \geq 0$ and

$$\int_Z \|\nabla\overline{w}\|_{\mathbb{R}^N}^{p-2}(\nabla\overline{w},\nabla y)_{\mathbb{R}^N}\,dz - \lambda_1\int_Z |\overline{w}|^{p-2}\overline{w}y\,dz \geq \int_Z \overline{g}(z,\overline{w}(z))y(z)\,dz$$

for all $y \in W_0^{1,p}(Z)$, such that $y \geq 0$.

(b) A function $\underline{w} \in W^{1,p}(Z)$ is a *"lower solution"* of (HVI), if $\underline{w}|_\Gamma \leq 0$ and

$$\int_Z \|\nabla\underline{w}\|_{\mathbb{R}^N}^{p-2}(\nabla\underline{w},\nabla y)_{\mathbb{R}^N}\,dz - \lambda_1\int_Z |\underline{w}|^{p-2}\underline{w}y\,dz \leq \int_Z \underline{g}(z,\underline{w}(z))y(z)\,dz$$

for all $y \in W_0^{1,p}(Z)$, such that $y \geq 0$.

Now by virtue of hypothesis H(j)(v), we can find $\delta > 0$, such that for almost all $z \in Z$ and all $u(z,\zeta) \in \partial j(z,\zeta)$, we have

$$u(z,\zeta) \geq 0 \quad \text{if } 0 < \zeta \leq \delta \tag{3.2}$$

$$u(z,\zeta) \leq 0 \quad \text{if } -\delta \leq \zeta < 0. \tag{3.3}$$

Let $u_1 \in C^{1,\alpha'}(\overline{Z})$ (with $0 < \alpha' < 1$), be the principal eigenfunction of $(-\Delta_p, W_0^{1,p}(Z))$. Recall that $u_1(z) > 0$ for all $z \in Z$. We can find $0 < c_1 < 1$, such that $c_1u_1(z) \leq \delta$ for all $z \in \overline{Z}$. Also since by Proposition 3.3, we have that $\overline{\phi} \in C^{1,\alpha}(\overline{Z})$ is such that $\overline{\phi}(z) > 0$ for all $z \in Z$

and $\frac{\partial\overline{\phi}}{\partial n}(z') < 0$ for all $z' \in \Gamma$ where $\overline{\phi}(z') = 0$, we infer that there exists $c_2 > 1$, such that $c_1 u_1(z) < c_2\overline{\phi}(z)$ for all $z \in Z$. So $\frac{c_1}{c_2}u_1(z) < \overline{\phi}(z)$ and $\frac{c_1}{c_2}u_1(z) \leq \delta$ for all $z \in Z$ (recall that $0 < c_1 < 1 < c_2$). Let us set $\underline{w} \stackrel{\mathrm{df}}{=} \frac{c_1}{c_2}u_1 \in C^{1,\alpha'}(\overline{Z})$. For all $y \in W_0^{1,p}(Z)$, such that $y \geq 0$, we have

$$\int_Z \|\nabla\underline{w}\|^{p-2}\big(\nabla\underline{w}, \nabla y\big)_{\mathbb{R}^N} dz - \lambda_1 \int_Z |\underline{w}|^{p-2}\underline{w} y\, dz = 0 \leq \int_Z \underline{g}(z, \underline{w}(z)) y(z) dz$$

since $0 < \underline{w}(z) \leq \delta$ for all $z \in Z$ (see (3.2)). So the function $\underline{w} \in C^{1,\alpha'}(\overline{Z})$ is a positive lower solution of (HVI).

On the other hand, since $\overline{\phi} \in C^{1,\alpha}(\overline{Z})$ is a solution of ($\mathrm{HVI}_{\beta\gamma}$) with $\overline{\phi}(z) > 0$ for all $z \in Z$, so for all $y \in W_0^{1,p}(Z)$, such that $y \geq 0$, we have

$$\begin{aligned} &\int_Z \|\nabla\overline{\phi}\|_{\mathbb{R}^N}^{p-2}\big(\nabla\overline{\phi}, \nabla y\big)_{\mathbb{R}^N}\, dz - \lambda_1 \int_Z |\overline{\phi}|^{p-2}\overline{\phi} y\, dz \\ &\quad = \int_Z \left(\vartheta(z) + \frac{\beta\lambda_1}{2}\right) |\overline{\phi}|^{p-2}\overline{\phi} y\, dz + \int_Z \gamma y\, dz \geq \int_Z \overline{g}\big(z, \overline{\phi}(z)\big) y(z)\, dz \end{aligned}$$

(see Proposition 3.2 and recall that γ was chosen for $\varepsilon = \frac{\beta\lambda_1}{2}$). Therefore $\overline{\phi} \in C^{1,\alpha}(\overline{Z})$ is a positive upper solution of (HVI) and $\underline{w} < \overline{\phi}$.

