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A Lévy-characterization for Gaussian processes on matrix groups

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Abstract. In this paper we extend the classical Lévy characterization of Brownian motions on $\mathbb{R}^n$ to real matrix groups by using group representations. More precisely, we show: Let $(\mu_t)_{t\geq 0}$ be a Gaussian convolution semigroup on a locally compact group G which admits a faithful finite-dimensional real representation ρ, and let $(X_t)_{t\geq 0}$ be an a.s. continuous process on G. If for $\pi \in \{\rho, \rho\otimes\rho\}$, the matrix-valued processes $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ are martingales with the invertible matrices $\tilde{\pi}(\mu_t) := \int_G \pi(g)\, d\mu_t(g)$, then $(X_t)_{t\geq 0}$ is a Gaussian process associated with $(\mu_t)_{t\geq 0}$.

1. Introduction

A classical result of P. Lévy states that an a.s. continuous process $(B_t)_{t\geq 0}$ on $\mathbb{R}$ is a Brownian motion if and only if $(B_t)_{t\geq 0}$ and $(B_t^2 - t)_{t\geq 0}$ are martingales. This characterization is usually extended to Markov processes in terms of the martingale problem; see, for instance, [4], [12]. A variant of the martingale problem for certain diffusions on $\mathbb{R}$ more closely related with the Lévy characterization of Brownian motion was given in [14]. Moreover, a modified version of the martingale problem was presented in [15] for Lévy processes on locally compact groups G in terms

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of unitary representations. In particular, for Brownian motions on compact Lie groups, this martingale characterization was extended in [15] to a Lévy-type characterization in terms of finitely many finite-dimensional unitary representations of G. In this paper we present a variant of this characterization for Brownian motions on non-compact matrix groups or, which is almost equivalent, on locally compact second countable groups having a not necessarily unitary, but finite-dimensional faithful real representation. We shall show that for $G=\mathbb{R}$, this characterization in fact leads to the classical Lévy characterization. Before we state the main result of this paper in Theorem 2.9, we recapitulate some notions and basic facts in the followig section. The proof of the main result as well as of some further technical statements will be given in Section 3. The final Section 4 contains an example based on the Heisenberg group as well as some further comments and conclusions.

2. Statement of the main results

Throughout this paper let G be a locally compact, second countable group. Let $M_b(G)$ and $M^1(G)$ be the spaces of all signed regular Borel measures and probability measures on G respectively. The space $M_b(G)$ together with the convolution $*$ as multiplication and the total variation norm is a Banach algebra. Moreover, let $C_0(G)$ be the space of all continuous functions on G vanishing at infinity.

2.1. Convolution semigroups

(1) A family $(\mu_t)_{t\geq 0}\subset M^1(G)$ is a convolution semigroup if $\mu_s*\mu_t=\mu_{s+t}$ for $s,t\geq 0$, if $\mu_0=\delta_e$, and if $[0,\infty[\ \to M^1(G)$, $t\mapsto \mu_t$, is weakly continuous. Moreover, a G-valued process $(X_t)_{t\geq 0}$ is called a Lévy process related with a convolution semigroup $(\mu_t)_{t\geq 0}$ on G, if for $n\in\mathbb{N}$ and $0\leq t_0\leq t_1\leq\cdots\leq t_n$, the increments

$$X_{t_1}X_{t_0}^{-1}, X_{t_2}X_{t_1}^{-1}, \ldots, X_{t_n}X_{t_{n-1}}^{-1}$$

are independent, and if $X_{t_1}X_{t_0}^{-1}$ is $\mu_{t_1-t_0}$-distributed. Recall that a Lévy process always admits a version with càdlàg paths.

(2) A convolution semigroup $(\mu_t)_{t\geq 0}$ on G is called Gaussian if

$$\lim_{t\to 0}\frac{1}{t}\mu_t(G\setminus U_e)=0 \quad \text{for all neighborhoods } U_e \text{ of } e.$$

A Lévy process on G associated with a Gaussian convolution semigroup $(\mu_t)_{t\geq 0}$ is called a Gaussian process. Recapitulate that $(\mu_t)_{t\geq 0}$ is Gaussian if and only if each associated Lévy process admits a continuous version (see Section I.9 of [2]). These continuous versions are called Brownian motions on G.

(3) We define the generator of a convolution semigroup $(\mu_t)_{t\geq 0}$ on G by

$$\begin{aligned} Lf(x) &:= \lim_{t\to 0}\frac{1}{t}\big(\mu_t^- * f(x) - f(x)\big)\\ &= \lim_{t\to 0}\frac{1}{t}\left(\int_G f(yx)\,d\mu_t(y) - f(x)\right) \end{aligned} \tag{2.1}$$

for $x\in G$ and f in the domain $D(L)$ of L which is $\|\,.\,\|_\infty$-dense in $C_0(G)$. This definition differs for non-commutative groups from that in Section 4.1 of [7], where the product is taken from the other side. As both definitions are essentially equivalent (note also our definition of a Lévy process), and as the definition above is more convenient later on from an operator theory point of view, we use the above notion. We here finally notice that for a Lie group G, the space $C_0^2(G)$ of all 2-times (from the left or from the right) differentiable $C_0(G)$-functions is contained in $D(L)$ for any convolution semigroup $(\mu_t)_{t\geq 0}$ on G; see Theorems 4.1.14 and 4.1.16 of [7].

