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Additive preservers on Banach algebras

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Abstract. It is shown that an additive, surjective mapping $\Phi : \mathrm{soc}(\mathcal{A}) \to \mathrm{soc}(\mathcal{A})$, preserving rank-one idempotents and their linear spans in both directions, is a real-linear Jordan isomorphism provided that $\mathcal{A}$ is a semiprime Banach algebra with no nonzero central elements in its socle.

0. Introduction

The problem of determining all linear maps on a given algebra, preserving certain algebraic properties, was first considered long ago. Historically, the authors focused their attention on matrix algebras $\mathcal{M}_n(\mathbb{F})$; recently, however, there seems to be an increasing interest in the investigation of more general algebras, especially $\mathscr{B}(\mathcal{X})$, i.e., the algebra of bounded linear operators on a Banach space. As a sample result we mention that a surjective mapping $\Phi : \mathscr{B}(\mathcal{X}) \to \mathscr{B}(\mathcal{X})$, preserving the spectrum, takes just two forms: Either $\Phi(x) = a^{-1}xa$ or else $\Phi(x) = c^{-1}x^*c$; here, $a \in \mathscr{B}(\mathcal{X})$, and $c \in \mathscr{B}(\mathcal{X}, \mathcal{X}^*)$, respectively (cf. [5, Thm. 2.5] and references therein). We remark that both forms satisfy the equation $\Phi(x^2) = \Phi(x)^2$; such mappings are called *Jordan homomorphisms*.

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Some of the techniques used to attack a given “preserver problem” are collected in a survey article [5]: Often, one finds that it also preserves *rank-one elements* of some special kind. After this is established, one “looks at a database of rank-one preservers”, which usually consists of just a few entries, and then tries to generalize to the whole algebra. It is of course natural to try to relax the assumptions on a preserver Φ to as few as possible. This was done, say, in [11] where the authors showed that a surjective, *additive mapping* Φ, defined on the ideal $\mathscr{F}(\mathcal{X})$ of finite-rank operators in $\mathscr{B}(\mathcal{X})$, and preserving rank-one idempotents and their linear spans, is real-linear Jordan isomorphism, provided that $\dim \mathcal{X} = \infty$ (however, cf. also [9]).

As it was indicated in [13], [3], [4], the concept of rank can be extended to the socle of semiprime algebras. This enables us to extend the result of [11] to the additive mappings, defined on a socle of *any semiprime Banach algebra* $\mathcal{A}$. First, we demonstrate that such algebra splits if it has a nonzero central element in its socle. Later, this result is used to show that an additive mapping $\Phi : \operatorname{soc}(\mathcal{A}) \to \operatorname{soc}(\mathcal{A})$, preserving rank-one idempotents and their linear spans *in both directions* is a real-linear Jordan isomorphism if $\operatorname{soc}(\mathcal{A}) \cap Z(\mathcal{A}) = 0$. An example is provided showing that the results are no longer true if Φ preserves such idempotents only in one direction. We remark that the techniques used, and the results, could easily be extended to the case $\Phi : \operatorname{soc}(\mathcal{A}) \to \operatorname{soc}(\mathcal{B})$.

1. Preliminaries

Unless explicitly otherwise stated, $\mathcal{A}$ will denote a (possibly nonunital) *semiprime* Banach algebra (i.e., $a \in \mathcal{A}\backslash\{0\}$ implies $a\mathcal{A}a \neq 0$) over the field of complex numbers. Its socle, $\operatorname{soc}(\mathcal{A})$, is the sum of the minimal left ideals; equivalently, it is an ideal of $\mathcal{A}$, generated by the set of all the minimal idempotents (an idempotent $p \neq 0$ is minimal if $p\mathcal{A}p = \mathbb{C}p$). It is known (cf. [13]) that the relation $e \sim f \iff eAf \neq 0$ is an equivalence in this set; the corresponding quotient set of all equivalence classes will be denoted by $\Xi = \Xi(\mathcal{A})$. As usual, the central elements are denoted by $Z = Z(\mathcal{A})$, and the spectrum of an element $a \in \mathcal{A}$ by $\sigma_{\mathcal{A}}(a)$ (shortly: $\sigma(a)$ if algebra is known). We note that in nonunital algebras, $\sigma(a) := \sigma_{\hat{\mathcal{A}}}(a)$, where $\hat{\mathcal{A}}$ is the *unitization of* $\mathcal{A}$. Moreover, the fact that algebras (vector spaces, ...) $\mathcal{X}, \mathcal{Y}$ are isomorphic will be denoted by $\mathcal{X} \simeq \mathcal{Y}$.

Suppose that $\mathcal{A} := \mathscr{B}(\mathcal{X})$ is the algebra of bounded operators on a Banach space $\mathcal{X}$. The common way to introduce the rank of $a \in \mathcal{A}$ is $\operatorname{rank}(a) := \dim(\operatorname{Im} a)$. As already mentioned in the Introduction, this concept was recently extended in various equivalent ways to elements of *semisimple, unital Banach algebras*. Since our interest will also include *semiprime Banach algebras*, the definition working best for our purposes seems to be the following: For $a \in \mathcal{A}$ we say that $\operatorname{rank}(a) = n$, if a is in the sum of n minimal left ideals, and is not in the sum of $n-1$ minimal left ideals. We define $\operatorname{rank}(0) := 0$, and put $\operatorname{rank}(a) = \infty$ if $a \notin \operatorname{soc}(\mathcal{A})$. Note that each minimal left ideal equals $\mathcal{A}e$ for some minimal idempotent e cf. [12]). It is known that when $\mathcal{A}$ is a semiprime algebra, the unitization of the closure of its socle, $\mathcal{B} := \widehat{\overline{\operatorname{soc}(\mathcal{A})}}$ is a semisimple (unital) algebra, (cf. [12, Prop. 4.4.4.(b), Prop. 8.7.3]). We may consequently consider the rank of a relative to algebra $\mathcal{B}$; it turns out that it equals $\operatorname{rank}(a)$. This and some other immediate observations regarding the rank are listed below.

Lemma 1.1. *Suppose* $\mathcal{B} := \widehat{\overline{\operatorname{soc}(\mathcal{A})}}$ *and* $a \in \operatorname{soc}(\mathcal{A})$. *Then the following hold:*

i. $\dim a\mathcal{A}a < \infty$. *Moreover, if* $\dim x\mathcal{A}x < \infty$ *for some* $x \in \mathcal{A}$ *then* $x \in \operatorname{soc}(\mathcal{A})$.

ii. $\operatorname{rank}_{\mathcal{B}}(a) = \operatorname{rank}_{\mathcal{A}}(a)$.

iii. $\operatorname{rank}(a) = \sup_{x \in \mathcal{A}} \#\big(\sigma(xa) \backslash \{0\}\big)$.

iv. *If* $\mathcal{A} = \mathscr{B}(\mathcal{X})$ *then* $\operatorname{rank}(a) = \dim(\operatorname{Im} a)$.

v. *There exist only finitely many primitive ideals* $P_1, \dots, P_k$ *not containing* a. *If, moreover,* $\pi_i : \mathcal{A} \to \mathscr{B}(\mathcal{X}_i)$ *are the corresponding irreducible representations on Banach spaces* $\mathcal{X}_i$, *then* $\operatorname{rank}(a) = \sum \operatorname{rank}\big(\pi_i(a)\big)$.

vi. $\operatorname{rank}(a) = 1$ *iff there is only one primitive ideal avoiding* a, *and* $\pi(a)$ *is a rank-one operator. Moreover,* p *is a minimal idempotent iff* $\operatorname{rank}(p) = 1$ *and* $p^2 = p$.

vii. *If* $b \in \operatorname{soc}(\mathcal{A})$ *and* $a\mathcal{A}b = 0 = b\mathcal{A}a$, *then* $\operatorname{rank}(a+b) = \operatorname{rank}(a) + \operatorname{rank}(b)$.