Similarly, we can find $0 < c_3 < 1$ such that $-\delta \leq c_3(-u_1)(z) < 0$ for all $z \in Z$. Also because of Proposition 3.4, we can find $c_4 > 1$, such that $c_4\underline{\phi}(z) < c_3(-u_1)(z)$ for all $z \in Z$. Set $\overline{w}(z) \stackrel{\mathrm{df}}{=} -\frac{c_3}{c_4}u_1(z)$. Evidently $-\delta \leq \overline{w}(z) < 0$ for all $z \in Z$. Therefore, for all $y \in W_0^{1,p}(Z)$, such that $y \geq 0$, we have

$$\begin{aligned} \int_Z \|\nabla\overline{w}\|^{p-2}\big(\nabla\overline{w}, \nabla y\big)_{\mathbb{R}^N} dz - \lambda_1 \int_Z |\overline{w}|^{p-2}\overline{w} y\, dz = 0 \\ \geq \int_Z \overline{g}\big(z, \overline{w}(z)\big) y(z) dz \end{aligned}$$

(see (3.3)). Hence $\overline{w} \in C^{1,\alpha'}(\overline{Z})$ is a negative upper solution of (HVI).

On the other hand, since $\underline{\phi} \in C^{1,\alpha}(\overline{Z})$ is a solution of ($\mathrm{HVI}'_{\beta\gamma}$) with $\underline{\phi}(z) < 0$ for all $z \in Z$, so for all $y \in W_0^{1,p}(Z)$, such that $y \geq 0$, we have

$$\begin{aligned} &\int_Z \|\nabla\underline{\phi}\|_{\mathbb{R}^N}^{p-2}\big(\nabla\underline{\phi}, \nabla y\big)_{\mathbb{R}^N}\, dz - \lambda_1 \int_Z |\underline{\phi}|^{p-2}\underline{\phi} y\, dz \\ &\quad = \int_Z \left(\vartheta(z) + \frac{\beta\lambda_1}{2}\right) |\underline{\phi}|^{p-2}\underline{\phi} y\, dz - \int_Z \gamma y\, dz \leq \int_Z \underline{g}\big(z, \underline{\phi}(z)\big) y(z)\, dz \end{aligned}$$

(see Proposition 3.2). Therefore $\underline{\phi} \in C^{1,\alpha}(\overline{Z})$ is a negative lower solution of (HVI) and $\underline{\phi} < \overline{w}$.

4. Positive and negative solutions

In this section, using the method of upper and lower solutions for the ordered upper-lower solution pairs $\{\overline{\phi}, \underline{w}\}$ and $\{\overline{w}, \underline{\phi}\}$, we will produce two bounded solutions of (HVI), one positive and the other negative.

Theorem 4.1. *If hypotheses* H(j) *hold, then problem* (HVI) *has a solution* $\overline{x} \in W_0^{1,p}(Z) \cap L^\infty(Z)$, *such that* $\overline{x}(z) > 0$ *for all* $z \in Z$.

Proof. We introduce the truncation map $\tau : W_0^{1,p}(Z) \longmapsto W_0^{1,p}(Z)$, defined by

$$\tau(x)(z) \stackrel{\mathrm{df}}{=} \begin{cases} \overline{\phi}(z) & \text{if } \overline{\phi}(z) \leq x(z) \\ x(z) & \text{if } \underline{w}(z) \leq x(z) \leq \overline{\phi}(z) \\ \underline{w}(z) & \text{if } x(z) \leq \underline{w}(z). \end{cases}$$