It is well-known that Lévy processes on Lie groups admit the following martingale characterization; see [5] and Theorems 4.1.7 and 4.4.1 in [4]:

2.2. Proposition. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on a Lie group G with generator L. Then a càdlàg process $(X_t)_{t\geq 0}$ on G is a Lévy process associated with $(\mu_t)_{t\geq 0}$ if and only if for each $f\in C_0^2(G)$ the process $(f(X_t)-\int_0^t Lf(X_s)\,ds)_{t\geq 0}$ is a martingale.*

We here note that for all Lévy processes $(X_t)_{t\geq 0}$ on Lie groups and $f\in C_0^2(G)$, the martingales $(f(X_t)-\int_0^t Lf(X_s)\,ds)_{t\geq 0}$ can be described explicitly as stochastic integrals in terms of classical Brownian motions and independent Poisson random measures; see Section 3 of [1].

In this paper we are interested in a variant of the martingale characterization 2.2 that involves representations of G. We first recapitulate some notations and facts on representations.

2.3. Representations of locally compact groups

(1) Let H be a real or complex separable Hilbert space H, and let the space $B(H)$ of all bounded linear operators on H be equipped with the weak operator topology. A representation π of G on H then is a continuous homomorphism $\pi : G \to B(H)$ with $\pi(e)$ as identity operator. Notice, that in particular for a representation π and all $a, b \in H$, the mappings $g \mapsto \langle \pi(g)a, b\rangle$ on G are continuous.

(2) A representation π of G on H is called faithful, if $\pi : G \to B(H)$ is injective. Moreover, π will be called real, if H is a real Hilbert space or if there is a orthonormal basis of H such that for all $x \in G$ the operators $\pi(x)$ can be written as real (possibly infinite) matrices with respect to this basis (which means that π may be regarded as a representation on a real Hilbert space).

(3) A representation π is called unitary if $\pi(g)$ is unitary for each $g \in G$. It is well-known that a unitary representation π can always be extended uniquely to a strongly continuous Banach algebra homomorphism $\tilde{\pi} : M_b(G) \to B(H)$, where the operator $\tilde{\pi}(\mu) = \int_G \pi(g)\, d\mu(g)$ is characterized by

$$\langle \tilde{\pi}(\mu)a, b\rangle = \int_G \langle \pi(g)a, b\rangle \, d\mu(g) \qquad (a, b \in H); \tag{2.2}$$

for the proof and further details see [6].

(4) Now let π be a not necessarily unitary representation of H. In order to ensure in this case that the operators $\tilde{\pi}(\mu_t) \in B(H)$ exist we need additional "moment conditions" for the convolution semigroup $(\mu_t)_{t\geq 0}$. The following result shows that we are always on the safe way for Gaussian semigroups. The proof of this result will be postponed to Section 3.

2.4. Proposition. *Let π be a representation of a locally compact second countable group G on some separable Hilbert space H, and let $(\mu_t)_{t\geq 0}$ be a Gaussian semigroup on G. Then there exist unique operators*

$\tilde{\pi}(\mu_t) \in B(H)$ *for which equation* (2.2) *holds. These operators form a weakly continuous one-parameter semigroup* $(\tilde{\pi}(\mu_t))_{t\geq 0} \subset B(H)$.

2.5. Remark. Let π be a representation of G on H and $(\mu_t)_{t\geq 0}$ a convolution semigroup on G. Assume that $(\tilde{\pi}(\mu_t))_{t\geq 0} \subset B(H)$ exists which is the case for unitary representations and all $(\mu_t)_{t\geq 0}$ by Section 2.3(3) as well as for Gaussian semigroups $(\mu_t)_{t\geq 0}$ and arbitrary representations by Proposition 2.4. In order to state our martingale characterizations of Lévy processes associated with $(\mu_t)_{t\geq 0}$, we need in addition that the operators $\tilde{\pi}(\mu_t)^{-1} \in B(H)$ exist. If H is finite-dimensional, then the weak topology on $B(H)$ is equivalent to the norm topology, and, as all operators in a neighborhood of the identity admit inverse operators, it follows readily that $\tilde{\pi}(\mu_t)^{-1} \in B(H)$ exists for all $t \geq 0$. We conclude that in particular for Gaussian semigroups $(\mu_t)_{t\geq 0}$ and finite-dimensional representations $\tilde{\pi}(\mu_t)^{-1} \in B(H)$ exists for each $t \geq 0$.

As a final preparation we introduce the following simple notion of operator-valued martingales (for general Banach space-valued martingales see [9]):

2.6. Operator-valued martingales. Let H be a separable Hilbert space. Let $(Z_t)_{t\geq 0}$ be a $B(H)$-valued stochastic process with filtration $(\mathcal{F}_t)_{t\geq 0}$. Then $(Z_t)_{t\geq 0}$ is called a $B(H)$-valued martingale (w.r.t. $(\mathcal{F}_t)_{t\geq 0}$), if for all $a, b \in H$, the $\mathbb{C}$-valued processes $(\langle Z_t a, b\rangle)_{t\geq 0}$ are martingales (w.r.t. $(\mathcal{F}_t)_{t\geq 0}$). Clearly, a similar notion is available for local L^2-martingales and so on.

We next recapitulate Lemma 2.5(1) of [15]; it is stated there for unitary representations only, but its proof obviously extends to the following setting.

2.7. Lemma. *Let* π *be a representation of* G *on some Hilbert space* H. *Let* $(\mu_t)_{t\geq 0}$ *be a convolution semigroup on* G *such that the operators* $\tilde{\pi}(\mu_t) \in B(H)$ *exist and are invertible for all* $t \geq 0$. *If* $(X_t)_{t\geq 0}$ *is a Lévy process associated with* $(\mu_t)_{t\geq 0}$ *and with filtration* $(\mathcal{F}_t)_{t\geq 0}$, *then* $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ *is a* $B(H)$*-valued martingale w.r.t.* $(\mathcal{F}_t)_{t\geq 0}$.

