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Perturbations of nonlinear evolution equations

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Abstract. Existence results are given for the evolution inclusions $x'(t) + A(t, x(t)) + G(t, x(t)) \ni f(t)$ with $A(t, \cdot)$ a monotone mapping and G a set-valued bounded or Lipschitz mapping.

1. Introduction

Consider the existence of solutions to the evolution inclusion

$$x'(t) + A(t, x(t)) + G(t, x(t)) \ni f(t) \text{ a.e. on } [0, T], \quad x(0) = x_0$$

in a evolution triple (V, H, V^*) with $A(t, \cdot)$ a monotone mapping and G a set-valued mapping.

As a perturbation to the classical equation $x'(t) + A(x(t)) = f(t)$, this problem is very important not only in evolution equation theory but also in other subjects such as distributed parameter control systems (see [1], [2], [11]). So, it has been recently studied in many publications under different conditions (see [3], [6], [8], [9], [12] and the references therein). In [3], the coerciveness assumptions made to A involves the norm of H and G is assumed to satisfy a convergence condition; In [8] and [9], $v \mapsto G(t, v)$ is supposed to be an upper semicontinuous mapping with closed convex values and satisfy a growth condition. In [6], G can be an nonconvex-valued mapping but it is supposed to be integrablly bounded. We notice

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although a wrong imbedding result is used in [6], the main conclusion is not effected due to the reason stated in [8].

In this paper, we will give two new existence results for the above problem. In one of our result, we just suppose that the mapping $v \mapsto G(t,v)$ is upper semicontinuous and bounded (maps bounded sets into bounded sets), do not impose growth condition, and consider the local existence. In another result, $v \mapsto G(t,v)$ is supposed to be Lipschitz with the constant depending on t and the values of G can be nonconvex. A continuity theorem is also presented, which is a modification to a similar one in [8].

2. Preliminaries

In this paper, we always suppose (V,H,V^*) is an evolution triple, that is, H is Hilbert space, V is a separable reflexive Banach space with dual V^* and $V \hookrightarrow H \hookrightarrow V^*$ densely and continuously. The inner product of H as well as the duality pairing between V and V^* are denoted by $(\cdot,\cdot)$. We also suppose that $\infty > p \geq 2$ is a real number and $q = p/(p-1)$. $(\!(\cdot,\cdot)\!)$ stands for the duality pairing between $L^p(0,T;V)$ and $L^q(0,T;V^*)$. The norm in any Banach space X involved is denoted by $\|\cdot\|_X$. The space X endowed with weak topology is denoted by X_w, the weak convergence (in X) is denoted by "$x_n \rightharpoonup x$", and the functional space $L^r(0,T;X)$ with $r > 0$ will be abbreviated to $L^r(X)$. For a sequence of subsets $D_n \subset X$, we denote by

$$w\text{-}\limsup_{n\to\infty} D_n = \{x \in X: \text{ there exist } n_k \text{ and } x_{n_k} \in D_{n_k} \text{ with } x_{n_k} \rightharpoonup x\}.$$

For a set-valued mapping $G:[0,T] \to X$, we denote by

$$S_G^1 = \{x \in L^1(X) : x(t) \in G(t) \text{ a.e.}\}.$$

A known result is that $S_G^1 \neq \emptyset$ if $\inf\{\|u\|_X : u \in G(t) \text{ a.e.}\} \in L^1(X)$.

Let $W(0,T) = \{x \in L^p(V) : x' \in L^q(V^*)\}$. It is known that $W(0,T)$ is a reflexive Banach space endowed with the norm $\|x\|_W := \|x\|_{L^p(V)} + \|x'\|_{L^q(V^*)}$, $W(0,T) \hookrightarrow C(0,T;H)$ continuously and, if the imbedding of V into H is compact, then $W(0,T) \hookrightarrow L^p(H)$ compactly.