Sketch of the proof. (i) This is known (cf. [1, Thm. 7.2]).

(ii) Suppose $a \in \mathcal{A}e_1 + \cdots + \mathcal{A}e_n$, where $e_1, \ldots, e_n$ are minimal idempotents. Then we have $a \in \sum \mathcal{A}e_i = \sum (\mathcal{A}e_i)e_i \subseteq \sum \mathrm{soc}(\mathcal{A})e_i \subseteq \sum \mathcal{B}e_i \subseteq \sum \mathcal{A}e_i$, and hence the result follows.

(iii) Using that $\mathrm{soc}(\mathcal{B}) \ni xa$ is von Neumann regular (cf. [3, Cor. 2.10]), and that $\sigma_{\mathcal{B}}(xa) = \sigma_{\mathcal{A}}(xa)$ (cf. [2, Cor. 3.2.14]), this can be deduced from the previous item by one of the equivalent definitions of the rank on semisimple, unital algebra $\mathcal{B}$ (cf. [3], [4]).

(v) By [2, Thm. 4.2.1.(iii)], we have $\sigma(xa) = \bigcup_\pi \sigma\big(\pi(xa)\big)$, where π runs over irreducible representations of $\mathcal{A}$. The extended Jacobson density theorem (cf. [6, p. 283]) combined with the previous two items give us the result (cf. also [4]).

The rest is straightforward. $\square$

We continue by repeating briefly some basic definitions: An additive operator $A : \mathcal{X} \to \mathcal{Y}$ between complex vector spaces $\mathcal{X}$, $\mathcal{Y}$ will be called an *h-quasilinear operator* if there exists a ring homomorphism (additive and multiplicative function) $h : \mathbb{C} \to \mathbb{C}$ such that $A(\lambda \mathbf{x}) = h(\lambda)A\mathbf{x}$ for every $(\lambda, \mathbf{x}) \in \mathbb{C} \times \mathcal{X}$.

An additive mapping $\Phi : \mathcal{A} \to \mathcal{C}$ between algebras $\mathcal{A}$, $\mathcal{C}$ is said to *decrease rank-one* if $\mathrm{rank}\big(\Phi(a)\big) \le 1$ whenever $\mathrm{rank}(a) = 1$. It is said to *preserve minimal (= rank-one; cf. Lemma 1.1.vi) idempotents* if $\Phi(p)$ is a minimal idempotent whenever p is a minimal idempotent. And finally, Φ *preserves linear spans (of rank-one elements/of minimal idempotents)* if $\Phi\big(\mathbb{C}\, a\big) \subseteq \mathbb{C}\, \Phi(a)$ for all (rank-one/minimal idempotent) $a \in \mathcal{A}$.

Finally, we state four results that will be used later. To begin with, suppose that $\mathcal{X}$ is a Banach space with a dual $\mathcal{X}^*$. In the paper [9], a series of lemmas was proved concerning the attributes of additive mappings $\Phi : \mathrm{soc}\big(\mathscr{B}(\mathcal{X})\big) \to \mathrm{soc}\big(\mathscr{B}(\mathcal{X})\big)$, decreasing operators of rank-one. The main tool in investigation was the fact that $\mathrm{soc}\big(\mathscr{B}(\mathcal{X})\big) = \mathcal{X} \otimes \mathcal{X}^*$. The arguments, and conclusions, however, are valid in a more general setting of $\Phi : \mathcal{X} \otimes \mathfrak{S} \to \mathcal{Y} \otimes \mathfrak{T}$ where $\mathfrak{S}$ and $\mathfrak{T}$ are arbitrary, at least two-dimensional, subspaces of $\mathcal{X}^*$ (respectively, $\mathcal{Y}^*$). Before stating the result, we emphasize that the inclusion $\mathcal{X} \otimes \mathfrak{S} \subset \mathscr{B}(\mathcal{X})$ enables us to define the rank of a tensor, as the dimension of the image of the corresponding operator. The proof of the theorem below can be found in [9] (cf. also [7]).

Theorem 1.2. *Suppose $\Phi : \mathcal{X} \otimes \mathfrak{S} \to \mathcal{Y} \otimes \mathfrak{T}$ is an additive mapping, decreasing rank-one. Then, Φ takes one of the following forms:*

i. $\Phi(\mathbf{x} \otimes f) = A\mathbf{x} \otimes Cf$ *for some h-quasilinear operators $A : \mathcal{X} \to \mathcal{Y}$, $C : \mathfrak{S} \to \mathfrak{T}$,*

ii. $\Phi(\mathbf{x} \otimes f) = Cf \otimes A\mathbf{x}$ *for some h-quasilinear operators $A : \mathcal{X} \to \mathfrak{T}$, $C : \mathfrak{S} \to \mathcal{Y}$,*

iii. $\Phi(\mathbf{x}\otimes f) = \mathfrak{A}(\mathbf{x}\otimes f)\otimes g_0$ *for some $g_0 \in \mathfrak{T}$ and additive $\mathfrak{A} : \mathcal{X}\otimes\mathfrak{S} \to \mathcal{Y}$, or*

iv. $\Phi(\mathbf{x}\otimes f) = \mathbf{x}_0\otimes\mathfrak{C}(\mathbf{x}\otimes f)$ *for some $\mathbf{x}_0 \in \mathcal{Y}$ and additive $\mathfrak{C} : \mathcal{X}\otimes\mathfrak{S} \to \mathfrak{T}$.*

Still more, if $\dim\bigl(\mathrm{lin}(\mathrm{Im}\,\Phi)\bigr) \geq 2$ and Φ preserves linear spans of rank-one operators, then Φ is h-quasilinear even in cases (iii) *and* (iv).

The last three claims of this section are of crucial importance for our subsequent work, and we give a complete proof. Their main strength is that they will enable us to use the powerful technique of tensors in investigating additive preservers *on socle of a semiprime algebra.* We emphasize that in any Banach algebra $\mathcal{A}$, the sets $\mathcal{A}e$ and $e\mathcal{A}$ are Banach spaces whenever $e \in \mathcal{A}$ is an idempotent.

Lemma 1.3. *Suppose $e \in \Xi(\mathcal{A})$. If $\pi : \mathcal{A} \to \mathscr{B}(\mathcal{X})$ is an irreducible representation with $\pi(e) \neq 0$, then its restriction, $\pi_e := \pi|_{\mathcal{A}e\mathcal{A}}$ is an algebra isomorphism of $\mathcal{A}e\mathcal{A}$ onto a subalgebra $\mathcal{X} \otimes \mathfrak{S} \leq \mathrm{soc}\bigl(\mathscr{B}(\mathcal{X})\bigr)$, where $\mathfrak{S}$ is a w^* dense subspace of $\mathcal{X}^*$. Given any set of $n+1$ linearly independent vectors $\mathbf{x}_i, \mathbf{y} \in \mathcal{X}$; $(i = 1, \ldots, n)$, there exists $f \in \mathfrak{S}$ with $\langle \mathbf{x}_i, f\rangle = 0 \neq \langle \mathbf{y}, f\rangle$.*

Moreover, $\pi_e : \mathcal{A}e\mathcal{A} \to \mathscr{B}(\mathcal{X})$ is also an irreducible representation.