In [15] several versions of converse statements of Lemma 2.7 of the following kind were derived: If $(\mu_t)_{t\geq 0}$ is a convolution semigroup and

$(X_t)_{t\geq 0}$ an arbitrary stochastic process on G such that $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is an operator-valued martingale for "sufficiently" many unitary representations, then $(X_t)_{t\geq 0}$ must be a Lévy process associated with $(\mu_t)_{t\geq 0}$. In the case of Gaussian semigroups on compact Lie groups, we derived the following Lévy-type characterization in [15]:

2.8. Theorem. *Let $(\mu_t)_{t\geq 0}$ be a Gaussian semigroup on a compact Lie group G, and let ρ be a faithful finite-dimensional unitary representation of G. Then the following statements are equivalent for a stochastic process $(X_t)_{t\geq 0}$ on G with filtration $(\mathcal{F}_t)_{t\geq 0}$:*

(1) *$(X_t)_{t\geq 0}$ is a Gaussian process associated with $(\mu_t)_{t\geq 0}$.*

(2) *The process $(X_t)_{t\geq 0}$ admits a continuous version, and for $\pi \in \{\rho, \rho\otimes\rho\}$, the process $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is a matrix-valued martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.*

(3) *For $\pi \in \{\rho,\ \rho\otimes\rho,\ \rho\otimes\overline{\rho}\}$, the process $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is a martingale.*

The following variant of the equivalence (1) $\Longleftrightarrow$ (2) in Theorem 2.8 is the main result of this paper; it will be proved in Section 3.

2.9. Theorem. *Let $(\mu_t)_{t\geq 0}$ be a Gaussian semigroup on a locally compact second countable group G, and let ρ be a faithful finite-dimensional real representation of G. Then the following statements are equivalent for a continuous process $(X_t)_{t\geq 0}$ on G with filtration $(\mathcal{F}_t)_{t\geq 0}$:*

(1) *$(X_t)_{t\geq 0}$ is a Gaussian process associated with $(\mu_t)_{t\geq 0}$;*

(2) *For $\pi \in \{\rho, \rho\otimes\rho\}$, the process $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is a matrix-valued $(\mathcal{F}_t)_{t\geq 0}$-martingale.*

Notice that Part (2) of the theorem makes sense by Proposition 2.4 and Remark 2.5. Here is the most important example:

2.10. Example. The usual Lévy-characterization of the n-dimensional Brownian motion on $\mathbb{R}^n$ can be obtained from Theorem 1.8 as follows: Embed $(\mathbb{R}^n, +)$ into $GL(2n, \mathbb{R})$ via the representation

$$\rho : (x_1, \ldots, x_n) \longmapsto \begin{pmatrix} B(x_1) & & 0 \\ & \ddots & \\ 0 & & B(x_n) \end{pmatrix} \quad \text{with} \quad B(x_i) := \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}.$$

Condition (2) of Theorem 2.9 for the standard Gaussian semigroup on $\mathbb{R}^n$ then just means that the processes $(X_t^i)_{t\geq 0}$ and $(X_t^i X_t^j - \delta_{i,j}t)_{t\geq 0}$ are martingales for $i, j = 1, \ldots, n$ as claimed. In other words, Theorem 2.9 yields the usual Lévy-characterization.

2.11. Remark. The condition of π being a real representation of G is essential in Theorem 2.9. Here is an example:

Let $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{R}$ be equipped with the usual group structures. Then the Lie group $G := \mathbf{T} \times \mathbb{R}$ admits the one-dimensional faithful complex representation $\pi : G \to \mathbb{C} \setminus \{0\}$ with $\pi(z, x) := z \cdot e^x$. Let $B := (B_t^1, B_t^2)_{t\geq 0}$ be a two-dimensional Brownian motion on $\mathbb{R}^2$ (with independent component and $B_0^1 = B_0^2 = 0$). Then, the process $X := (e^{iB_t^1}, B_t^2)_{t\geq 0}$ is a non-degenerated Gaussian process on G. On the other hand, as for each $c \geq 0$, the processes $(e^{icB_t^1 + c^2t/2})_{t\geq 0}$ and $(e^{cB_t^2 - c^2t/2})_{t\geq 0}$ are independent $\mathbb{C}$-valued martingales, it follows that

$$\Big(\pi(X_t) = e^{iB_t^1 + B_t^2} = e^{iB_t^1 + t/2} \cdot e^{B_t^2 - t/2}\Big)_{t\geq 0}$$

and

$$\Big((\pi \otimes_{\mathbb{C}} \pi)(X_t) = e^{2(iB_t^1 + B_t^2)} = e^{2iB_t^1 + 2t} \cdot e^{2B_t^2 - 2t}\Big)_{t\geq 0}$$

are $\mathbb{C}$-valued martingales with respect to the Brownian filtration of B (see, for instance, the proof of Theorem 7.3.16 in [16]). If Theorem 2.9 would be available in this case, we could conclude for the degenerate convolution Gaussian semigroup $(\mu_t := \delta_{(1,0)})_{t\geq 0}$ that $X_t = (1, 0)$ for all $t \geq 0$ almost surely which is not the case.

On the other hand, we always may regard any n-dimensional complex representation $\rho : G \to GL(n, \mathbb{C})$ of G as a $2n$-dimensional real representation $\rho : G \to GL(2n, \mathbb{R})$ to which Theorem 2.9 can be applied. But in this case we have to use the real tensor product $\rho \otimes_{\mathbb{R}} \rho$ in Condition (2) of Theorem 2.9 instead of the complex one which is not sufficient.

