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Another characterization of the gamma function

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Abstract. The function $\frac{\log\Gamma(x)}{\log x}$ is characterized to be the only convex solution of the functional equation

$$f(x+1)=\frac{\log x}{\log(x+1)}(f(x)+1),\quad x\in(0,\infty).$$

Some relations to the function $\log\Gamma(x+1)/x^a$, $0<a\leq 1$ are shown.

0. Introduction

In this paper we examine the behavior of the Euler gamma function Γ in the logarithmically scaled coordinate system. More exactly, we show that the function $g:(0,\infty)\to\mathbb{R}$ defined by

$$g(x):=\begin{cases}\dfrac{\log\Gamma(x)}{\log x} & \text{for } x\neq 1,\\ -\gamma & \text{for } x=1,\end{cases}$$

(where γ is the Euler gamma constant) is increasing, convex, $g(0+)=-1$ and $g(2)=0$. The main result of our paper states that the function g is the only convex solution of the functional equation

$$f(x+1)=\frac{\log x}{\log(x+1)}(f(x)+1),\quad x\in(0,\infty). \tag{1}$$

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One can weaken the supposition of convexity of the solution in this way that only convexity is supposed in a neighborhood of infinity.

Note that no initial condition is required.

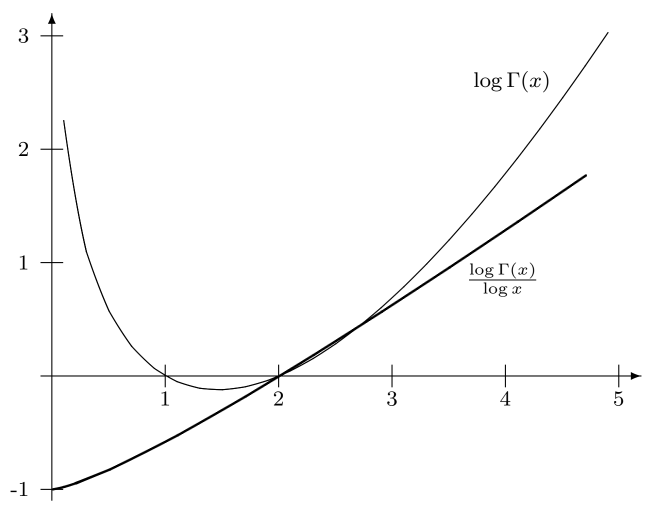
We would like to remark that this characterization of the gamma function is not a consequence of the famous Bohr–Mollerup characterization of the gamma function (see e.g. [2] or [3], p. 288) as the only log-convex solution of the functional equation

$$f(x+1) = x \cdot f(x),\ x \in (0,\infty)\,; \quad f(1) = 1. \tag{2}$$

And, the more, it cannot be derived from the recent generalization of the Bohr–Mollerup theorem [5] that says that the gamma function is the only solution of (2), which is geometrically convex on a neighborhood of infinity. In these characterizations the initial condition is indispensable.

As an interesting consequence we infer that the function $G(x) = \frac{\log\Gamma(\exp x)}{\log x}$ is strictly increasing and strictly convex on $\mathbb{R}$.

We further consider the functions $\frac{\log\Gamma(x+1)}{\log x^\alpha}$ and $\frac{\log\Gamma(x)}{\log x^\alpha}$ for a fixed real α, $0 < \alpha \leq 1$, relating it with a recent paper of Grabner *et al.* [4].



1. Some properties of $\log\Gamma(x)/\log x$

The function $g(x) := \frac{\log\Gamma(x)}{\log x}$ is analytic on $(0,\infty)$ with a removable singularity at $x=1$, $g(1) = -0.577215\cdots = -\gamma$
(where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$, the Euler gamma constant). We further have $g(0+) = \lim_{x\to 0+} g(x) = -1$, and $g'_+(0) = 0$.