Proof. By a well-known result of Johnson, π is continuous. Since $\overline{\mathcal{A}e\mathcal{A}}$ is topologically simple (cf. [12, Prop. 8.7.3]), one has $\mathrm{Ker}\,\pi\cap\overline{\mathcal{A}e\mathcal{A}}=0$; thus π_e is one-to-one. As $0<\mathrm{rank}(\pi(e))\leq\mathrm{rank}(e)=1$, we have $\pi(e) = \mathbf{x}_e \otimes f_e$ with $\langle \mathbf{x}_e, f_e\rangle = 1$. But $\pi(\mathcal{A})$ acts densely on $\mathcal{X}$, implying that $\pi(\mathcal{A}e) = \mathcal{X} \otimes \{f_e\}$. Thus, π induces a vector space isomorphism between $\mathcal{A}e$ and $\mathcal{X}$. On the other hand, it follows from

$$\pi(e\mathcal{A}) = \{\mathbf{x}_e \otimes T^* f_e;\ T \in \pi(\mathcal{A})\}$$

that π induces a one-to-one vector space homomorphism of $e\mathcal{A}$ onto a vector subspace $\mathfrak{S} \leq \mathcal{X}^*$. This readily implies that $\pi_e : \mathcal{A}e\mathcal{A} \simeq \mathcal{X} \otimes \mathfrak{S}$.

Since $\pi(\mathcal{A})$ acts densely on $\mathcal{X}$, we have $\pi(\mathcal{A}e\mathcal{A})\mathcal{Y} = \mathcal{X}$ whenever $0 \neq \mathcal{Y} \leq \mathcal{X}$. Thus, π_e is irreducible, and $\mathfrak{S} \simeq \pi(e\mathcal{A})$ separates the given vectors $\mathbf{x}_i, \mathbf{y} \in \mathcal{X}$ by annihilating all $\mathbf{x}_i$. To show that $\mathfrak{S}$ is a w^* dense subspace of $\mathcal{X}^*$, we form

$$\mathcal{N} := \bigcap_{f \in \mathfrak{S}} \operatorname{Ker} f.$$

It is obvious that $\mathcal{N}$ is a proper, closed subspace of $\mathcal{X}$, invariant for $\pi(\mathcal{A}e\mathcal{A})$, since $\pi(\mathcal{A}e\mathcal{A})\mathcal{N} = 0$. Thus, $\mathcal{N} = 0$ by the irreducibility of π_e. Now suppose $g \in \mathcal{X}^* \backslash \overline{\mathfrak{S}}^{w^*}$. By virtue of the theorem on separation of convex sets we can find a w^*-continuous functional F with $\langle g, F\rangle = 1$ and $\langle \mathfrak{S}, F\rangle = 0$. Since F is w^*-continuous, it equals some $F_{\mathbf{x}} \in \kappa(\mathcal{X}) \leq \mathcal{X}^{**}$, where $\kappa : \mathcal{X} \to \mathcal{X}^{**}$ is the natural embedding. Then, however, $\langle \mathbf{x}, \mathfrak{S}\rangle = 0$, implying that $\mathbf{x} \in \mathcal{N} = 0$, which is a contradiction with the fact that $1 = \langle g, F_{\mathbf{x}}\rangle = \langle \mathbf{x}, g\rangle$. □

Corollary 1.4. *The mappings π_e compose an algebra isomorphism*

$$\underline{\pi} : \operatorname{soc}(\mathcal{A}) \simeq \bigoplus_{e \in \Xi} \mathcal{X}_e \otimes \mathfrak{S}_e, \tag{1}$$

where $\underline{\pi} = \bigoplus_{e\in\Xi} \pi_e$, and where each $\mathfrak{S}_e$ is a w^* dense subspace of $\mathcal{X}_e^*$. Furthermore, under this isomorphism, $a \in \operatorname{soc}(A)$ has rank-one iff its image is contained in $\mathcal{X}_e \otimes \mathfrak{S}_e$ and equals some $\mathbf{x} \otimes f \in \mathcal{X}_e \otimes \mathfrak{S}_e \setminus \{0\}$.

Proof. In view of Lemma 1.3 and Lemma 1.1.vi, it suffices to show that $\operatorname{soc}(\mathcal{A}) = \bigoplus_{e\in\Xi} \mathcal{A}e\mathcal{A}$. We know that, by definition, $\operatorname{soc}(\mathcal{A}) = \sum\limits_{e\in\mathcal{P}} \mathcal{A}e\mathcal{A}$, where $\mathcal{P}$ is the set of all minimal idempotents. Hence, all we have to see is that $e \sim q$ iff $\mathcal{A}e\mathcal{A} = \mathcal{A}q\mathcal{A}$; once this is established it implies that $\operatorname{soc}(\mathcal{A}) = \sum_{e\in\Xi} \mathcal{A}e\mathcal{A}$ and the sum is direct since each $\mathcal{A}e\mathcal{A} \simeq \mathcal{X}_e \otimes \mathfrak{S}_e$ is a simple algebra.

If $\mathcal{A}q\mathcal{A} = \mathcal{A}e\mathcal{A}$, then $q = q^3 \in \mathcal{A}q\mathcal{A} = \mathcal{A}e\mathcal{A} \simeq \mathcal{X}_e \otimes \mathfrak{S}_e$. Obviously, this is a prime algebra so we have $0 \neq q(\mathcal{A}e\mathcal{A})e = q\mathcal{A}(\mathbb{C}\, e) = q\mathcal{A}e$; hence $e \sim q$. Conversely, if $q\mathcal{A}e \neq 0$, then

$$0 \neq q^2\mathcal{A}e^3 \subseteq (\mathcal{A}q\mathcal{A}) \cdot (\mathcal{A}e\mathcal{A}).$$

Since $\mathcal{A}e\mathcal{A}$, as well as $\mathcal{A}q\mathcal{A}$, are simple algebras it follows that $\mathcal{A}q\mathcal{A} = \mathcal{A}e\mathcal{A}$, as claimed. □

Corollary 1.5. *Suppose $z \in \overline{\mathcal{A}e\mathcal{A}}$. Then $z = 0$ iff $qzq = 0$ for every minimal idempotent $q \in \mathcal{A}e\mathcal{A}$.*

Proof. Let π be an irreducible representation from Lemma 1.3; in view of its proof, $\pi|\overline{\mathcal{A}e\mathcal{A}}$ is one-to-one. Now, if $\pi(z) = T \in \mathscr{B}(\mathcal{X}_e)$ is nonzero, then there exists an $\mathbf{x}$ with $T\mathbf{x} \neq 0$, and as $\pi(\mathcal{A}e\mathcal{A})$ acts densely, there exists an $f \in \mathfrak{S}_e$ with $\langle T\mathbf{x}, f\rangle \neq 0$ and $\langle \mathbf{x}, f\rangle = 1$. Hence, if $q \in \mathcal{A}e\mathcal{A}$ satisfies $\pi(q) = \mathbf{x} \otimes f$, then q is a minimal idempotent with $qzq \neq 0$. $\square$

2. Central elements in the socle

In this section, which may be of independent interest, a characterization of the existence of nonzero central elements in the socle of semiprime algebras is given. Consequently, it is shown that such algebras split.