3. Proof of the main results

3.1. Proof of Proposition 2.4. Denoting the operator norm of $A \in B(H)$ by $\|A\|$, we first observe that the function $f : G \to \,]0, \infty[$ with

$f(x) := \|\pi(x)\|$ is submultiplicative, i.e., $f(xy) \leq f(x)f(y)$ for $x, y \in G$. Moreover, as, by the Riesz–Fréchet theorem,

$$\|\pi(x)\| = \sup\{|\pi(x)a| : a \in E\} = \sup\{|\langle\pi(x)a, b\rangle| : a, b \in E\}$$

for all $x \in G$ and any countable dense subset E of the unit sphere in H, f is also measurable on G. We therefore may apply Theorem 1 of Siebert [10] to the Gaussian semigroup $(\mu_t)_{t\geq 0}$ whose Lévy measure is equal to 0, and we conclude that for all $t \geq 0$,

$$\sup_{0\leq s\leq t} \int_G \|\pi(x)\| d\mu_s(x) =: M_{t,\pi} < \infty.$$

As for $a, b \in H$ and $x \in G$, $|\langle\pi(x)a, b\rangle\,| \leq \|\pi(x)\|\,\|a\|\,\|b\|$, the equation $T_b(a) := \int_G \langle\pi(g)a, b\rangle\, d\mu_t(g)$ defines bounded linear functionals. Hence, by the Riesz–Fréchet theorem, there are unique linear operators $\tilde{\pi}(\mu_t)$ on H for $t \geq 0$ with

$$\langle\tilde{\pi}(\mu_t)a, b\rangle = \int_G \langle\pi(g)a, b\rangle\, d\mu_t(g) \quad (a, b \in H).$$

Moreover, as for $a, b \in H$,

$$|\langle\tilde{\pi}(\mu_t)a, b\rangle| \leq \int_G |\langle\pi(x)a, b\rangle|\, d\mu_t(x) \leq \|a\|\,\|b\|\, M_{t,\pi},$$

these operators are bounded with

$$\|\tilde{\pi}(\mu_t)\| = \sup\bigl\{|\langle\tilde{\pi}(\mu_t)a, b\rangle|/(\|a\|\,\|b\|) : a, b \in H \setminus \{0\}\bigr\} \leq M_{t,\pi}.$$

Moreover, for $s, t \geq 0$ and $a, b \in H$ we have

$$\begin{aligned}\langle\tilde{\pi}(\mu_{s+t})a, b\rangle &= \int_G \langle\pi(z)a, b\rangle\, d\mu_{s+t}(z) = \int_G\int_G \langle\pi(x)\pi(y)a, b\rangle\, d\mu_s(x)d\mu_t(y)\\ &= \int_G \langle\tilde{\pi}(\mu_s)\pi(y)a, b\rangle\, d\mu_t(y) = \int_G \langle\pi(y)a, \tilde{\pi}(\mu_s)^*b\rangle\, d\mu_t(y)\\ &= \langle\tilde{\pi}(\mu_t)a, \tilde{\pi}(\mu_s)^*b\rangle = \langle\tilde{\pi}(\mu_s)\tilde{\pi}(\mu_t)a, b\rangle,\end{aligned}$$

where $.^*$ denotes the adjoint operator. This shows that $(\tilde{\pi}(\mu_t))_{t\geq 0} \in B(H)$ is a one-parameter semigroup in $B(H)$ where $\tilde{\pi}(\mu_0) = \pi(e)$ is the identity.

We next check that this semigroup is weakly continuous. For this take $a, b \in H$, $t \in [0, \infty[$, and a sequence $(t_n)_{n\geq 0} \subset [0, \infty[$ which converges to

t. In order to prove $\lim_n \langle \tilde{\pi}(\mu_{t_n})a, b\rangle = \langle \tilde{\pi}(\mu_t)a, b\rangle$, we use the continuous function $f : G \to \mathbb{C}$ with $f(x) := \langle \pi(x)a, b\rangle$, and observe that for all $s \geq 0$,

$$\langle \tilde{\pi}(\mu_s)a, b\rangle = \int_G f \, d\mu_s = \int_{\mathbb{C}} x \, df(\mu_s)(x),$$

where the images $f(\mu_{t_n}) \subset M^1(\mathbb{C})$ tend weakly to $f(\mu_t) \subset M^1(\mathbb{C})$. We shall prove that

$$\lim_n \int_{\mathbb{C}} x \, df(\mu_{t_n})(x) = \int_{\mathbb{C}} x \, df(\mu_t)(x),$$

which then yields the claim.

For this, we fix some orthonormal basis $(e_i)_{i\in I}$ of H with index set $I \subset \mathbb{N}$. Each vector $c \in H$ may be uniquely written as $c = \sum_{i\in I} c_i e_i$ with $c_i \in \mathbb{C}$; therefore, the complex conjugate $\overline{c} := \sum_{i\in I} \overline{c}_i e_i$ of c w.r.t. $(e_i)_{i\in I}$ is well defined. In a similar way, we may define a kind of contragredient representation $\overline{\pi}$ of G on H w.r.t. $(e_i)_{i\in I}$ by taking complex conjugate entries in the representation of $\pi(x)$ for $x \in G$ in the matrix representation of $\pi(x)$ w.r.t. $(e_i)_{i\in I}$. Obviously, $\overline{\pi}$ is again a representation of G on H. Now consider the tensor product representation $\pi \otimes \overline{\pi}$ of G on the Hilbert space $H \otimes H$. Then for $s \geq 0$,

$$\begin{aligned}
&\int_G \langle (\pi(x) \otimes \overline{\pi}(x))(a \otimes \overline{a}), (b \otimes \overline{b})\rangle \, d\mu_s(x) \\
&\quad = \int_G \Big(\langle \pi(x)a, b\rangle \cdot \langle \overline{\pi}(x)\overline{a}, \overline{b}\rangle \Big) \, d\mu_s(x) \\
&\quad = \int_G |\langle \pi(x)a, b\rangle|^2 \, d\mu_s(x) = \int_G |f|^2 \, d\mu_s = \int_{\mathbb{C}} |x|^2 \, df(\mu_s)(x),
\end{aligned}$$

and for all $0 \leq s \leq T$

$$\begin{aligned}
&\Big| \int_G \langle (\pi(x) \otimes \overline{\pi}(x))(a \otimes \overline{a}), (b \otimes \overline{b})\rangle \, d\mu_s(x) \Big| \\
&\qquad \leq \|a\|^2 \|b\|^2 \int_G \| \, \pi(x) \otimes \overline{\pi}(x)\| \, d\mu_s(x) \\
&\qquad \leq \|a\|^2 \|b\|^2 M_{T,\pi\otimes\overline{\pi}} < \infty.
\end{aligned}$$