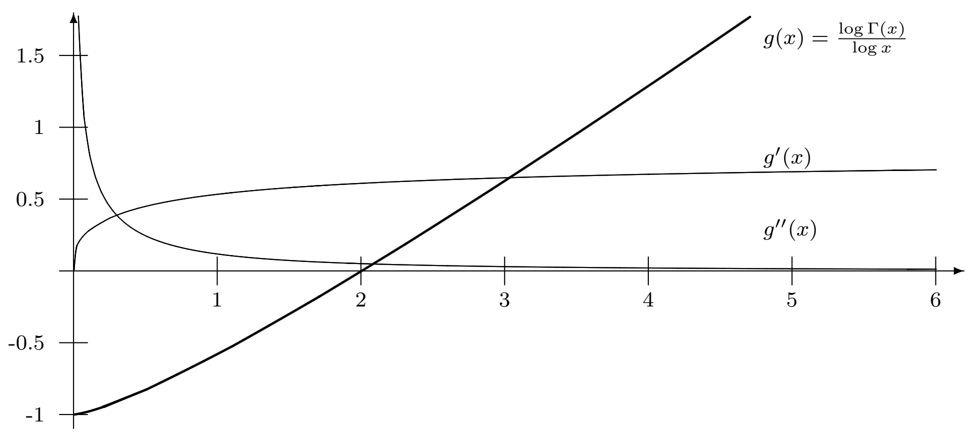
To show this the following representations are useful (see [3], p. 287f.):

$$\log\Gamma(x) = -\log x - \gamma x - \sum_{n=1}^{\infty}\left(\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right), \quad x > 0;$$

$$\Psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty}\frac{x}{(x+n)n}, \quad x > 0.$$

Proposition. *The function* $g(x) = \frac{\log\Gamma(x)}{\log x}$ *is strictly monotone increasing and strictly convex on* $(0,\infty)$.

Proof. We show that g' and g'' are positive on $(0,\infty)$. To do this we use asymptotic expansions for $\log\circ\Gamma$, Ψ and Ψ' which will show us, that $g'(x)$ and $g''(x)$ are positive for large x. For smaller x one can see this from the graph of these functions:



The following asymptotic formulas are valid for positive x:

$$\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} - \frac{1}{360x^3} + \dots \tag{3}$$

$$\Psi(x) = \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \dots \tag{4}$$

$$\Psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \dots \tag{5}$$

If we only take a partial sum of one of these series then the error will be less than the first term neglected and has the same sign ([1], p. 257f.).

1. Note that

$$g'(x) = \frac{\Psi(x)}{\log x} - \frac{\log \Gamma(x)}{x(\log x)^2}, \quad x > 0,$$

with a removable singularity at $x = 1$. By (3) and (4) we have

$$\log \Gamma(x) < (x - 1/2x) \log x \quad \text{and} \quad \Psi(x) > \log x - \frac{1}{2x} - \frac{1}{12x^2}.$$

Hence

$$\begin{aligned} g'(x)x(\log x)^2 &= x(\log x)\Psi(x) - \log \Gamma(x) \\ &> x(\log x)\left(\log x - \frac{1}{2x} - \frac{1}{12x^2}\right) - \left(x - \frac{1}{2}\right)\log x > 0 \end{aligned}$$

if $x > 2.7484\dots$.

2. $g''(x) = \frac{\Psi'(x)}{\log x} - \frac{2\Psi(x)}{x(\log x)^2} + \left(\frac{1}{x^2(\log x)^2} + \frac{2}{x^2(\log x)^3}\right) \log \Gamma(x)$ for $x > 0$, again with a removable singularity at $x = 1$. We get

$$\begin{aligned} x^2(\log x)^3 g''(x) = x^2(\log x)^2\Psi'(x) + (\log x + 2)\log \Gamma(x) \\ - 2x \log x \dot{\Psi}(x). \end{aligned} \tag{6}$$

Here we use the inequalities following from (5), (3) (where $\frac{1}{2}\log(2\pi) = 0.9189\dots$) and (4):

$$\Psi'(x) > \frac{1}{x} + \frac{1}{2x^2}, \ \log \Gamma(x) > \left(x - \frac{1}{2}\right)\log x - x + 0.9 \quad \text{and} \quad \Psi(x) < \log x.$$

Herewith we get a lower bound for (6) by

$$\left(x - \frac{1}{10}\right)\log x - \frac{10x - 9}{5},$$

which is positive for say $x \geq 6$ (more exactly $x > 5.491776524\dots$). Thus also $g''(x) > 0$ for at least $x > 5.491776524\dots$. □

Remark 1. In the previous proof we found it more appropriate to use computer aided calculations to show the convexity of $g(x)$ for small x than to tackle complicate inequalities. For those who are not convinced by these methods we have a second version of our main result (see Theorem 2 below).