Lemma 2.1. *Let $\mathcal{A}$ be a semiprime Banach algebra and $a \in \mathcal{A}$. Then the following are equivalent.*

i. $\dim \mathcal{A}a < \infty$.

ii. $\dim a\mathcal{A} < \infty$.

iii. *There exists a central idempotent $p \in \operatorname{soc} \mathcal{A}$ and an ideal $\mathcal{A}_1$ in $\mathcal{A}$, such that $\mathcal{A} = \mathcal{A}_1 \oplus p\mathcal{A}p$ and $a \in p\mathcal{A}p$.*

Proof. Obviously, (iii) implies (i) and (ii) since, by Lemma 1.1.i, $\dim p\mathcal{A}p < \infty$. Hence we are done once the validity of (i) $\Longrightarrow$ (iii) and of (ii) $\Longrightarrow$ (iii) is checked. We proceed with the former only. So suppose that $\mathcal{A}a = \operatorname{lin}\{x_1 a, \ldots, x_n a\}$. Then obviously $\dim a\mathcal{A}a \leq \dim \mathcal{A}a$ and thus, by Lemma 1.1.i, $a \in \operatorname{soc}(\mathcal{A})$. The same lemma implies that there are only finitely many primitive ideals $P_1, \ldots, P_n$ avoiding a; let $\pi_i : \mathcal{A} \to \mathscr{B}(\mathcal{X}_i)$ be the corresponding irreducible representations on Banach spaces $\mathcal{X}_i$. We take a closer look at P_1 for a moment: As $r := \operatorname{rank}\bigl(\pi_1(a)\bigr) \leq \operatorname{rank}(a) < \infty$, we have:

$$\pi_1(a) = \mathbf{x}_1 \otimes f_1 + \cdots + \mathbf{x}_r \otimes f_r,$$

where $\bigl(\mathbf{x}_j\bigr)_{1\leq j\leq r}$, as well as $\bigl(f_j\bigr)_{1\leq j\leq r}$ are linearly independent. Consequently, there exists $\mathbf{z} \in \bigcap_2^r \operatorname{Ker} f_i \setminus \operatorname{Ker} f_1$. The density theorem gives

$\pi_1(\mathcal{A}a)\mathbf{z} = \mathcal{X}_1$, and as $\dim\bigl(\pi_1(\mathcal{A}a)\bigr) \le \dim(\mathcal{A}a) < \infty$ we have that $\mathcal{X}_1$ is finite-dimensional.

Similarly, we proceed with $P_2, \ldots, P_n$. Consequently, the representation $\underline{\pi} := \pi_1 \oplus \cdots \oplus \pi_n$ maps $\mathcal{A}$ into the algebra $\mathcal{M} := \mathcal{M}_{k_1} \oplus \cdots \oplus \mathcal{M}_{k_n}$ where each $\mathcal{M}_{k_i}$ is the full matrix algebra on the finite-dimensional space $\mathcal{X}_i$. Next, as each restriction $\pi_i|_{\mathcal{A}a\mathcal{A}}$ is also irreducible, the extended Jacobson density theorem shows us that the representation $\underline{\pi}|_{\mathcal{A}a\mathcal{A}} : \mathcal{A}a\mathcal{A} \to \mathcal{M}$ is surjective. Thus there exists an element $p = \sum_{i=1}^{k} x_i a y_i \in \mathcal{A}a\mathcal{A} \le \operatorname{soc}(\mathcal{A})$, with $\underline{\pi}(p) = \mathbf{1} \in \mathcal{M}$.

To prove that this p is the one we are after, we first note that p belongs to every primitive ideal $P \notin \{P_1, \ldots, P_n\}$, since this is true for a. Moreover, $\underline{\pi}(p^2 - p) = 0$ and thus $p^2 - p \in \operatorname{soc}(\mathcal{A}) \cap \operatorname{Rad}(\mathcal{A}) = 0$. Similarly, for arbitrary $x \in \mathcal{A}$ one has $px - xp \in \operatorname{soc}(\mathcal{A}) \cap \operatorname{Rad}(\mathcal{A}) = 0$. This implies that $p \in \operatorname{soc}(\mathcal{A})$ is a central idempotent. Same arguments give $pa - a \in \operatorname{soc}(\mathcal{A}) \cap \operatorname{Rad}(\mathcal{A}) = 0$, consequently, $a \in p\mathcal{A}p$. Therefore, $\mathcal{A}a\mathcal{A} \subseteq \mathcal{A}\, p\mathcal{A}p\, \mathcal{A} = p\mathcal{A}^3 p \subseteq p\mathcal{A}p$. Finally, by letting $\mathcal{A}_1 := \operatorname{Ker} \underline{\pi} = P_1 \cap \cdots \cap P_n$ and recalling that $\underline{\pi}|_{p\mathcal{A}p}$ is surjective (since $\pi|_{\mathcal{A}a\mathcal{A}}$ is also surjective), it is easy to check that $\mathcal{A} = \mathcal{A}_1 \oplus p\mathcal{A}p$. $\square$

Remark 2.2. Actually, $\dim a\mathcal{A} = \dim \mathcal{A}a$. This is an immediate consequence of the fact that $p\mathcal{A}p \simeq \mathcal{M}_{k_1} \oplus \cdots \oplus \mathcal{M}_{k_n}$, and, if $\mathcal{A} = \mathcal{M}_n$, then $\dim \mathcal{A}a = \dim(\mathcal{A}a)^t = \dim a^t\mathcal{A}$ where t denotes the transpose. Since a^t is equivalent to a (think of Jordan forms!), we have $\dim a^t\mathcal{A} = \dim a\mathcal{A}$.

Lemma 2.3. *If $z \in \operatorname{soc}(\mathcal{A}) \cap Z(\mathcal{A})$, then there exists a central idempotent p of the same rank as z, and an ideal $\mathcal{A}_1$ in $\mathcal{A}$, such that $\mathcal{A} = \mathcal{A}_1 \oplus p\mathcal{A}p$ and $z \in p\mathcal{A}p$.*

Proof. We proceed as before: If $P_1, \ldots, P_n$ are all primitive ideals avoiding z, then $\pi_i(z) \in \mathscr{B}(\mathcal{X}_i)$ is a finite-rank operator that commutes with $\pi_i(\mathcal{A})$. By Schurr's lemma, $\pi_i(z) = \lambda_i \operatorname{Id}_{\mathcal{X}_i}$ for some $\lambda_i \in \mathbb{C}\backslash\{0\}$ and hence, $\dim \mathcal{X}_i < \infty$. As before, $\underline{\pi} := \pi_1 \oplus \cdots \oplus \pi_n : \mathcal{A} \to \mathscr{B}(\mathcal{X}_1) \oplus \cdots \oplus \mathscr{B}(\mathcal{X}_n)$ is surjective. Pick a polynomial $P \in \mathbb{C}[X]$ with $P(\lambda_i) := 1/\lambda_i$ and let $p := z\,P(z)$. Then, $\underline{\pi}(p) = \operatorname{Id}_{\mathcal{X}_1} \oplus \cdots \oplus \operatorname{Id}_{\mathcal{X}_n}$ and by Lemma 1.1.v, $\operatorname{rank} p = \sum \operatorname{rank}(\operatorname{Id}_{\mathcal{X}_i}) = \sum \operatorname{rank}(\lambda_i \operatorname{Id}_{\mathcal{X}_i}) = \operatorname{rank} z$. Obviously $\pi_i|_{p\mathcal{A}p}$ are still irreducible, so arguments from Lemma 2.1 finish the proof. $\square$

The above two lemmas combined give us the following theorem:

Theorem 2.4. *A semiprime, Banach algebra $\mathcal{A}$ has a nonzero central element z of finite-rank iff* $\dim \mathcal{A}a < \infty$ *for some nonzero $a \in \mathcal{A}$. In this case, there exists an idempotent $p \in \operatorname{soc}(\mathcal{A}) \cap Z(\mathcal{A})$ with $pzp = z$ and such that $\mathcal{A} = \mathcal{A}_1 \oplus p\mathcal{A}p$. Moreover,* $\dim(p\mathcal{A}p) < \infty$.