It is easy to see that τ is continuous and bounded. Also it is such as treated as a map from $L^p(Z)$ into itself and for any $x \in W_0^{1,p}(Z)$, we have that

$$\|\tau(x)\|_p^p \leq \|x\|_p^p + c_5, \tag{4.1}$$

with $c_5 \stackrel{\mathrm{df}}{=} \|\underline{w}\|_p^p$. Next, let $\pi : Z \times \mathbb{R} \longmapsto \mathbb{R}$ be the penalty function, defined by

$$\pi(z,\zeta) \stackrel{\mathrm{df}}{=} \begin{cases} (\zeta - \overline{\phi}(z))^{p-1} & \text{if } \overline{\phi}(z) \leq \zeta \\ 0 & \text{if } \underline{w}(z) \leq \zeta \leq \overline{\phi}(z) \\ -(\underline{w}(z) - \zeta)^{p-1} & \text{if } \zeta \leq \underline{w}(z). \end{cases}$$

From the above definition, it is clear that $\pi(z,\zeta)$ is a Caratheodory function (i.e. measurable in $z \in Z$ and continuous in $\zeta \in \mathbb{R}$), nondecreasing in $\zeta \in \mathbb{R}$.

Let $A : W_0^{1,p}(Z) \longmapsto W^{-1,p'}(Z)$ be the maximal monotone operator introduced in the proof of Proposition 3.3. Also let J_1, $N_\pi : W_0^{1,p}(Z) \longmapsto L^{p'}(Z)$, be defined by

$$J_1 x(\cdot) \stackrel{\mathrm{df}}{=} \lambda_1 |\tau(x)(\cdot)|^{p-2} \tau(x)(\cdot), \quad N_\pi x(\cdot) \stackrel{\mathrm{df}}{=} \pi(\cdot, x(\cdot)).$$

Both these operators are continuous and bounded (recall that τ is continuous). From Hölder inequality and Young inequality for any $x \in W_0^{1,p}(Z)$, we have

$$\begin{aligned}
\big(J_1 x, x\big)_{pp'} &= \lambda_1 \int_Z |\tau(x)(z)|^{p-2}\tau(x)(z)x(z)\,dz \\
&\leq \lambda_1 \left(\int_Z |\tau(x)(z)|^{(p-1)p'}\,dz\right)^{\frac{1}{p'}} \left(\int_Z |x(z)|^p\,dz\right)^{\frac{1}{p}} \\
&\leq \frac{\lambda_1}{p'}\|\tau(x)\|_p^p + \frac{\lambda_1}{p}\|x\|_p^p
\end{aligned}$$

and so using also (4.1), we get

$$(J_1 x, x)_{pp'} \leq \lambda_1\|x\|_p^p + c_6 \quad \forall x \in W_0^{1,p}(Z), \tag{4.2}$$

with $c_6 \overset{\text{df}}{=} \frac{\lambda_1 c_5}{p'}$. Next for any $x \in W_0^{1,p}(Z)$, we have

$$\begin{aligned}
\big(N_\pi x, x\big)_{pp'} &= \int_{\{\overline{\phi}\leq x\}} \big(x(z) - \overline{\phi}(z)\big)^{p-1} x(z)\,dz \\
&\quad - \int_{\{x\leq \underline{w}\}} \big(\underline{w}(z) - x(z)\big)^{p-1} x(z)\,dz \\
&\geq \int_{\{\overline{\phi}\leq x\}} \big(x(z) - \overline{\phi}(z)\big)^{p}\,dz \\
&\quad - \int_{\{0<x\leq \underline{w}\}} |\underline{w}(z)|^p\,dz + \int_{\{x\leq 0\}} |x(z)|^p\,dz \\
&\quad + \frac{1}{2^{p-1}}\int_{\{0<x<\overline{\phi}\}} |x(z)|^p\,dz - \frac{1}{2^{p-1}}\int_{\{0<x<\overline{\phi}\}} |\overline{\phi}(z)|^p\,dz,
\end{aligned}$$

so, using the inequality $|a-b|^p \geq \frac{1}{2^{p-1}}|a|^p - |b|^p$ (valid for all $a, b \in \mathbb{R}$), we get

$$\big(N_\pi x, x\big)_{pp'} \geq \frac{1}{2^{p-1}}\|x\|_p^p - c_7 \quad \forall x \in W_0^{1,p}(Z), \tag{4.3}$$

with $c_7 \overset{\text{df}}{=} \|\overline{\phi}\|_p^p + \|\underline{w}\|_p^p$.