Therefore, $\sup_{0\leq s\leq T} \int_{\mathbb{C}} |x|^2 \, df(\mu_s)(x) < \infty$ for all $T \geq 0$. This fact and the weak convergence of the measures $f(\mu_{t_n}) \in M^1(\mathbb{C})$ to $f(\mu_t)$ now imply that

$$\lim_n \int_{\mathbb{C}} x\, df(\mu_{t_n})(x) = \int_{\mathbb{C}} x\, df(\mu_t)(x);$$

see, for instance, Corollary 7 in Section 8.1 of [3]. This completes the proof of the proposition. □

3.2. Remarks. (1) Instead of the result of Siebert on submultiplicative functions, one may also use moment estimates of Ibero [8] for Gaussian processes on Lie groups in the proof above. Ibero's results, however, can be applied to finite-dimensional representations only.

(2) Let π be a representation of a locally compact second countable group G on some separable Hilbert space H, and let $(\mu_t)_{t\geq 0}$ be an arbitrary convolution semigroup on G. Along the lines of the preceding proof, the results in [10], [11] lead to jump conditions on the Lévy measure of $(\mu_t)_{t\geq 0}$ and the representation π which ensure that $(\tilde{\pi}(\mu_t))_{t\geq 0} \subset B(H)$ exists as a weakly continuous one-parameter semigroup.

We next turn to the proof of Theorem 2.9. We here need some preparations. We begin with a result which gives a connection between the classical form of the martingale problem and a martingale characterization in the spirit of Part (2) of Theorems 2.8 and 2.9. We omit the proof which is completely analogous to that of Proposition 2.8 of [15].

3.3. Proposition. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup and π be a finite-dimensional representation of G on some Hilbert space H such that $(\tilde{\pi}(\mu_t))_{t\geq 0}$ exists as a one-parameter semigroup. Let*

$$F := \lim_{t\to 0} \frac{1}{t}\big(\tilde{\pi}(\mu_t) - Id\big) \in B(H)$$

be its generator. Then for each càdlàg-process $(X_t)_{t\geq 0}$ on G, $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is a $B(H)$-valued local L^2-martingale if and only if so is $\big(\pi(X_t) - F\cdot \int_0^t \pi(X_s)\, ds\big)_{t\geq 0}$.

We also need the following technical result (which is likely to be known).

3.4. Lemma. *Let G, H be second countable locally compact spaces and $f : G \to H$ continuous and injective. Then, for all probability measures $\mu, \nu \in M^1(G)$, $f(\mu) = f(\nu)$ implies $\mu = \nu$.*

Proof. If $K \subset G$ is compact, then $f(K)$ is compact with $f^{-1}(f(K)) = K$. Hence, $\mu(K) = f(\mu)(f(K)) = f(\nu)(f(K)) = \nu(K)$. As μ and ν are regular, $\mu = \nu$ follows. □

3.5. Proposition. *Let G, H be second countable locally compact groups and $\pi : G \to H$ a continuous injective group homomorphism. Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on G and $(X_t)_{t\geq 0}$ a G-valued stochastic process. If $(\pi(X_t))_{t\geq 0}$ is a Lévy process on H associated with the convolution semigroup $(\pi(\mu_t))_{t\geq 0}$ on H, then $(X_t)_{t\geq 0}$ is a Lévy process on G associated with $(\mu_t)_{t\geq 0}$.*

Proof. We have to show that for any $n \in \mathbb{N}$ and $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ the probability measure $\mu := \mu_{t_n - t_{n-1}} \otimes \cdots \otimes \mu_{t_1 - t_0} \in M^1(G^n)$ is equal to the distribution $\nu \in M^1(G^n)$ of the random variable $(X_{t_n} X_{t_{n-1}}^{-1}, \ldots, X_{t_1} X_{t_0}^{-1})$. As by our assumption, $\pi^n(\mu) = \pi^n(\nu) \in M^1(H)$ holds for the continuous injective homomorphism $\pi^n : G^n \to H^n$ with $\pi^n((x_1, \ldots, x_n)) := (\pi(x_1), \ldots, \pi(x_n))$, the result follows from the preceding lemma. □

3.6. Proof of Theorem 2.9. The conclusion (1) $\Longrightarrow$ (2) follows from Lemma 2.7, as this lemma can be applied to the representations ρ and $\rho \otimes \rho$ by Proposition 2.4.

To prove (2) $\Longrightarrow$ (1), we regard ρ as a continuous faithful group homomorphism $\rho : G \to GL(n, \mathbb{R})$ for some $n \in \mathbb{N}$. By taking images of measures, this homomorphism induces a Banach algebra homomorphism $\rho : M_b(G) \to M_b(GL(n, \mathbb{R}))$ of measures such that, obviously, $(\rho(\mu_t))_{t\geq 0}$ is now a Gaussian semigroup on $GL(n, \mathbb{R})$. If the theorem is proved for the group $GL(n, \mathbb{R})$ with the identity as representation, we conclude that the process $(\rho(X_t))_{t\geq 0}$ is a Gaussian process on $GL(n, \mathbb{R})$ associated with $(\rho(\mu_t))_{t\geq 0}$, and the claim follows from Proposition 3.5.