The last result of the present section was inspired by [10, Prop. 1.1].

Corollary 2.5. *$a \in \mathcal{A}$ is a rank-one central element iff* $\dim \mathcal{A}a = 1$. *Then, $\mathcal{A} \simeq \mathcal{A}_1 \oplus \mathbb{C}$.*

Proof. If $\dim(\mathcal{A}a) = 1$, then $\mathcal{A}a = \mathbb{C}(x_0 a)$ and thus $a \in \operatorname{soc} \mathcal{A}$ with $\operatorname{rank}(a) = 1$, by (i) and (iii) of Lemma 1.1. Hence, we have only one primitive ideal P not containing a, and, from the proof of Lemma 2.1, it is immediate that the corresponding Banach space $\mathcal{X}$ is one-dimensional. Thus, $a = \lambda p$ with p being a central, minimal idempotent and $\mathcal{A} = \mathcal{A}_1 \oplus \mathbb{C}a \simeq \mathcal{A}_1 \oplus \mathbb{C}$. The other implication is a consequence of Lemma 2.3. $\square$

3. Additive preservers

In the final section, additive mappings decreasing rank-one, and later, preserving rank-one idempotents will be characterized. Such mappings are slightly less well-behaved in finite-dimensional algebras than in the infinite ones – namely, in the former ones, they can well be discontinuous. For a typical example one could consider the *discontinuous ring automorphism* $h : \mathbb{C} \to \mathbb{C}$ and let $\Phi\big((a_{ij})\big) := \big(h(a_{ij})\big)$ where $(a_{ij}) \in \mathcal{M}_n(\mathbb{C})$. In this particular example, Φ also preserves elements of trace one, so it preserves rank-one idempotents (and their linear spans), as well. The results from the previous section, combined with the Theorem 3.4, will show that such anomalies cannot occur if $\operatorname{soc}(\mathcal{A}) \cap Z(\mathcal{A}) = 0$ (equivalently, if $\mathcal{A}$ has no finite-dimensional direct summands).

The following notation will be used from now on: If $A : \mathcal{X} \to \mathcal{Y}$ is a continuous, conjugate-linear operator, then we denote by $A' : \mathcal{Y}^* \to \mathcal{X}^*$ the mapping sending g to $(A'g) : \mathbf{x} \mapsto \overline{\langle A\mathbf{x}, g\rangle}$. In this way A' is distinguished from the ordinary adjoint B^* of *a linear operator* B.

Theorem 3.1. *Suppose* $\Phi : \operatorname{soc}(\mathcal{A}) \to \operatorname{soc}(\mathcal{A})$ *is an additive mapping, decreasing rank-one. Then, for each* $e \in \Xi$ *there exists a* $q \in \Xi$ *such that one of the following holds for* $xey \in \mathcal{A}e\mathcal{A}$*:*

i. $\Phi(xey) = A_e(xe) \cdot C_e(ey)$ *for some* h_e*-quasilinear operators* $A_e : \mathcal{A}e \to \mathcal{A}q$ *and* $C_e : e\mathcal{A} \to q\mathcal{A}$*,*
ii. $\Phi(xey) = C_e(ey) \cdot A_e(xe)$ *for some* h_e*-quasilinear operators* $C_e : e\mathcal{A} \to \mathcal{A}q$ *and* $A_e : \mathcal{A}e \to q\mathcal{A}$*,*
iii. $\Phi(\mathcal{A}e\mathcal{A}) \subseteq \mathcal{A}q'$*, for some minimal idempotent* q' *with* $\mathcal{A}q'\mathcal{A} = \mathcal{A}q\mathcal{A}$*, or*
iv. $\Phi(\mathcal{A}e\mathcal{A}) \subseteq q'\mathcal{A}$ *for some minimal idempotent* q' *with* $\mathcal{A}q'\mathcal{A} = \mathcal{A}q\mathcal{A}$*.*

Furthermore, if in the last two possibilities Φ *preserves linear spans of rank-one elements and* $\dim\big(\operatorname{lin}\Phi(\mathcal{A}e\mathcal{A})\big) \geq 2$*, then the restriction,* $\Phi|_{\mathcal{A}e\mathcal{A}}$ *is also* h_e*-quasilinear for some ring homomorphism* $h_e : \mathbb{C} \to \mathbb{C}$*.*

Remark 3.2. We may, conversely, have given h_e-quasilinear $A_e : \mathcal{A}e \to \mathcal{A}q$ and $C_e : e\mathcal{A} \to q\mathcal{A}$. By Lemma 1.3, $\pi_e : \mathcal{A}e\mathcal{A} \simeq \mathcal{X}_e \otimes \mathfrak{S}_e$ with $\pi_e(\mathcal{A}e) = \mathcal{X}_e \otimes \{f_e\}$, and $\pi_e(e\mathcal{A}) = \{\mathbf{x}_e\} \otimes \mathfrak{S}_e$, and $\langle \mathbf{x}_e, f_e \rangle = 1$. Let $i_e : \mathbf{x} \otimes f_e \mapsto \mathbf{x}_e$ be the natural isomorphism, and $\hat{A}_e := i_q(\pi_q|_{\mathcal{A}_q}) A_e (\pi_e|_{\mathcal{A}_e})^{-1} i_e^{-1} : \mathcal{X}_e \to \mathcal{X}_q$; similarly for $\hat{C}_e \to \mathfrak{S}_e \to \mathfrak{S}_q$. It is plian, therefore, that the mapping $\pi_q^{-1}(\hat{A}_e \otimes \hat{C}_e)\pi_e : \mathcal{A}e\mathcal{A} \to \mathcal{A}q\mathcal{A}$ is well defined, and maps $xey = xe \cdot ey$ to $A_e(xe) \cdot C_e(ey)$.

We will not distinguish between the two sides of equation (1) in the sequel.

Proof of Theorem 3.1. By Theorem 1.2, it is enough to prove that $\Phi(\mathcal{A}e\mathcal{A})$ is entirely contained in some $\mathcal{A}q\mathcal{A}$, where $q \in \Xi$.