We also recall that a set-valued mapping F between Hausdorff spaces X and Y is said to be upper semicontinuous (u.s.c.) if $F^{-1}(D) := \{x \in X : F(x) \cap D \neq \emptyset\}$ is closed for each closed subset $D \subset Y$; F is said to be hemicontinuous if $t \mapsto F(x+ty)$ is u.s.c. If X is a reflexive Banach space, $Y = X^*$ and $(y_1 - y_2, x_1 - x_2) \geq 0$ for all $x_i \in X, y_i \in F(x_i), i = 1, 2$, then F is called monotone on X.

Now, we consider evolution equation

$$x'(t) + A(t, x(t)) = f(t) \text{ a.e. on } [0,T], \quad x(0) = x_0 \in H \tag{2.1}$$

under the following assumptions.

(H1) $A : [0,T] \times V \to V^*$ is an operator with $t \mapsto A(t,v)$ measurable, $v \mapsto A(t,v)$ hemicontinuous and monotone.

(H2) There exist $a_1 \geq 0$, $a_2 \in L^q(0,T)$ such that

$$\|A(t,v)\|_{V^*} \leq a_1 \|v\|_V^{p-1} + a_2(t), \quad \text{for all } v \in V,\ t \in [0,T].$$

(H3) There exist $a_3 > 0$, $a_4 \in L^1(0,T)$ such that

$$(A(t,v), v) \geq a_3 \|v\|_V^p - a_4(t), \quad \text{for all } v \in V,\ t \in [0,T].$$

(H4) $V \hookrightarrow H$ compactly.

It is well known (see [7] or [12]) that if (H1)–(H3) are satisfied, then, for each $x_0 \in H$ and each $f \in L^q(V^*)$, equation (2.1) has a unique solution in $W(0,T)$, which will be always denoted in the following by x_f, and, if D is a bounded subset of $L^q(V^*)$, then the solution set $\{x_f : f \in D\}$ is bounded in $W(0,T)$. In fact, there exists $c > 0$ such that $\|x_f\|_{W(0,T)} \leq c + c\|f\|_{L^q(V^*)}$. (This is also true for some implicit problems, see [4].) Moreover, the solution mapping $f \mapsto x_f$ has a property as stated below.

Proposition 2.1. *Suppose* (H1)–(H4) *are satisfied. Then the solution mapping* $f \mapsto x_f$ *of equation* (2.1) *is continuous from* $L^q(H)_w$ *to* $C(0,T;H)$, *monotone on* $L^q(V^*)$ *and*

$$\|x_f(t) - x_g(t)\|_H \leq \int_0^t \|f(s) - g(s)\|_H ds \quad \text{for all } f, g \in L^q(H). \tag{2.2}$$

Proof. Let $f_n \rightharpoonup f$ in $L^q(H)$. Then $\{f_n\}$ is bounded in $L^q(V^*)$. From the remarks we made above, we know that $\{x_{f_n}\}$ is bounded in $W(0,T)$.

So, by passing to a subsequence, we may assume that $x_{f_n} \rightharpoonup y$ in $W(0,T)$. Since $W(0,T) \hookrightarrow C(0,T;H)$ continuously, $\{x_{f_n}\}$ is bounded in $C(0,T;H)$. Since $x_{f_n}(0) = x_f(0) = x_0$ and

$$x'_{f_n}(t) + A(t, x_{f_n}(t)) = f_n(t), \quad x'_f(t) + A(t, x_f(t)) = f(t) \quad \text{a.e.,}$$

$$\big(x'_{f_n}(t) - x'_f(t), x_{f_n}(t) - x_f(t)\big) = \frac{1}{2}\frac{d}{dt}\|x_{f_n}(t) - x_f(t)\|_H^2,$$

by the monotonicity of $A(t,\cdot)$, we have

$$\frac{1}{2}\frac{d}{dt}\|x_{f_n}(t) - x_f(t)\|_H^2 \le \big(f_n(s) - f(s), x_{f_n}(s) - x_f(s)\big).$$