It therefore suffices to check the theorem for $G = GL(n, \mathbb{R})$ and ρ as identity. In this case, take a continuous process $(X_t)_{t\geq 0}$ on G which satisfies Condition (2) of the theorem. Realize the tensor product representation $\rho \otimes \rho$ on $\mathbb{R}^{n^2}$ as matrices with $(\rho \otimes \rho(x))^{(ik)(jl)} = x^{ij} x^{kl}$. Moreover, denote the generators of the semigroups $(\tilde{\rho}(\mu_t))_{t\geq 0} \subset GL(n, \mathbb{R})$ and $(\widetilde{\rho \otimes \rho}(\mu_t))_{t\geq 0} \subset GL(n^2, \mathbb{R})$ by F_1 and F_2 respectively. Let $i, j, k, l \in \{1, \ldots, n\}$. It follows from Condition (2) that the coordinates $(X_t^{ij})_{t\geq 0}$

are semimartingales. Itô's formula hence yields

$$d(X_t^{ij}X_t^{kl}) = X_t^{ij}dX_t^{kl} + X_t^{kl}dX_t^{ij} + d[X^{ij}, X^{kl}]_t$$

where $[\,.\,,\,.\,]$ as usual denotes the quadratic variation. Therefore,

$$\begin{aligned} d(X_t^{ij}X_t^{kl}) - (F_2\cdot(X_t\otimes X_t))^{(i,k)(j,l)}dt - X_t^{ij}\big(dX_t^{kl} - (F_1\cdot X_t)^{kl}dt\big) \\ -X_t^{kl}\big(dX_t^{ij} - (F_1\cdot X_t)^{ij}dt\big) = H_{ijkl}(X_t)dt + d[X^{ij}, X^{kl}]_t \end{aligned} \tag{3.1}$$

with

$$H_{ijkl}(x) := x^{ij}(F_1\cdot x)^{kl} + x^{kl}(F_1\cdot x)^{ij} - (F_2\cdot(x\otimes x))^{(i,k)(j,l)} \quad (x\in G).$$

The left hand side of (3.1) is the differential of a local martingale by Condition (2) and Proposition 3.3. Moreover, as a stochastic integral with respect to a process with paths with a.s. locally finite variation is again a process of this kind see, for example, Proposition 5.3.5 in [16]), the right hand side of (3.1) has a.s. locally finite variation. Hence, both sides of (3.1) are a.s. equal to zero (see, for instance, Theorem 5.3.2 in [16]). We thus conclude that

$$d[X^{ij}, X^{kl}]_t = -H_{ijkl}(X_t)dt. \tag{3.2}$$

Now take $h\in C_0^2(\mathbb{R}^{n^2})$, and denote the partial derivative w.r.t. the variable x^{ij} by $\partial_{i,j}$. Then by Itô's formula and equation (3.2),

$$\begin{aligned} dh(X_t) = &\sum_{i,j=1}^n (\partial_{i,j}h)(X_t)\big(dX_t^{ij} - (F_1\cdot X_t)^{ij}dt\big) \\ &+ \sum_{i,j=1}^n (\partial_{i,j}h)(X_t)(F_1\cdot X_t)^{ij}dt \\ &- \frac{1}{2}\sum_{i,j,k,l=1}^n (\partial_{k,l}\partial_{i,j}h)(X_t)\cdot H_{ijkl}(X_t)dt. \end{aligned} \tag{3.3}$$

By our discussion in Section 2.1(3), $h|_G\in C_0^2(G)$ is in the domain of the generator L associated with $(\mu_t)_{t\geq 0}$. As

$$\sum_{i,j=1}^n (\partial_{i,j}h)(X_t)\big(dX_t^{ij} - (F_1\cdot X_t)^{ij}dt\big)$$

is the differential of a local martingale by Condition (2) and Proposition 3.3, equation (3.3) implies that

$$dh(X_t) - Lh(X_t)\,dt \tag{3.4}$$

is the differential of a local martingale if and only if

$$\begin{aligned}\Bigg\{\sum_{i,j=1}^{n} (\partial_{i,j}h)(X_t)(F_1\cdot X_t)^{ij} - Lh(X_t)\\ - \frac{1}{2}\sum_{i,j,k,l=1}^{n} (\partial_{k,l}\partial_{i,j}h)(X_t)\cdot H_{ijkl}(X_t)\Bigg\}dt\end{aligned} \tag{3.5}$$

is the differential of a local martingale. The latter is possible if and only if (3.5) is equal to zero a.s., which means that the integrand there disappears.

In order to prove that (3.5) is equal to zero a.s., we replace the process $(X_t)_{t\geq 0}$ above by some Gaussian process $(Y_t)_{t\geq 0}$ related with $(\mu_t)_{t\geq 0}$ such that the support of the initial distribution of Y_0 is equal to G. By Proposition 2.2, the associated differential (3.4) belongs to a local martingale. Hence, by the conclusions above,

$$\begin{aligned}\sum_{i,j=1}^{n} (\partial_{i,j}h)(Y_t)(F_1\cdot Y_t)^{ij} - Lh(Y_t)\\ - \frac{1}{2}\sum_{i,j,k,l=1}^{n} (\partial_{k,l}\partial_{i,j}h)(Y_t)\cdot H_{ijkl}(Y_t) = 0\end{aligned} \tag{3.6}$$

for all $t\geq 0$ a.s. It follows that for all $x\in G$,

$$\begin{aligned}\sum_{i,j=1}^{n} (\partial_{i,j}h)(x)(F_1\cdot x)^{ij} - Lh(x)\\ - \frac{1}{2}\sum_{i,j,k,l=1}^{n} (\partial_{k,l}\partial_{i,j}h)(x)\cdot H_{ijkl}(x) = 0.\end{aligned} \tag{3.7}$$

Therefore, the differential (3.5) is equal to zero for $(X_t)_{t\geq 0}$, and the differential (3.4) belongs to a local martingale for each $h\in C_0^2(\mathbb{R}^{n^2})$. Proposition 2.2 now completes the proof.