Since Φ is additive we only have to check this for elementary tensors. So, suppose that $\mathbf{x} \otimes f$, $\mathbf{y} \otimes g \in \mathcal{A}e\mathcal{A}$ are two nonzero tensors, and suppose moreover that $0 \neq \Phi(\mathbf{x} \otimes f)$. As Φ decreases rank-one we have $\Phi(\mathbf{x} \otimes f) \in \mathcal{A}q\mathcal{A}$ for some $q \in \Xi$. Now, $(\mathbf{x} + \mathbf{y}) \otimes f$ is of rank at most one and the same must be true for its Φ-image; it is therefore necessarily the case that $\Phi(\mathbf{y} \otimes f) = 0$, or else $\Phi(\mathbf{y} \otimes f) \in \mathcal{A}q\mathcal{A}$ as well. If $\Phi(\mathbf{y} \otimes f)$ is nonzero, we may repeat the arguments with $\mathbf{y} \otimes (f + g)$ to see what we are after: $\Phi(\mathbf{y} \otimes g) \in \mathcal{A}q\mathcal{A}$. One proceeds similarly when $\Phi(\mathbf{y} \otimes f) = 0$ but $\Phi(\mathbf{x} \otimes g) \neq 0$. Finally, if both are zero then the result is obtained by considering the rank-one tensor $(\mathbf{x} + \mathbf{y}) \otimes (f + g)$. $\square$

Lemma 3.3. *Suppose $\Phi : \mathrm{soc}(\mathcal{A}) \to \mathrm{soc}(\mathcal{A})$ is an additive mapping, preserving idempotents of rank-one, and their linear spans. Then Φ decreases rank-one, and maps nilpotents of rank at most one to themselves.*

Proof. It follows from the fact that $\Phi\big(\mathbb{C}\, p\big) \subseteq \mathbb{C}\,\Phi(p)$ that Φ decreases rank-one *nonnilpotents*. If, on the other hand, $n = \mathbf{x} \otimes f \in \mathcal{A}e\mathcal{A}$ is a rank-one nilpotent then $n = p_1 - p_2$ where $p_1 = \mathbf{y} \otimes f \in \mathcal{A}e\mathcal{A}$, and $p_2 = (\mathbf{y} - \mathbf{x}) \otimes f \in \mathcal{A}e\mathcal{A}$ are idempotents of rank-one. By assumption, we have $\Phi(p_i) = \mathbf{z}_i \otimes g_i \in \mathcal{A}q_i\mathcal{A}$; $(i = 1, 2)$ where $q_i \in \Xi$ and $\langle \mathbf{z}_i, g_i\rangle = 1$. Now, as $\frac{1}{2}(p_1 + p_2)$ is a rank-one idempotent, the same must be true of its Φ-image $\frac{1}{2}(\mathbf{z}_1 \otimes g_1 + \mathbf{z}_2 \otimes g_2) \in \mathcal{A}q_1\mathcal{A} \oplus \mathcal{A}q_2\mathcal{A}$. Thus, $q_1 = q_2$, and either $\mathbf{z}_1, \mathbf{z}_2$ are linearly dependent, or else g_1, g_2 are. In either case, by absorbing the appropriate scalar in the other term of the tensor product, we may assume that either $\mathbf{z}_1 = \mathbf{z}_2$ or else $g_1 = g_2$. Hence, $\Phi(n)$ is a nilpotent of rank at most one, which proves the lemma. $\square$

Theorem 3.1, combined with this lemma, now implies that, for each $e \in \Xi$, there exists a $q \in \Xi$ with $\Phi(\mathcal{A}e\mathcal{A}) \subseteq \mathcal{A}q\mathcal{A}$. This enables us to prove the main theorem of this paper:

Theorem 3.4. *Let $\mathcal{A}$ be a semiprime Banach algebra, and $\Phi\colon \mathrm{soc}(\mathcal{A}) \to \mathrm{soc}(\mathcal{A})$ a surjective, additive mapping, preserving idempotents of rank-one and their linear spans. Suppose moreover, that $\Phi(a)$ is a rank-one idempotent only if $\mathrm{rank}(a) = 1$, and that $\mathrm{soc}(\mathcal{A}) \cap Z(\mathcal{A}) = 0$. Then Φ is a real-linear Jordan isomorphism.*

Proof. Several steps are considered:

Step 1. Suppose that $\Phi(a) = 0$ for some nonzero $a = a_1 \oplus a_2$, where $a_1 \in \mathcal{A}e\mathcal{A}$; $(e \in \Xi)$, and where $a_2\mathcal{A}e = 0 = e\mathcal{A}a_2$. Since $\mathrm{soc}(\mathcal{A}) \cap Z(\mathcal{A}) = 0$, Theorem 2.4 implies that $\mathcal{A}e = \mathcal{X}_e \otimes \{f_e\} \simeq \mathcal{X}_e$, as well as $e\mathcal{A} = \{\mathbf{x}_e\} \otimes \mathfrak{S}_e \simeq \mathfrak{S}_e$, are infinite-dimensional. Therefore, we can find a minimal idempotent $p = \mathbf{x} \otimes f \in \mathcal{A}e\mathcal{A} = \mathcal{X}_e \otimes \mathfrak{S}_e$ such that $\mathrm{rank}(p + a) = \mathrm{rank}(a) + 1$. This, however, is a contradiction since then, $\Phi(a + p)$ is a minimal idempotent although $a + p$ has rank greater than one. Thus, Φ is one-to-one, and consequently, bijective.

Step 2. Lemma 3.3 implies that Φ decreases rank-one. By Theorem 3.1, for any $e \in \Xi$ one has $\Phi(\mathcal{A}e\mathcal{A}) \subseteq \mathcal{A}q\mathcal{A}$ with $\Phi(e) = q = \mathbf{y}_q \otimes g_q$; $\langle \mathbf{y}_q, g_q\rangle = 1$. We claim that $\Phi(\mathcal{A}e\mathcal{A})$ belongs neither to the left ideal $\mathcal{A}q$

nor to $q\mathcal{A}$. Assume to the contrary that $\Phi(\mathcal{A}e\mathcal{A}) \subseteq \mathcal{A}q = \mathcal{X}_q \otimes \{g_q\}$. By Lemma 1.3, there would exist $0 \neq h \in \mathfrak{S}_q$ annihilating $\mathbf{y}_q$; clearly, $\mathbf{y}_q \otimes h \notin \mathcal{X}_q \otimes \{g_q\}$ and $\mathbf{y}_q \otimes (h + g_q)$ is a minimal idempotent. Consequently, by the surjectivity of Φ, we could find some rank-one element $n \in \mathcal{A}e'\mathcal{A}$; $(e' \in \Xi\backslash\{e\})$ with $\Phi(n) = \mathbf{y}_q \otimes (h + g_q)$. Then, however, the element $s := 2n - e$ is of rank at least two since $e\mathcal{A}n = 0 = n\mathcal{A}e$, and is mapped by Φ onto a minimal idempotent $\mathbf{y}_q \otimes (2h + g_q)$, a contradiction. One proceeds similarly when $\Phi(\mathcal{A}e\mathcal{A}) \subseteq q\mathcal{A}$.

The above arguments, together with Theorem 3.1, imply that for $\mathbf{x} \otimes f \in \mathcal{A}e\mathcal{A}$ either

$$\Phi(\mathbf{x} \otimes f) = A_e\mathbf{x} \otimes C_e f \quad \text{or else} \quad \Phi(\mathbf{x} \otimes f) = C_e f \otimes A_e\mathbf{x} \tag{2}$$

for appropriate one-to-one h_e-quasilinear mappings A_e, C_e.