Also, let $G : W_0^{1,p}(Z) \longmapsto 2^{L^{r'}(Z)} \setminus \{\emptyset\}$ be defined by

$$Gx \overset{\text{df}}{=} \big\{u \in L^{r'}(Z) : u(z) \in \partial j(z, \tau(x)(z)) \text{ for a.a. } z \in Z\big\}.$$

Since the multifunction $Z \ni z \longmapsto \partial j(z, \tau(x)(z)) \in 2^{L^{r'}(Z)}$ is graph measurable, invoking the Yankov–von Neumann–Aumann selection theorem (see Hu–Papageorgiou [11], p. 158) and using hypothesis H(j)(iii), we have that for all $x \in W_0^{1,p}(Z)$, set Gx is nonempty, convex and w-compact in $L^{r'}(Z)$. Since $L^{r'}(Z)$ is embedded continuously in $W^{-1,p'}(Z)$, we have that Gx is also nonempty, convex and w-compact in $W^{-1,p'}(Z)$.

Using estimate analogous as in Proposition 3.2 and estimate (4.1), for any $x \in W_0^{1,p}(Z)$ and any $u \in Gx$ (note that from hypotheses H(j)(iii) and (iv), we get that $u \in L^{p'}(Z)$), we have

$$\begin{aligned}\|u\|_{p'}^{p'} &\le 2^{p'-1}\int_Z \left(|\vartheta(z)-1|^{p'}|\tau(x)(z)|^p + |\gamma(z)|^{p'}\right)dz\\ &\le c_8\|x\|_p^p + c_9,\end{aligned}$$

where $c_8 \stackrel{\mathrm{df}}{=} 2^{p'-1}\|\vartheta-1\|_\infty^{p'}$ and $c_9 \stackrel{\mathrm{df}}{=} c_8c_5 + 2^{p'-1}|Z|\|\gamma\|_\infty^{p'}$ and also

$$(u,x)_{pp'} \le \|u\|_{p'}\|x\|_p \le \frac{1}{p'}\|u\|_{p'}^{p'} + \frac{1}{p}\|x\|_p^p,$$

so finally, we obtain

$$(u,x)_{pp'} \le c_{10}\|x\|_p^p + c_{11}, \tag{4.4}$$

where $c_{10} \stackrel{\mathrm{df}}{=} \frac{c_8}{p'} + \frac{1}{p}$ and $c_{11} \stackrel{\mathrm{df}}{=} \frac{c_9}{p'}$.

Let $\mu \stackrel{\mathrm{df}}{=} 2^{p-1}(\lambda_1 + c_{10})$. We consider the following auxiliary nonlinear hemivariational inequality:

$$\begin{cases} -\mathrm{div}(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)) - \lambda_1|\tau(x)(z)|^{p-2}\tau(x)(z) & \\ \qquad +\mu\pi(z,x(z)) \in \partial j(z,\tau(x)(z)) & \text{a.e. on } Z \\ x|_\Gamma = 0. & \end{cases} \tag{HVI_π}$$

Now, let $K : W_0^{1,p}(Z) \longmapsto 2^{W^{-1,p'}(Z)} \setminus \{\emptyset\}$ be the multifunction with convex and w-compact values, defined by

$$Kx = Ax - J_1x + \mu N_\pi x - Gx.$$

We will show that K is pseudomonotone and coercive. Since K is everywhere defined and bounded, to show the pseudomonotonicity of K, it suffices to show that K is generalized pseudomonotone (see Section 2).

To this end let $x_n \longrightarrow x$ weakly in $W_0^{1,p}(Z)$ and $x_n^* \longrightarrow x^*$ weakly in $W^{-1,p'}(Z)$, with $x_n^* \in Kx_n$ for $n \geq 1$ and assume that

$$\limsup_{n\to+\infty} \langle x_n^*, x_n - x\rangle \leq 0.$$

By definition, for every $n \geq 1$, we have that

$$x_n^* = Ax_n - J_1x_n + \mu N_\pi x_n - u_n \quad \text{with } u_n \in Gx_n.$$

From the compactness of the embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$ and $L^r(Z)$, we see that $x_n \longrightarrow x$ in $L^p(Z)$ and $L^r(Z)$ and passing to a subsequence if necessary, we have that $x_n(z) \longrightarrow x(z)$ almost everywhere on Z and $|\tau(x_n)(z)| \leq \chi(z)$ almost everywhere on Z with $\chi \in L^s(Z)$, with $s = \max\{r,p\}$. So, we have that