3.7. Remark. Equation (3.7) above yields the well-known fact due to Hunt that the generator L of a Gaussian semigroup on $GL(n,\mathbb{R})$ is a second-order differential operator; see Ch. 4 of [7].

Notice also that equation (3.7) allows an easy computation of L from the matrices F_1, F_2 above. Conversely, if L is known, then F_1, F_2 can be computed via

$$F_1 = (L\rho_{i,j}(I))_{i,j=1,\dots,n}, \quad F_2 = (L(\rho\otimes\rho)_{(i,k)(j,l)}(I))_{i,j,k,l=1,\dots,n} \tag{3.8}$$

with I the identity matrix. We give an example in the following section.

4. Examples and further conclusions

4.1. Example. The Heisenberg group $H := \{(x_1,x_2,x_3) : x_1,x_2,x_3 \in \mathbb{R}\}$ with multiplication

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3) := (x_1+y_1, x_2+y_2, x_3+y_3+(x_1y_2-x_2y_1)/2)$$

is isomorphic with the matrix group

$$\tilde{H} := \left\{ M(x,y,z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}$$

via the faithful representation

$$\rho : (x_1,x_2,x_3) \longmapsto M(x_1,x_2,x_3+x_1x_2/2). \tag{4.1}$$

It is well known that the sub-Laplacian

$$L := \frac{1}{2}\Big(\partial_1 + \frac{x_2}{2}\partial_3\Big)^2 + \frac{1}{2}\Big(\partial_2 - \frac{x_1}{2}\partial_3\Big)^2 \tag{4.2}$$

is the generator of a Gaussian convolution semigroup on H. If we transfer this object to $\tilde{H}$ by using ρ, we get the generator

$$\tilde{L} = \frac{1}{2}(\partial_x^2 + \partial_y^2 + y^2\partial_z^2 + 2y\partial_x\partial_z). \tag{4.3}$$

It is now straightforward to compute the generator matrices F_1, F_2 from (3.8) and (4.3), and thus the matrices $\tilde{\rho}(\mu_t)$ and $\widetilde{\rho\otimes\rho}(\mu_t)$. In this way, Theorem 2.9 leads to the following Lévy-characterization:

4.2. Corollary. *Let $(X_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (Z_t)_{t\geq 0}$ be continuous $\mathbb{R}$-valued processes. Then the $\tilde{H}$-valued process $(M(X_t, Y_t, Z_t))_{t\geq 0}$ is a Gaussian process associated with $\tilde{L}$ if and only if the processes*

$$(X_t)_{t\geq 0}, \quad (Y_t)_{t\geq 0}, \quad (Z_t)_{t\geq 0}, \quad (X_t^2 - t)_{t\geq 0}, \quad (Y_t^2 - t)_{t\geq 0}, \ (X_tY_t)_{t\geq 0},$$

$$(Z_tY_t)_{t\geq 0}, \quad (X_tZ_t - tY_t)_{t\geq 0}, \quad \text{and} \quad (Z_t^2 - tY_t^2 + t^2/2)_{t\geq 0}$$

are (local) martingales.

Let $(W_{1,t}, W_{2,t})_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^2$. Itô's calculus immediately implies that the martingale condition of the corollary holds for the process

$$\left(M\left(W_{1,t}, W_{2,t}, \int_0^t W_{2,s}\, dW_{1,s}\right)\right)_{t\geq 0}. \tag{4.4}$$

Corollary 4.2 therefore implies the well-known fact that the process (4.4) is a left-invariant Gaussian process on $\tilde{H}$ associated with the Gaussian convolution semigroup with generator $\tilde{L}$. Note that by Itô's formula, this is equivalent to the fact that

$$\left(W_{1,t}, W_{2,t}, \frac{1}{2}\int_0^t W_{2,s}\, dW_{1,s} - W_{1,s}\, dW_{2,s}\right)_{t\geq 0} \tag{4.5}$$

is a left-invariant Gaussian process on H associated with L. Clearly, this example can be extended to further Gaussian semigroups on H or on arbitrary simply connected nilpotent Lie groups (as these groups always admit a faithful representation as upper triangular matrices).

4.3. Remark. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$ with standard Gaussian semigroup $(\nu_t)_{t\geq 0}$. Consider the representation

$$\rho : x \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

of $\mathbb{R}$ as in Section 2.10. Then for each $n \in \mathbb{N}$, $\rho^{\otimes,n}(x)$ is a $(2^n \times 2^n)$-matrix with rows and columns labelled, say, by $(a_1, \ldots, a_n)$ with $a_i \in \{1, 2\}$ in the obvious way. In particular,

$$\rho^{\otimes,n}(x)_{(a_1,\ldots,a_n),(b_1,\ldots,b_n)} = \begin{cases} 0 & \text{if } a_i > b_i \text{ for some } i, \\ x^{|\{i: a_i < b_i\}|} & \text{otherwise.} \end{cases}$$

Using the moments of normal distributions, we thus obtain

$$\widetilde{\rho^{\otimes,n}}(\nu_t)_{(a_1,\ldots,a_n),(b_1,\ldots,b_n)} = \begin{cases} 0 & \text{if } a_i > b_i \text{ for some } i \text{ or} \\ & \text{if } |\{i : a_i < b_i\}| \text{ is odd,} \\ \dfrac{t^r(2r)!}{2^r r!} & \text{if } r := \frac{1}{2}|\{i : a_i < b_i\}| \in \mathbb{Z}. \end{cases}$$