Therefore

$$\frac{1}{2}\|x_{f_n}(t) - x_f(t)\|_H^2 \le \int_0^t \big(f_n(s) - f(s), x_{f_n}(s) - x_f(s)\big)ds \tag{2.3}$$

$$= \int_0^t \big(f_n(s) - f(s), x_{f_n}(s) - y(s)\big)ds \tag{2.4}$$

$$+ \int_0^t \big(f_n(s) - f(s), y(s) - x_f(s)\big)ds.$$

Since $W(0,T) \hookrightarrow L^p(H)$ compactly, we may suppose that $x_{f_n} \to y$ strongly in $L^p(H)$. So from the boundedness of $\{f_n\}$, it follows that

$$\int_0^t \big(f_n(s) - f(s), x_{f_n}(s) - y(s)\big)ds \le \|f_n - f\|_{L^q(H)}\|x_{f_n} - y\|_{L^p(H)} \to 0.$$

By letting $\chi(s) = 1$ for $s \le t$ and $\chi(s) = 0$ for $s > t$, we see

$$\int_0^t \big(f_n(s) - f(s), y(s) - x_f(s)\big)ds$$

$$= \int_0^T \big(f_n(s) - f(s), \chi(s)(y(s) - x_f(s))\big)ds \to 0.$$

So, from (2.3) and (2.4), it follows that $\|x_{f_n}(t) - x_f(t)\|_H \to 0$ for each t. Together with the boundedness of $\{x_{f_n}\}$ in $C(0,T;H)$, we see that $x_{f_n} \to x_f$ in $L^p(H)$ and therefore, by (2.3) and Hölder's Inequality, we see

$$\|x_{f_n}(t) - x_f(t)\|_H^2 \le 2\|f_n - f\|_{L^q(H)}\|x_{f_n} - x_f\|_{L^p(H)} \to 0.$$

That is, $x_{f_n}(t) \to x_f(t)$ in H uniformly. This proves the continuity of $f \mapsto x_f$ from $L^q(H)_w$ to $C(0,T;H)$.

Using the same method as used to obtain (2.3), we can prove, for all $f, g \in L^q(V^*)$, that

$$\frac{1}{2}\|x_f(t) - x_g(t)\|_H^2 \le \int_0^t \big(f(s) - g(s), x_f(s) - x_g(s)\big) ds, \quad t \in [0,T]. \quad (2.5)$$

Let $t = T$, we see that $(\!(f - g, x_f - x_g)\!) \ge 0$ which implies the monotonicity of $f \mapsto x_f$. If, in (2.5), let $f, g \in L^q(H)$, then we obtain

$$\frac{1}{2}\|x_f(t) - x_g(t)\|_H^2 \le \int_0^t \|f(s) - g(s)\|_H \|x_f(s) - x_g(s)\|_H ds.$$

Applying the extended Gronwall's inequality (see [5] or [13]), we have

$$\|x_f(t) - x_g(t)\|_H \le \int_0^t \|f(s) - g(s)\|_H ds.$$

This proves (2.2) and completes the proof. □

Remark 2.2. The continuity of $f \mapsto x_f$ from $L^q(H)_w$ to $C(0,T;H)$ was also claimed in Proposition 1 of [8] where a_2 is a constant and $a_4 \equiv 0$. Moreover, our method is different.

3. Existence results

In this section, under (H1)–(H4), we suppose $G(t,\cdot)$ is either a bounded or a Lipschitz mapping on H, and consider the existence of solutions of the inclusion

$$x'(t) + A(t, x(t)) + G(t, x(t)) \ni f(t) \text{ a.e. on } [0,T], \quad x(0) = x_0 \in H. \quad (3.1)$$

Theorem 3.1. *Under assumptions* (H1)–(H4), *suppose* $b \in L^q(0,T)$ *is a given function. Let* $G : [0,T] \times H \to 2^H$ *be a set-valued mapping with closed convex values,* $t \mapsto G(t,v)$ *be measurable and* $v \mapsto G(t,v)$ *be u.s.c. from* H *to* H_w. *If for any bounded subset* $D \subset H$, *there exists* $M > 0$ *such that*

$$\sup\big\{\|G(t,v)\|_H : v \in D\big\} \le M + b(t) \quad \text{a.e.,}$$

then problem (3.1) *admits solutions on* $[0, T_0]$ *for some* $T_0 \in (0,T]$.