$$\begin{aligned}
\langle J_1x_n, x_n - x\rangle &= \big(J_1x_n, x_n - x\big)_{pp'} \longrightarrow 0\\
\langle N_\pi x_n, x_n - x\rangle &= \big(N_\pi x_n, x_n - x\big)_{pp'} \longrightarrow 0\\
\langle u_n, x_n - x\rangle &= \big(u_n, x_n - x\big)_{rr'} \quad\;\; \longrightarrow 0
\end{aligned}$$

(recall that $\tau(\cdot)$ is continuous and bounded and $\{u_n\}_{n\geq 1} \subseteq L^{r'}(Z)$ is bounded). We obtain that

$$\limsup_{n\to+\infty} \langle Ax_n, x_n - x\rangle \leq 0.$$

But A being maximal monotone, is also generalized pseudomonotone and so, we have that $Ax_n \longrightarrow Ax$ weakly in $W^{-1,p'}(Z)$ and $\langle Ax_n, x_n\rangle \longrightarrow \langle Ax, x\rangle$. Note that

$$\langle J_1x_n, x\rangle = \big(J_1x_n, x\big)_{pp'} \longrightarrow \big(J_1x, x\big)_{pp'} = \langle J_1x, x\rangle,$$

$$\langle N_\pi x_n, x\rangle = \big(N_\pi x_n, x\big)_{pp'} \longrightarrow \big(N_\pi x, x\big)_{pp'} = \langle N_\pi x, x\rangle.$$

Then

$$u_n = -x_n^* + Ax_n - J_1x_n + \mu N_\pi x_n \;\longrightarrow\; -x^* + Ax - J_1x + \mu N_\pi x = u$$

weakly in $W^{-1,p'}(Z)$. Moreover, since $\{u_n\}_{n\geq 1} \subseteq L^{r'}(Z)$ is bounded (see hypothesis H(j)(iii)), passing to a subsequence if necessary, we have that

$u_n \longrightarrow u$ weakly in $L^{r'}(Z)$. Invoking Proposition VII.3.9, p. 694 of Hu–Papageorgiou [11], we have that

$$u(z) \in \overline{\operatorname{conv}} \limsup_{n\to+\infty} \partial j(z,\tau(x_n)(z)) \subseteq \partial j(z,\tau(x)(z)) \quad \text{a.e. on } Z,$$

where the last inclusion is a consequence of the fact that $\operatorname{Gr} \partial j(z,\cdot)$ is closed in $\mathbb{R}\times\mathbb{R}$. Hence $u \in Gx$ and so $x^* \in Kx$ and $\langle x_n^*, x_n\rangle \longrightarrow \langle x^*, x\rangle$. This proves the generalized pseudomonotonicity, thus the pseudomonotonicity of K.

Next, we will show that operator K is coercive. To this end, let $x \in W_0^{1,p}(Z)$ and $x^* \in Kx$. We have

$$x^* = Ax - J_1 x + \mu N_\pi x - u \quad \text{with } u \in Gx.$$

Using (4.2), (4.3) and (4.4), we have

$$\begin{aligned}
\langle x^*, x\rangle &= \langle Ax, x\rangle - \big(J_1 x, x\big)_{pp'} + \mu\big(N_\pi x, x\big)_{pp'} - (u,x)_{rr'} \\
&\geq \|\nabla x\|_p^p - \lambda_1\|x\|_p^p - c_6 + \frac{\mu}{2^{p-1}}\|x\|_p^p - \mu c_7 - c_{10}\|x\|_p^p - c_{11} \\
&= \|\nabla x\|_p^p - c_6 - \mu c_7 - c_{11},
\end{aligned}$$