As this matrix is an upper triangular which remains invariant under permutations in $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, a straightforward, but tedious induction on n yields that

$$(\widetilde{\rho^{\otimes,n}}(\nu_t))^{-1}_{(1,\ldots,1),(b_1,\ldots,b_n)} = \begin{cases} (-1)^r \dfrac{t^r(2r)!}{2^r r!} & \text{if } r := \frac{1}{2}|\{i : 1 < b_i\}| \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

We now use the usual normalization of the Hermite polynomials H_n (see, for instance, [13]) such that these polynomials are orthogonal w.r.t. the weight function e^{-x^2}. We then obtain from the well-known power series for the H_n (see equation (5.5.4) in [13]) that

$$\Big(\big(\widetilde{\rho^{\otimes,n}}(\nu_t)\big)^{-1} \rho^{\otimes,n}(x) \Big)_{(1,\ldots,1),(2,\ldots,2)} = \frac{1}{2^{n/2}} t^{n/2} H_n\big(x/\sqrt{t/2}\,\big). \tag{4.6}$$

Consequently, the trivial part of Theorem 1.8 implies that $(R_n(t, B_t))_{t\geq 0}$ is a martingale for each heat polynomial $R_n(t, x) := t^{n/2} H_n(x/\sqrt{t/2})$ with $n \geq 0$. We expect that equation (4.6) admits some physical meaning.

On the other hand it was observed by Wesolowski [17] that if $(X_t)_{t\geq 0}$ is an arbitrary process on $\mathbb{R}$ such that $(R_n(t, X_t))_{t\geq 0}$ is a martingale for $n \in \{1, 2, 3, 4\}$, then $(X_t)_{t\geq 0}$ is already a Brownian motion in distribution (in fact, the existence of a continuous version of $(X_t)_{t\geq 0}$ follows from Kolmogorov's criterion, and then Levy's classical martingale characterization applies). In view of (4.6) this observation may be restated as follows:

4.4. Theorem. *Let $(X_t)_{t\geq 0}$ be an arbitrary process on $\mathbb{R}$ such for $n \in \{1, 2, 3, 4\}$ and ρ as above, the processes $((\widetilde{\rho^{\otimes,n}}(\nu_t))^{-1} \rho^{\otimes,n}(X_t))_{t\geq 0}$ are martingales. Then $(X_t)_{t\geq 0}$ is a Brownian motion in distribution.*

This result can be extended to $\mathbb{R}^n$ in the obvious way. We expect that such a characterization holds more generally. More precisey, we have the following conjecture:

4.5. Conjecture. *Let $(\mu_t)_{t\geq 0}$ be a Gaussian semigroup on a locally compact second countable group G, and let ρ be a faithful finite-dimensional real representation of G. Let $(X_t)_{t\geq 0}$ be a G-valued process such that for each $\pi \in \{\rho^{\otimes,n} : n = 1,2,3,4\}$, the process $(\tilde{\pi}(\mu_t)^{-1}\pi(X_t))_{t\geq 0}$ is a martingale. Then $(X_t)_{t\geq 0}$ is a Gaussian process associated with $(\mu_t)_{t\geq 0}$.*

By Proposition 3.5, the conjecture can be easily reduced to the case $G = GL(n, \mathbb{R})$.

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References

[1] D. APPLEBAUM and H. KUNITA, Lévy flows on manifolds and Lévy processes on Lie groups, *J. Math. Kyoto Univ.* **33** (1993), 1103–1123.

[2] R. M. BLUMENTHAL and R. K. GETOOR, Markov Processes and Potential Theory, *Academic Press*, 1968.

[3] Y. S. CHOW and H. TEICHER, Probability Theory: Independence, Interchangeability, Martingales, *Springer*, 1978.

[4] S. N. ETHIER and T. G. KURTZ, Markov Processes, Characterization and Convergence, *Wiley*, 1986.

[5] P. FEINSILVER, Processes with independent increments on a Lie group, *Trans. Amer. Math. Soc.* **242** (1978), 73–121.

[6] E. HEWITT and K. A. ROSS, Abstract Harmonic Analysis I, *Springer*, 1979.

[7] H. HEYER, Probability Measures on Locally Compact Groups, *Springer*, 1977.

[8] M. IBERO, Intégrales stochastiques multiplicatives et construction de diffusions sur un groupe de Lie, *Bull. Soc. Math. France* **100** (1976), 175–191.

[9] M. METIVIER and J. PELLAUMAIL, Stochastic Integration, *Academic Press*, 1980.

[10] E. SIEBERT, Continuous convolution semigroups integrating a submultiplicative function, *Man. Math.* **37** (1982), 383–391.

[11] E. SIEBERT, Jumps of stochastic processes with values in a topological group, *Probab. Math. Stat.* **5** (1985), 197–209.

[12] D. STROOCK and S. R. S. VARADHAN, Multidimensional Diffusion Processes, *Springer*, 1979.

[13] G. SZEGŐ, Orthogonal Polynomials, (4th edn), *AMS*, 1975.

[14] M. VOIT, A Lévy-characterization of one-dimensional diffusions, *Arch. Math.* **70** (1998), 235–239.

[15] M. Voit, Martingale characterizations of stochastic processes on compact groups, *Probab. Math. Stat.* **19** (1999), 389–405.

[16] H. von Weizsäcker and G. Winkler, Stochastic Integrals, *Vieweg*, 1990.

[17] J. Wesolowski, A martingale characterization of the Poisson process, *Bull. Polish Acad. Sci. Math.* **38** (1990), 49–53.

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