Remark 2. The function $G : \mathbb{R} \to \mathbb{R}$ defined by $G = g \circ \exp$, i.e.

$$G(x) = \frac{\log \Gamma(\exp x)}{x},$$

as a composition of two strictly increasing and strictly convex functions is again *strictly increasing and strictly convex on* $\mathbb{R}$.

2. The functional equation

The function g satisfies the functional equation

$$f(x+1) = \frac{\log x}{\log(x+1)}(f(x)+1), \quad x \in (0,\infty). \tag{7}$$

If $f : (0,\infty) \to \mathbb{R}$ is an arbitrary solution of (7), then (7) with $x = 1$ yields

$$f(2) = 0. \tag{8}$$

Thus, the initial condition (8) is forced by the functional equation (7) itself. Furthermore we have

$$f(n) = \frac{\log \Gamma(n)}{\log n}, \quad n \in \mathbb{N},\ n \geq 2.$$

and also

$$f(x+n) = \frac{\log x}{\log(x+n)} f(x) + \frac{\log\big[(x+n-1)\dots x\big]}{\log(x+n)}, \quad x \in (0,\infty),\ n \in \mathbb{N}.$$

Hence also for $x \in (0,\infty),\ n \in \mathbb{N}$,

$$f(x+n+1) = \frac{\log x}{\log(x+n+1)} f(x) + \frac{\log\big[(x+n)\dots(x)\big]}{\log(x+n+1)}, \tag{9}$$

Theorem 1. *The only solution of* (7), *convex on* $(0,\infty)$ *is*

$$g(x) = \frac{\log\Gamma(x)}{\log x}.$$

Proof. Let $f:(0,\infty)\to\mathbb{R}$ a solution of (7), convex on $(0,\infty)$. Then we have automatically (8) and

$$f(n) = g(n) = \frac{\log(n-1)!}{\log n}, \quad n\in\mathbb{N},\ n\geq 2. \tag{10}$$

For $x\in(0,1]$, $n\in\mathbb{N}$, $n\geq 2$ the convexity condition yields:

$$f(n+1)-f(n) \leq \frac{f(x+n+1)-f(n+1)}{x} \leq f(n+2)-f(n+1), \tag{11}$$

whence

$$\begin{aligned} 0 &\leq \frac{1}{x}\left[f(x+n+1)-f(n+1)-x\left(f(n+1)-f(n)\right)\right] \\ &\leq f(n+2)+f(n)-2f(n+1). \end{aligned}$$

Hence, applying in turn: relation (9) and (10), multiplication by $\log(x+n+1)>0$, the monotonicity of log and $\frac{\log\Gamma}{\log}$, and, finally the Stirling formula which implies that $\log n! < (n+\frac{1}{2})\log n - n + 1$ (see [1], p. 257), we obtain

$$\begin{aligned} 0 &\leq \frac{1}{x}\Bigg[\log x\cdot f(x) + \log[(x+n)\dots x] \\ &\qquad - x\log(x+n+1)\left(\frac{\log n!}{\log(n+1)} - \frac{\log(n-1)!}{\log n}\right)\Bigg] \\ &\leq \log(x+n+1)\left[\frac{\log(n+1)!}{\log(n+2)} + \frac{\log(n-1)!}{\log n} - 2\frac{\log n!}{\log(n+1)}\right] \\ &\leq \log(n+2)\left[\frac{\log(n+1)!}{\log(n+2)} + \frac{\log(n-1)!}{\log n} - 2\frac{\log n!}{\log(n+1)}\right] \\ &= \log(n+2)\left[\frac{\log(n+1)!}{\log(n+2)} - \frac{\log n!}{\log(n+1)}\right] \\ &\quad + \log(n+2)\left[\frac{\log(n-1)!}{\log n} - \frac{\log n!}{\log(n+1)}\right] \end{aligned}$$

$$
\begin{aligned}
&< \log(n+2)\left[\frac{\log(n+1)!}{\log(n+2)} - \frac{\log n!}{\log(n+1)}\right] \\
&\quad + \log(n+1)\left[\frac{\log(n-1)!}{\log n} - \frac{\log n!}{\log(n+1)}\right] \\
&= \log(n+1)! - \log n! - \frac{\log(n+2)}{\log(n+1)}\log n! + \frac{\log(n+1)}{\log n}\log(n-1)! \\
&= \log(n+1) - \frac{\log(n+1)}{\log n}\left(\log n! - \log(n-1)!\right) \\
&\quad + \left(\frac{\log(n+1)}{\log n} - \frac{\log(n+2)}{\log(n+1)}\right)\log n! \\
&= \left[\frac{\log(n+1)}{\log n} - \frac{\log(n+2)}{\log(n+1)}\right]\log n! \\
&< \left[\frac{\log(n+1)}{\log n} - \frac{\log(n+2)}{\log(n+1)}\right]\left[\left(n+\frac{1}{2}\right)\log n - n + 1\right] =: \theta(n).
\end{aligned}
$$