Proof. Let

$$d = \left(\|x_0\|_H^2 + 2\|a_4\|_{L^1(0,T)} + \frac{2}{q(pa_3)^{q/p}} \|f\|_{L^q(V^*)}^q \right)^{1/2} + \int_0^T b(t)dt,$$

$$D = \{u \in H : \|u\|_H \le d + k\} \quad \text{with } k > 0 \text{ a given number.}$$

By our assumptions on G, there exists $M > 0$ such that

$$\sup\big\{\|u\|_H : u \in G(t,v),\ v \in D\big\} \le M + b(t) \quad \text{a.e. on } [0,T]. \tag{3.2}$$

We choose $T_0 \in (0,T]$ such that $T_0 M \le k$ (that is, $T_0 = \min\{T, k/M\}$) and denote by

$$D_1 = \big\{g \in L^q(H) : \|g(t)\|_H \le M + b(t) \text{ a.e. on } [0,T_0]\big\},$$

$$F(g) = S^1_{G(\cdot, x_{f-g}(\cdot))} \quad \text{for } g \in D_1.$$

Then D_1 is a bounded, closed and convex subset of $L^q(0,T_0;H)$, $F(g)$ is a nonempty, closed, bounded and convex subset for each $g \in D_1$.

Take $g \in D_1$ and write $x = x_{f-g}$ for convenience. Then

$$(x'(t), x(t)) + (A(t,x(t)), x(t)) = (f(t) - g(t), x(t)) \quad \text{a.e.}$$

From (H3), the fact that $(x'(t), x(t)) = \frac{1}{2}\frac{d}{dt}\|x(t)\|_H^2$ and Young's inequality, it follows that

$$\begin{aligned}
&\|x(t)\|_H^2 + 2a_3 \int_0^t \|x(s)\|_V^p ds \le \|x_0\|_H^2 + 2\int_0^t a_4(s)ds \\
&\quad + 2\int_0^t \|f(s)\|_{V^*}\|x(s)\|_V + 2\int_0^t \|g(s)\|_H \|x(s)\|_H ds \le \|x_0\|_H^2 \\
&\quad + 2\|a_4\|_{L^1(0,T)} + 2a_3 \int_0^t \|x(s)\|_V^p ds + \frac{2}{q(pa_3)^{q/p}} \int_0^t \|f(s)\|_{V^*}^q ds \\
&\quad + 2\int_0^t \|g(s)\|_H \|x(s)\|_H ds.
\end{aligned}$$

By the extended Gronwall's Inequality ([5] or [13]), we have

$$\|x(t)\|_H \le \left(\|x_0\|_H^2 + 2\|a_4\|_{L^1(0,T)} + \frac{2}{q(pa_3)^{q/p}} \|f\|_{L^q(V^*)}^q \right)^{1/2}$$

$$+\int_0^t \|g(s)\|_H ds \le d + T_0 M \le d + k \quad \text{a.e. on } [0, T_0].$$

So $x(t) = x_{f-g}(t) \in D$ for each $t \in [0, T_0]$ and, therefore, $\|z(t)\|_H \le M + b(t)$ for each $z \in F(g)$ and each $t \in [0, T_0]$ because of (3.2). This means that F maps D_1 into itself as a set-valued mapping.