(recall that $\mu = 2^{p-1}(\lambda_1 + c_{10})$). It follows, that K is coercive. Recall that pseudomonotone, coercive map is surjective. Thus, we can find $\overline{x} \in W_0^{1,p}(Z)$, such that $0 \in K\overline{x}$. As in the proof of Proposition 3.3, we can check that $\overline{x}$ is a solution of (HVI_π). So for some $u^* \in G\overline{x}$, we have that

$$\begin{cases}
-\operatorname{div}(\|\nabla\overline{x}(z)\|_{\mathbb{R}^N}^{p-2}\nabla\overline{x}(z)) - \lambda_1|\tau(\overline{x})(z)|^{p-2}\tau(\overline{x})(z) & \\
\qquad +\mu\pi(z,\overline{x}(z)) = u^*(z) & \text{a.e. on } Z \\
\overline{x}|_\Gamma = 0 & \\
u^*(z) \in \partial j(z,\tau(\overline{x})(z)) & \text{a.e. on } Z.
\end{cases} \tag{HVI'_π}$$

On the other hand, since $\underline{w} \in C^{1,\alpha'}(\overline{Z})$ is a lower solution of (HVI), we have that

$$\begin{aligned}
&\int_Z \|\nabla\underline{w}\|_{\mathbb{R}^N}^{p-2}\big(\nabla\underline{w},\nabla y\big)_{\mathbb{R}^N}\,dz - \lambda_1\int_Z |\underline{w}|^{p-2}\underline{w}y\,dz \\
&\qquad \leq \int_Z \underline{g}\big(z,\underline{w}(z)\big)y(z)\,dz \quad \forall y \in W_0^{1,p}(Z),\ y \geq 0.
\end{aligned} \tag{4.5}$$

From (HVI'_π) and (4.5) and using as test function

$$y = (\underline{w} - \overline{x})^+ = \max\{\underline{w} - \overline{x}, 0\} \in W_0^{1,p}(Z)$$

(see GILBARG–TRUDINGER [8], p. 145), we obtain

$$\begin{aligned} &\int_Z \big(\|\nabla \underline{w}\|_{\mathbb{R}^N}^{p-2} \nabla \underline{w} - \|\nabla \overline{x}\|_{\mathbb{R}^N}^{p-2} \nabla \overline{x}, \nabla(\underline{w} - \overline{x}\,)^+\big)_{\mathbb{R}^N}\, dz \\ &\qquad - \lambda_1 \int_Z \big(|\underline{w}|^{p-2}\underline{w} - |\tau(\overline{x})|^{p-2}\tau(\overline{x})\big)(\underline{w} - \overline{x})^+\, dz \\ &\qquad - \mu \int_Z \pi(z, \overline{x}(z))(\underline{w} - \overline{x})^+(z)\, dz \\ &\quad \le \int_Z \big(\underline{g}(z, \underline{w}(z)) - u^*(z)\big)(\underline{w} - \overline{x})^+(z)\, dz. \end{aligned} \tag{4.6}$$

We know that

$$\nabla(\underline{w} - \overline{x})^+(z) = \begin{cases} \nabla(\underline{w} - \overline{x})(z) & \text{a.e. on } \{\underline{w} > \overline{x}\} \\ 0 & \text{a.e. on } \{\underline{w} \le \overline{x}\}. \end{cases}$$

So

$$\int_Z \big(\|\nabla \underline{w}\|_{\mathbb{R}^N}^{p-2} \nabla \underline{w} - \|\nabla \overline{x}\|_{\mathbb{R}^N}^{p-2} \nabla \overline{x}, \nabla(\underline{w} - \overline{x}\,)^+\big)_{\mathbb{R}^N}\, dz \ge 0.$$

From the definition of τ, we have

$$\begin{aligned} &\lambda_1 \int_Z \big(|\underline{w}|^{p-2}\underline{w} - |\tau(\overline{x})|^{p-2}\tau(\overline{x})\big)(\underline{w} - \overline{x})^+\, dz \\ &= \lambda_1 \int_{\{\underline{w} > \overline{x}\}} \big(|\underline{w}|^{p-2}\underline{w} - |\underline{w}|^{p-2}\underline{w}\big)(\underline{w} - \overline{x})\, dz = 0. \end{aligned}$$