With the aid of the expansions

$$
\log(n+1) = \log n - \log\left(1 - \frac{1}{n+1}\right) = \log n + \sum_{i=1}^{\infty}\frac{1}{i\cdot(n+1)^i},
$$

$$
\begin{aligned}
\log(n+2) &= \log(n+1) + \log\left(1 + \frac{1}{n+1}\right) \\
&= \log(n+1) + \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i\cdot(n+1)^i}
\end{aligned}
$$

we get

$$
\begin{aligned}
&\frac{\log(n+1)}{\log n} - \frac{\log(n+2)}{\log(n+1)} \\
&\qquad = \frac{1}{\log n}\cdot\sum_{i=1}^{\infty}\frac{1}{i\cdot(n+1)^i} - \frac{1}{\log(n+1)}\cdot\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i\cdot(n+1)^i} \\
&\qquad = \left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right)\frac{1}{n+1} \\
&\qquad\quad + \left(\frac{1}{\log n} + \frac{1}{\log(n+1)}\right)\frac{1}{2(n+1)^2} + \cdots = \mathrm{o}\left(\frac{1}{n\log n}\right).
\end{aligned}
$$

From this it is easy to see that $\theta(n)$ tends to 0 for $n \to \infty$. Therefore for $x \in (0,1]$ we have

$$f(x) = \lim_{n\to\infty} \left[x \log(x+n+1) \left(\frac{\log n!}{\log(n+1)} - \frac{\log(n-1)!}{\log n} \right) - \log[(x+n)\dots x] \right] \frac{1}{\log x}$$

that is, f is uniquely determined on $(0,1]$. According to well known uniqueness theorems on difference equations, we have that f, as a convex solution of (7), is uniquely determined on all of $(0,\infty)$. Since g is also a convex solution of (7), we have $f = g$. □

Theorem 2. *The only solution of* (7)*, convex on a neighborhood of infinity is* $g(x) = \frac{\log\Gamma(x)}{\log x}$.

Proof. Let $f : (0,\infty) \to \mathbb{R}$ a solution of (7), convex say for $x > a$ for some positive real a. We can proceed as in the proof of Theorem 1, except that we have suppose in (11) that $n > a$ holds. □

3. Final remarks

In a recent paper [4] P. Grabner *et al.* showed that the function $x \mapsto \frac{\log\Gamma(x+1)}{x}$ is concave on $(-1,\infty)$ and it is characterized as the only concave solution of its functional equation. In contrary to this the function $\frac{\log\Gamma(x+1)}{\log x} = \frac{\log\Gamma(x)}{\log x} + 1$ is convex on $(0,\infty)$.

In this connection the following question arises. What is the behavior of the function $x \mapsto \frac{\log\Gamma(x+1)}{x^\alpha}$ or $x \mapsto \frac{\log\Gamma(x)}{x^\alpha}$ for a fixed real α? Here the case $0 < \alpha < 1$ is of interest.

For $\alpha \in (0,1)$ the plotted graph of these functions looks like convex, at least for small x. Nevertheless it is easy to show that there is a constant c, depending on α, such that these functions are concave for $x > c$.

The function $x \mapsto \frac{\log\Gamma(x)}{x^\alpha}$ e.g. fulfills the functional equation

$$f(x+1) = f(x) \cdot \frac{x^\alpha}{(x+1)^\alpha} + \frac{\log(x)}{(x+1)^\alpha}, \quad x \in (0,\infty). \tag{12}$$

It is routine to show that $x \mapsto \frac{\log\Gamma(x)}{x^\alpha}$ is the only solution of (12), concave in a neighborhood of infinity, together with the side condition $f(1) = 0$. A similar statement can be given for the function $x \mapsto \frac{\log\Gamma(x+1)}{x^\alpha}$.

By this we receive different forms of characterizations of the gamma function.

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