Let $(g_n, z_n) \in \mathrm{Graph}(F)$ and $g_n \rightharpoonup g$, $z_n \rightharpoonup z$ in $L^q(0, T_0; H)$. By Proposition 2.1, $x_{f-g_n} \to x_{f-g}$ in $C(0, T_0; H)$ and, therefore, $x_{f-g_n}(t) \to x_{f-g}(t)$ in H for each $t \in [0, T_0]$. Since $G(t, \cdot)$ is u.s.c., we see

$$w\text{-}\limsup_{n\to\infty} G(t, x_{f-g_n}(t)) \subset G(t, x_{f-g}(t)) \quad \text{a.e.}.$$

Invoking Theorem 4.2 of [10], we have

$$z \in w\text{-}\limsup_{n\to\infty} F(g_n) \subset S^1_{w\text{-}\limsup_{n\to\infty} G(\cdot, x_{f-g_n}(\cdot))} \subset S^1_{G(\cdot, x_{f-g}(\cdot))} = F(g).$$

So $(g, z) \in \mathrm{Graph}\, F$, that is, F is closed under the weak topology. Since D_1 is weakly compact, we see that F is weakly u.s.c. under the weak topology. From Kakutani's fixed point theorem, it follows that F has fixed point, say g. By the meaning of the notion x_f we see that x_{f-g} is a solution of (3.1) on $[0, T_0]$. □

Remark 3.2. Suppose x_1 is a solution of (3.1) on $[0, T_0]$. Then, by the same method as used above, we can prove that there exist $T_1 \in (T_0, T]$ and $x_2 \in W(T_0, T_1)$ such that

$$x_2(T_0) = x_1(T_0), \quad x_2'(t) + A(t, x_2(t)) + G(t, x_2(t)) \ni f(t) \quad \text{a.e. on } [T_0, T_1].$$

This implies that the interval on which (3.1) has solutions can be extended. But, without further assumptions, we are not sure whether this interval can be extended to $[0, T]$.

Now, we consider the case when G is Lipschitz with nonconvex values.

Theorem 3.3. *Under assumptions* (H1)–(H4), *let* $G : [0, T] \times H \to 2^H$ *be a set-valued mapping with closed and bounded values,* $\sup\{\|u\|_H : u \in G(t, 0)\} \in L^q(0, T)$ *and* $t \mapsto G(t, v)$ *be measurable. Suppose there exists* $k \in L^q(0, T)$ *such that*

$$\mathcal{H}(G(t, v_1), G(t, v_2)) \le k(t)\|v_1 - v_2\|_H, \quad \textit{for all } t \in [0, T],\ v_1, v_2 \in H.$$

Here, $\mathcal{H}(\cdot,\cdot)$ means the Hausdorff distance on H. Then problem (3.1) *has solutions. If, in addition, G is single-valued, then the solution is unique.*

Proof. Let $f \mapsto x_f$ be the same operator as in Proposition 2.1 and let

$$F(g) = S^1_{G(\cdot,x_{f-g}(\cdot))} \quad \text{for } g \in L^q(H).$$

Then $F(g) \neq \emptyset$ for every $g \in L^q(H)$ and $F(g) \subset L^q(H)$ because of our assumptions on G. It is easy to see that $F(g)$ is closed and bounded.

Take $g_1, g_2 \in L^q(H)$ and let $\varepsilon > 0, z_1 \in F(g_1)$ be given. Since G is Lipschitz, there exists $z_2 \in F(g_2)$ such that

$$\|z_1(t) - z_2(t)\|_H \leq k(t)\|x_{f-g_1}(t) - x_{f-g_2}(t)\|_H + \varepsilon, \text{ a.e.}.$$

Let $l > 0$ be a real number such that $2T^{1/p}(2lq)^{-q} < 1$. For each $z \in L^q(H)$, let

$$\|z\|_l = \left(\int_0^T \exp(-2lqr(t))\|z(t)\|_H^q dt\right)^{1/q} \quad \text{with } r(t) = \int_0^t k^q(s)ds.$$