Step 3. As for (conjugate)linearity: Obviously, $\mathcal{A}e$ and $e\mathcal{A}$ are infinite-dimensional Banach spaces; hence so are their isomorphic images, $\mathcal{X}_e \simeq \mathcal{A}e$ and $\mathfrak{S}_e \simeq e\mathcal{A}$, in a topology, transferred to by the corresponding isomorphism (note, however, that this topology may differ from the standard one, induced by inclusion $\mathfrak{S}_e \subseteq \mathcal{X}_e^*$). Moreover, $\Phi_e : \mathcal{X}_e \otimes \mathfrak{S}_e \to \mathcal{X}_q \otimes \mathfrak{S}_q$ preserves rank-one idempotents and nilpotents. This, in turn, enables us to use the arguments from [11, p. 252–253] by which the ring homomorphisms h_e, making A_e and C_e h_e-quasilinear, must be continuous. Hence, A_e and C_e are (conjugate)linear, and thus so is Φ_e.

Step 4. Let us demonstrate that both $\operatorname{Im} A_e$ and $\operatorname{Im} C_e$ contain a linear subspace of codimension one for any $e \in \Xi$. We consider the instance when Φ_e is of the first form in equation (2) only; the proof of the second form goes similarly. Now, recall that $\Phi(e) = \mathbf{y}_q \otimes g_q$ and pick arbitrary $\mathbf{z} \in \operatorname{Ker} g_q$; then all elements $(i\,\mathbf{z} + \mathbf{y}_q) \otimes g_q$; $(i = 0, \frac{1}{2}, 1)$ are rank-one idempotents. By assumptions, there is a rank-one element $a \in \mathcal{A}$ with $\Phi(a) = (\mathbf{z}+\mathbf{y}_q)\otimes g_q$. Then, $(e+a)/2$ is mapped into a rank-one idempotent $(\mathbf{z}/2 + \mathbf{y}_q) \otimes g_q$, hence $\operatorname{rank}(e + a) = 1$, and hence $a \in \mathcal{A}e\mathcal{A}$. This shows that $\mathbf{z} \otimes g_q = \Phi(a - e) \in \operatorname{Im} \Phi_e = \operatorname{Im} A_e \otimes \operatorname{Im} C_e$; thus, $\operatorname{Ker} g_q \subseteq \operatorname{Im} A_e$. By similar arguments, $\operatorname{Ker} F_{\mathbf{y}_q} \subseteq \operatorname{Im} C_e$ where $F_{\mathbf{y}_q} : h \mapsto \langle \mathbf{y}_q, h\rangle$ is a functional on $\mathfrak{S}_q$.

Step 5. As $\dim \mathcal{X}_e = \infty = \dim \mathfrak{S}_e$, Steps 3 and 4 imply that A_e and C_e are surjective: If not, there would exist some nonzero $\mathbf{y}\otimes g \in \mathcal{A}q\mathcal{A}\backslash\Phi(\mathcal{A}e\mathcal{A})$

with $\langle \mathbf{y}, g\rangle = 1$. As Φ is surjective we could find an element (of rank-one!) $a' \in \mathcal{A}e'\mathcal{A}$; $(e' \in \Xi\backslash\{e\})$ with $\Phi(a') = \mathbf{y}\otimes g$. Hence, $\Phi(\mathcal{A}e'\mathcal{A}) \subseteq \mathcal{A}q\mathcal{A} \supseteq \Phi(\mathcal{A}e\mathcal{A})$, and there would exist nonzero $\mathbf{z} \in \operatorname{Im} A_e \cap \operatorname{Im} A_{e'}$ and nonzero $u \in \operatorname{Im} C_e \cap \operatorname{Im} C_{e'}$ (resp., nonzero $\mathbf{z} \in \operatorname{Im} A_e \cap \operatorname{Im} C_{e'}$ and nonzero $u \in \operatorname{Im} C_e \cap \operatorname{Im} A_{e'}$ if $\Phi_{e'}$ takes the second form in equation (2)). This, in turn would imply that some $a = a_e \oplus a_{e'} \in (\mathcal{A}e\mathcal{A}\backslash\{0\}) \oplus (\mathcal{A}e'\mathcal{A}\backslash\{0\})$ is mapped to zero, contrary to Step 1. Hence both A_e and C_e are bijections.

Step 6. Suppose $\Phi_e(\mathbf{x}\otimes f) = A_e\mathbf{x}\otimes C_e f$, and let $\mathbf{y} := A_e\mathbf{x}$. Since Φ_e preserves idempotents and nilpotents of rank-one, we have

$$\begin{gathered}\langle A_e^{-1}\mathbf{y}, f\rangle = \langle \mathbf{x}, f\rangle = \langle A_e\mathbf{x}, C_e f\rangle = \langle \mathbf{y}, C_e f\rangle \quad \text{or}\\ \langle A_e^{-1}\mathbf{y}, f\rangle = \overline{\langle A_e\mathbf{y}, C_e f\rangle};\end{gathered}\tag{3}$$

this, combined with the closed graph theorem, and the fact that the functionals $f \in \mathfrak{S}_e$ separate points in $\mathcal{X}_e$, implies at once that A_e^{-1} is continuous. If Φ_e is linear (and hence so are A_e and C_e), then $C_e = (A_e^{-1})^*|_{\mathfrak{S}_e}$ by the first equality in (3). If, on the other hand, Φ_e is conjugate-linear, then, from the second equality in (3), $C_e = (A_e^{-1})'|_{\mathfrak{S}_e}$. In either case, it is immediate that $\Phi_e : (\mathbf{x}\otimes f) \mapsto A_e(\mathbf{x}\otimes f)A_e^{-1}$; hence, Φ_e is a (conjugate)linear isomorphism.

Finally, suppose $\Phi_e(\mathbf{x}\otimes f) = C_e f \otimes A_e\mathbf{x}$. In resemblance to (3) we have:

$$\begin{gathered}\langle \mathbf{y}, A_e\mathbf{x}\rangle = \langle \mathbf{x}, C_e^{-1}\mathbf{y}\rangle = \langle C_e^{-1}\mathbf{y}, \kappa\mathbf{x}\rangle \quad \text{or} \quad \langle \mathbf{y}, A_e\mathbf{x}\rangle = \overline{\langle C_e^{-1}\mathbf{y}, \kappa\mathbf{x}\rangle},\\ (\mathbf{x}\in\mathcal{X}_e,\ \mathbf{y} := C_e f \in \mathcal{X}_q);\end{gathered}$$

here, $\kappa : \mathcal{X} \to \mathcal{X}^{**}$ is a natural embedding, and $C_e^{-1} : \mathcal{X}_q \to \mathfrak{S}_e \le \mathcal{X}_e^*$. Again, using the closed graph theorem, it follows that A_e and C_e^{-1} are continuous and that $A_e = (C_e^{-1})^*\kappa$ (respectively, $A_e = (C_e^{-1})'\kappa$). In either case, $\Phi_e : (\mathbf{x}\otimes f) \mapsto C_e(f\otimes\kappa\mathbf{x})C_e^{-1} = C_e(\mathbf{x}\otimes f)^* C_e^{-1}$ is a (conjugate)linear antiisomorphism. □

Remark 3.5. The above theorem can be viewed as a partial converse to KAPLANSKY's result [8]. Namely, the Main Theorem of [8], in conjunction with the fact that $\operatorname{soc}(\mathcal{A})\cap Z(\mathcal{A}) = 0$, states that for any ring isomorphism (additive and multiplicative mapping) $\Psi : \mathcal{A} \to \mathcal{A}$, the *semisimple Banach* *algebra* $\mathcal{A}$ splits into two parts: $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, and moreover, $\Psi|_{\mathcal{A}_1}$ is linear and $\Psi|_{\mathcal{A}_2}$ is conjugate-linear. Thus, if $\Psi : \mathcal{A} \to \mathcal{A}$ *is a ring isomorphism*, it is real-linear and therefore preserves minimal idempotents and their linear spans in both directions (recall that p is a minimal idempotent iff $p\mathcal{A}p = \mathbb{C}\,p$).