Since $u^*(z) \in \partial j(z, \tau(\overline{x})(z)) = \partial j(z, \underline{w}(z))$ almost everywhere on $\{\underline{w} > \overline{x}\}$, so $\underline{g}(z, \underline{w}(z)) \le u^*(z)$ and

$$\begin{aligned} &\int_Z \big(\underline{g}(z, \underline{w}(z)) - u^*(z)\big)(\underline{w} - \overline{x})^+(z)\, dz \\ &= \int_{\{\underline{w} > \overline{x}\}} \big(\underline{g}(z, \underline{w}(z)) - u^*(z)\big)(\underline{w} - \overline{x})(z)\, dz \le 0. \end{aligned}$$

Using these facts in (4.6), we obtain

$$-\mu \int_Z \pi(z, \overline{x}(z))(\underline{w} - \overline{x})^+(z)\, dz \le 0,$$

so from the definition of π, we have

$$\int_Z \left[(\underline{w} - \overline{x})^+(z)\right]^p \, dz \leq 0.$$

From this inequality, it follows that $\underline{w}(z) \leq \overline{x}(z)$ almost everywhere on Z. Similarly, we show that $\overline{x}(z) \leq \overline{\phi}(z)$ almost everywhere on Z. Then, we have that $\overline{x} \in W_0^{1,p}(Z) \cap L^\infty(Z)$ and $\tau(\overline{x}) = \overline{x}$, $\pi(z, \overline{x}(z)) = 0$, which imply that $\overline{x}$ is a bounded, positive solution on (HVI). $\square$

In a similar fashion, working with the ordered pair of upper-lower solutions $\{\overline{w}, \underline{\phi}\}$, we obtain the following result:

Theorem 4.2. *If hypotheses* H(j) *hold, then problem* (HVI) *has a solution* $\underline{x} \in W_0^{1,p}(Z) \cap L^\infty(Z)$, *such that* $\underline{x}(z) < 0$ *for all* $z \in Z$.

Putting together Theorems 4.1 and 4.2, we have the following multiplicity result for problem (HVI).

Theorem 4.3. *If hypotheses* H(j) *hold, then problem* (HVI) *has at least two bounded solutions* $\overline{x}, \underline{x} \in W_0^{1,p}(Z) \cap L^\infty(Z)$, *such that* $\underline{x}(z) < 0 < \overline{x}(z)$ *for all* $z \in Z$.

We conclude with a simple example of a nonsmooth potential which satisfies hypotheses H(j). Let $\vartheta \in L^\infty(Z)$ be such that $\vartheta(z) \leq 0$ almost everywhere on Z with strict inequality on a set of positive measure and let us define

$$j(z,\zeta) \stackrel{\text{df}}{=} \begin{cases} \dfrac{1}{p}|\zeta|^p & \text{if } |\zeta| \leq 1, \\ \dfrac{1}{p\zeta^2} + \frac{\vartheta(z)}{p}|\zeta|^p - \dfrac{\vartheta(z)}{p} & \text{if } |\zeta| > 1. \end{cases}$$

From Clarke [3], p. 34, we have that

$$\partial j(z,\zeta) = \begin{cases} |\zeta|^{p-2}\zeta & \text{if } |\zeta| < 1, \\ [\vartheta(z) - 2, 1] & \text{if } \zeta = 1, \\ [-1, 2 - \vartheta(z)] & \text{if } \zeta = -1, \\ -\dfrac{2}{p\zeta^3} + \vartheta(z)|\zeta|^{p-2}\zeta & \text{if } |\zeta| > 1. \end{cases}$$

So for every $u(z,\zeta) \in \partial j(z,\zeta)$, we have that

$$\limsup_{|\zeta|\to+\infty} \frac{u(z,\zeta)}{|\zeta|^{p-2}\zeta} = \limsup_{|\zeta|\to+\infty} \left[-\frac{2}{p|\zeta|^{p+2}} + \vartheta(z) \right] = \vartheta(z),$$

uniformly for almost all $z \in Z$. Also

$$\liminf_{\zeta\to 0} \frac{u(z,\zeta)}{|\zeta|^{p-2}\zeta} = \liminf_{\zeta\to 0} \frac{|\zeta|^{p-2}\zeta}{|\zeta|^{p-2}\zeta} = 1 > 0.$$

Thus hypotheses H(j) are satisfied with $r = p < p^*$.

Finally, we remark that our formulation incorporates problems with discontinuities such as the ones studied by Chang in [2].

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