Clearly, $\|\cdot\|_l$ is a norm on $L^q(H)$ and equivalent to the usual one. By Proposition 2.1, Hölder's Inequality and using the integration by parts, we obtain

$$\begin{aligned}
\|z_1 - z_2\|_l^q &= \int_0^T \exp(-2lqr(t))\|z_1(t) - z_2(t)\|_H^q dt \\
&\leq 2^q \int_0^T \exp(-2lqr(t)) \left(k(t)\int_0^t \|g_1(s) - g_2(s)\|_H ds\right)^q dt \\
&\quad + \varepsilon 2^q \int_0^T \exp(-2lqr(t)dt \\
&\leq 2^q T^{q/p}\int_0^T \exp(-2lqr(t))k^q(t)\int_0^t \|g_1(s) - g_2(s)\|_H^q dsdt + 2^q\varepsilon T \\
&= -2^q \frac{T^{q/p}}{2lq} \exp(-2lqr(t)) \int_0^t \|g_1(s) - g_2(s)\|_H^q ds\Big|_0^T \\
&\quad + 2^q \frac{T^{q/p}}{2lq} \int_0^T \exp(-2lqr(s))\|g_1(s) - g_2(s)\|_H^q ds + 2^q\varepsilon T \\
&\leq 2^q \frac{T^{q/p}}{2lq} \|g_1 - g_2\|_l^q + 2^q\varepsilon T.
\end{aligned}$$

We denote by $\mathcal{H}_l(\cdot,\cdot)$ the Hausdorff distance in $L^q(H)$ endowed with the new norm $\|\cdot\|_l$. Since g_1, g_2 are arbitrary, we see

$$\big(\mathcal{H}_l\big(F(g_1),F(g_2)\big)\big)^q \le 2^q \frac{T^{q/p}}{2lq}\|g_1-g_2\|_l^q + 2^q\varepsilon T.$$

By letting $\varepsilon \to 0$, we obtain

$$\mathcal{H}_l\big(F(g_1),F(g_2)\big) \le 2T^{1/p}(2lq)^{-q}\|g_1-g_2\|_l.$$

So F is a contraction on $L^q(H)$, and therefore F has a fixed point g. Obviously, $x = r(g)$ is a solution of (3.1). If, in addition, G is single-valued, then the solution is unique due to the uniqness of fixed point of F as a single-valued mapping. □

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References

[1] N. U. Ahmed, Optimal control of infinite dimensional systems governed by functional differential inclusions, *Discussiones Mathematicar, Differential Inclusions* **15** (1995), 75–94.

[2] N. U. Ahmed, Existence of optimal control for a class of systems governed by differential inclusions on a Banach space, *J. Optimization Theory Appl.* **50** (1986), 213–237.

[3] N. U. Ahmed, Compactness in certain abstract function spaces with application to differential inclusions, *Discussiones Mathematicar, Differential Inclusions* **15** (1995), 21–28.

[4] W. M. Bian, Solutions of implicit evolution inclusions with pseudo-monotone mappings, *Topological Methods Nonl. Anal.* **15** (2000), 101–113.

[5] C. M. Dafermos, The second law of thermodynamics and stability, *Arch. Rational Mech. Anal.* **70** (1979), 167–179.

[6] D. Kravaritis and G. Pantelidis, On evolution inclusions with nonconvex valued orientor fields, *Nonlinear Analysis, TMA* **26** (1996), 1–8.

[7] J. L. Lions, Quelques méthodes de résolution des problémes aux limites non linéairs, *Dunod*, *Paris*, 1969.

[8] S. Migórski, On an existence result for nonlinear evolution inclusions, *Proc. Edin. Math. Soc.* **39** (1996), 133–141.

[9] N. S. Papageorgiou, Continuous dependence results for evolution inclusions, *Proc. Edin. Math. Soc.* **35** (1992), 139–158.

[10] N. S. Papageorgiou, Convergence theorems for Banach valued integrable multifunctions, *Intern, J. Math. Math. Sci.* **10** (1987), 433–442.

[11] N. S. Papageorgiou, On Nonlinear nonconvex evolution inclusions, *Discussiones Mathematicar, Differential Inclusions* **15** (1995), 29–42.

[12] R. E. Showalter, Monotone Operator in Banach space and nonlinear partial differential equations, *Mathematical Surveys and Monographs* **49**, Amer. Math. Soc., Providence, R.I. (1997).

[13] J. R. L. Webb, An extension of Gronwall's inequality, Proceeding of Conference on Nonlinear Partial Differential Equations, *Kiev*, 1997.

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