At the end, we show in the examples that various assumptions of the Theorem 3.4 cannot be relaxed. The abbreviation $\mathbb{R}^- := (-\infty, 0)$ will be useful.

Example 3.6. Let ℓ^2 be a separable, infinite-dimensional Hilbert space, and $\mathcal{A}$ be a semisimple Banach algebra with $\operatorname{soc}(\mathcal{A}) = \bigoplus_{i\in\mathbb{R}} \operatorname{soc}\big(\mathscr{B}(\ell^2)\big)$ – a direct sum of $2^{\aleph_0}$ identical copies of $\operatorname{soc}\big(\mathscr{B}(\ell^2)\big) \simeq \ell^2 \otimes \ell^2$. (As for the existence of $\mathcal{A}$: take a completion of $\bigoplus_{i\in\mathbb{R}} \mathscr{B}(\ell^2)$ in the norm $\|a_1 \oplus a_2 \oplus \dots\| := \sup \|a_i\|$). Obviously, $\operatorname{soc}(\mathcal{A}) \cap Z(\mathcal{A}) = 0$. Now, pick a Hamel basis $\big(\mathbf{x}_t\big)_{t\in\mathbb{R}}$ of ℓ^2 and choose bijections $g : \mathbb{R}^- \to \mathbb{R}$ and $h : [0,\infty) \to \mathbb{R}\backslash\{0\}$. For fixed index $i \in \mathbb{R}$ we agree that $\big(\mathbf{x}_{t;i} \otimes \mathbf{x}_{s;i}\big) := (\mathbf{x}_t \otimes \mathbf{x}_s)_i$ is an element in the i'th summand $(\ell^2 \otimes \ell^2)_i \subseteq \operatorname{soc}(\mathcal{A})$. In this fashion, $\big(\mathbf{x}_{t;i} \otimes \mathbf{x}_{s;i}\big)_{s,t;i}$ is a Hamel basis for $\operatorname{soc}(\mathcal{A})$. Next, for each $i \in \mathbb{R}^-$ choose a bijection $\mathbf{f}_i : \mathbb{R}^2\backslash\{(0,0)\} \to \big\{\mathbf{x}_{t;0};\ t \in \mathbb{R}\backslash\{g(i)\}\big\} \subset (\ell^2)_0$, and let $\mathbf{f}_i(0,0) := \mathbf{0} \in (\ell^2)_0$. Define

$$\pi : \mathbf{x}_{t;i} \otimes \mathbf{x}_{s;i} \mapsto \begin{cases} \mathbf{x}_{g(i);0} \otimes \left(\dfrac{\langle \mathbf{x}_t, \mathbf{x}_s\rangle \,-\, \langle \mathbf{x}_{g(i)}\,,\, \mathbf{f}_i(t,s)\rangle}{\|\mathbf{x}_{g(i)}\|^2} \mathbf{x}_{g(i);0} + \mathbf{f}_i(t,s) \right) & i < 0 \\ \mathbf{x}_{t;h(i)} \otimes \mathbf{x}_{s;h(i)} & i \geq 0 \end{cases};$$

thus, for $i \geq 0$ fixed, π simply maps everything in i'th summand “identically” onto $h(i)$'th one, while for $i < 0$ fixed, everything in i'th summand is mapped into a subset of rank-one elements $\big(\{\mathbf{x}_{g(i)}\} \otimes \ell^2\big)_0 \subseteq (\ell^2 \otimes \ell^2)_0 \subseteq \operatorname{soc}(\mathcal{A})$. Moreover,

$$\langle \mathbf{x}_{t;i}, \mathbf{x}_{s;i}\rangle = \operatorname{Tr}(\mathbf{x}_{t;i} \otimes \mathbf{x}_{s;i}) \equiv \operatorname{Tr}\big(\pi(\mathbf{x}_{t;i} \otimes \mathbf{x}_{s;i})\big). \tag{4}$$

It is easy to see that π is a permutation between two Hamel bases of $\operatorname{soc}(\mathcal{A})$. Consequently, the linear mapping

$$\Phi : \sum_{s,t;i} \lambda_{s,t;i}\, \mathbf{x}_{t,i} \otimes \mathbf{x}_{s,i} \mapsto \sum_{s,t;i} \lambda_{s,t;i}\, \pi(\mathbf{x}_{t,i} \otimes \mathbf{x}_{s,i})$$

is a bijection; here, both sums are finite. Furthermore, if p is a minimal idempotent it completely belongs to one and only summand, say $p = \big(\lambda_1\mathbf{x}_{1;i} + \cdots + \lambda_n\mathbf{x}_{n;i}\big) \otimes \big(\xi_1\mathbf{x}_{1;i} + \cdots + \xi_n\mathbf{x}_{n;i}\big) \in (\ell^2 \otimes \ell^2)_i$. Hence,

$$\begin{aligned} 1 = \operatorname{Tr} p = \operatorname{Tr} \sum_{t,s} \lambda_t\xi_s\, \mathbf{x}_{t;i} \otimes \mathbf{x}_{s;i} = \sum_{t,s} \lambda_t\xi_s \langle \mathbf{x}_{t;i}, \mathbf{x}_{s;i}\rangle \\ = \sum_{t,s} \lambda_t\xi_s \cdot \operatorname{Tr}\big(\pi(\mathbf{x}_{t;i} \otimes \mathbf{x}_{s;i})\big) = \operatorname{Tr}\big(\Phi(p)\big); \end{aligned}$$

where the last but one equality follows by equation (4). Therefore, $\operatorname{Tr}(\Phi(p)) = 1$, proving that Φ preserves minimal idempotents.

This Φ is a linear bijection, preserving minimal idempotents. However, it is not Jordan: Indeed, let $p_\alpha = \mathbf{z}_\alpha \otimes \mathbf{z}_\alpha$; $(\alpha = 1, 2)$ be two idempotents in $(\ell^2 \otimes \ell^2)_{-1}$ with $(p_1 - p_2)^2 \neq 0$. Then, $\Phi(p_1 - p_2)$ is a rank-one nilpotent (both $\Phi(p_1)$ and $\Phi(p_2)$ are idempotents with the same image!) implying that $\Phi(p_1 - p_2)^2 = 0$. Hence, Φ cannot be both Jordan and bijective.

Example 3.7. Suppose that $\mathcal{A}$ is a Banach algebra of compact operators on ℓ^2 and that $\mathcal{B}$ is a Banach algebra of Hilbert–Schmidt operators on ℓ^2. Then $\operatorname{soc}(\mathcal{A}) = \mathscr{F}(\ell^2) = \operatorname{soc}(\mathcal{B})$, where $\mathscr{F}(\ell^2)$ is an ideal of finite-rank operators. However, the identity mapping $\operatorname{Id} : \operatorname{soc}(\mathcal{A}) \to \operatorname{soc}(\mathcal{B})$ *is not continuous*, although it preservers rank-one idempotents.

Hence, general Banach algebras are less well-behaved than $\mathscr{B}(\mathcal{X})$, where any additive mapping preserving idempotents of rank-one and their linear spans is continuous (cf. [11, Main Thm.])

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