Publ. Math. Debrecen
63/1-2 (2003), 145–156

Asymptotic stability of monomial functional equations

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Abstract. In this paper the asymptotic stability of monomial functional equations of any degree is proved. This kind of stability was investigated in the case of additive functions by F. Skof and for quadratic functionals by F. Skof and by S. M. Jung, and by A. Gilányi for higher orders.

Introduction

Throughout this paper X denotes a real normed space and Y denotes a Banach space. Let Y^X be the vector space of all functions from X to Y. For $y \in X$, the linear difference operator $\Delta_y : Y^X \to Y^X$ is defined by

$$\Delta_y f(x) = f(x+y) - f(x), \quad f \in Y^X,\ x \in X$$

and for $n \in \mathbb{N}$ by

$$\Delta_y^{n+1} f(x) = \Delta_y(\Delta_y^n f(x)), \quad f \in Y^X,\ x \in X.$$

It can be easily verified by induction that the n-th iterate satisfies

$$\Delta_y^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+ky), \quad f \in Y^X,\ x \in X. \tag{1}$$

Mathematics Subject Classification: 39B82.
Key words and phrases: functional equations stability monomial function, functional inequality, abelian group, normed space.

A function $f : X \to Y$ is called a monomial of degree $n \in \mathbb{N}$ if

$$\Delta_y^n f(x) = n! f(y), \quad x, y \in X.$$

The stability questions concerning polynomial or monomial function of higher orders have been extensively studied for many years, cf. e.g. D. H. Hyers [4], M. Albert and J. A. Baker [1]. Recently some important results were obtained by A. Gilányi [3], see Remark 3 below.

In this work we prove the Hyers–Ulam stability of the monomial functional equation on a special resticted domain and this result we apply to the study of an asymptotic behavior of that equation. The aim of this paper is to give the asymptotic stability of monomial functional equation in the following form:

Let X and Y be a real normed space and a Banach space, respectively, let n be a positive integer. Then the function $f : X \to Y$ satisfies the asymptotic condition:

$$\left\| \Delta_y^n f(x) - n! f(y) \right\| \to 0 \quad as \quad \|x\| + \|y\| \to \infty$$

if and only if it is a monomial of degree n.

This kind of stability was verified earlier in the case of additive functions by F. Skof [6], [8] and for quadratic functions by F. Skof [7], [9] and by S. M. Jung [5], cf. also A. Gilányi [3]. The present results were inspired by A. Gilányi [2], and in particular we will use in the sequel the following lemma, stated as Lemma 2 in [2].

Lemma. *Let $(G, +)$ be a group, $(S, +)$ be an abelian group, $f : G \to S$ be a function. Fix $n \in \mathbb{N}$, $\xi \in G$ and for the integers $i \in \mathbb{N} \cup \{0\}$ define the functions $F_i : G \to S$ by*

$$F_i(y) = \Delta_y^n f(\xi - iy) - n! f(y), \qquad y \in G. \tag{2}$$

Then for an arbitrary integer $l \geq 2$ and for the function $g : G \to S$ defined by

$$g(y) = \Delta_{ly}^n f(\xi - (l-1)ny) - n! f(ly), \qquad y \in G \tag{3}$$

there exist positive integers $k_1, \dots, k_{(l-1)n-1}$ for which

$$k_1 + \dots + k_{(l-1)n-1} = l^n - 2 \tag{4}$$

and

$$g(y) = F_0(y) + k_1 F_1(y) + \ldots + k_{(l-1)n-1} F_{(l-1)n-1}(y) \\ + F_{(l-1)n}(y) + l^n n! f(y) - n! f(ly), \qquad y \in G. \tag{5}$$

Results

Lemma 1. *Let X be a real normed space and Y be a Banach space. Let $\delta \geq 0$ and $d > 0$ be real numbers, n be a positive integer. If for a function $f : X \to Y$*

$$\left\| \Delta_y^n f(x) - n! f(y) \right\| \leq \delta, \quad x, y \in X, \ \|x\| + \|y\| \geq d \tag{6}$$

then

$$\|f(2x) - 2^n f(x)\| \leq \frac{2^n + 1}{n!} \delta, \qquad x \in X. \tag{7}$$

Proof. Let $n \in \mathbb{N}$, $d > 0$, $\delta \geq 0$ be fixed. Let X be a real normed space and Y be a Banach space. Assume that a mapping $f : X \to Y$ satisfies condition (6). Let $\xi \in X$ with $\|\xi\| = nd$ be fixed. We define the function $F_i : X \to Y$ for $i \in \{0, 1, \ldots, n\}$ and $g : X \to Y$ as in (2) and (3) with $l = 2$:

$$F_i(y) = \Delta_y^n f(\xi - iy) - n! f(y), \quad i \in \{0, 1, \ldots, n\}, \ y \in X$$

and

$$g(y) = \Delta_{2y}^n f(\xi - ny) - n! f(2y), \quad y \in X.$$

First we prove that

$$\|F_i(y)\| \leq \delta, \quad i \in \{0, 1, \ldots, n\}, \ y \in X \tag{8}$$

and

$$\|g(y)\| \leq \delta, \qquad y \in X. \tag{9}$$

We fix a $z \in X$ and we show the following inequalities:

$$\|\xi - iz\| + \|z\| \geq d, \qquad i \in \{0, 1, \ldots, n\} \tag{10}$$

$$\|\xi - nz\| + \|2z\| \geq d. \tag{11}$$

The inequality (10), for $i = 0$, takes the form

$$\|\xi\| + \|z\| = nd + \|z\| \geq nd \geq d.$$

Let $i = 1, \dots, n$ be fixed. Then

$$\|iz\| = \|\xi - (\xi - iz)\| \geq \|\xi\| - \|\xi - iz\|$$

and hence

$$\|z\| \geq \frac{1}{i}\|\xi\| - \frac{1}{i}\|\xi - iz\|. \tag{12}$$

Now, taking into account (12) we get for every $i \in \{1, \dots, n\}$

$$\begin{aligned}\|\xi - iz\| + \|z\| &\geq \|\xi - iz\| - \frac{1}{i}\|\xi - iz\| + \frac{1}{i}\|\xi\| \\ &\geq \left(1 - \frac{1}{i}\right)\|\xi - iz\| + \frac{1}{i}nd \geq \frac{nd}{i} \geq \frac{nd}{n} = d.\end{aligned}$$

Thus (10) is proved and the inequality (11) is an immediate consequence of (10) in the case $i = n$. Now, the inequality (6) together with (10) and (11) implies (8) and (9).

By Gilányi's lemma for $l = 2$ there exist positive integers $k_1, \dots, k_{n-1}$ such that

$$k_1 + \ldots + k_{n-1} = 2^n - 2 \tag{13}$$

and

$$\begin{aligned}n!(f(2y) - 2^n f(y)) = F_0(y) + k_1F_1(y) + \cdots + k_{n-1}F_{n-1}(y) \\ + F_n(y) - g(y), \qquad y \in X,\end{aligned} \tag{14}$$

(cf. (4) and (5)). From (14), (8), (9) and (13) we infer for every $y \in X$:

$$\begin{aligned}&\|f(2y) - 2^n f(y)\| \\ &= \frac{1}{n!}\|F_0(y) + k_1F_1(y) + \cdots + k_{n-1}F_{n-1}(y) + F_n(y) - g(y)\| \\ &\leq \frac{1}{n!}\left(\|F_0(y)\| + k_1\|F_1(y)\| + \cdots + k_{n-1}\|F_{n-1}(y)\| + \|F_n(y)\| + \|g(y)\|\right) \\ &\leq \frac{1}{n!}\left(\delta + (k_1 + \ldots + k_{n-1})\delta + 2\delta\right) = \frac{\delta}{n!}(3 + 2^n - 2) = \frac{2^n + 1}{n!}\delta, \quad y \in X\end{aligned}$$

that is, (7) holds. $\square$

Let us state our main result.

Theorem 2. *Let X be a real normed space and Y be a Banach space. Let $\delta \geq 0$, $d > 0$ be real numbers, n be a positive integer. If for a function $f : X \to Y$*

$$\left\|\Delta_y^n f(x) - n! f(y)\right\| \leq \delta, \quad x, y \in X, \ \|x\| + \|y\| \geq d \tag{15}$$

then there exists a unique monomial function $g : X \to Y$ of degree n such that

$$\|f(x) - g(x)\| \leq \frac{2^n+1}{2^{n-1} n!}\delta, \quad x \in X. \tag{16}$$

Proof. If a function $f : X \to Y$ satisfies a condition (15) then, by Lemma 1, we get

$$\|f(2x) - 2^n f(x)\| \leq \frac{2^n+1}{n!}\delta, \qquad x \in X. \tag{17}$$

Substituting $\frac{x}{2}$ for x in (17) and dividing this inequality by 2^n, we obtain:

$$\left\|\frac{1}{2^n} f(x) - f\left(\frac{x}{2}\right)\right\| \leq \frac{2^n+1}{n!}\,\frac{\delta}{2^n}, \qquad x \in X. \tag{18}$$

Using (18) we prove, by induction on k, that

$$\left\|\frac{1}{2^{nk}} f(x) - f\left(\frac{x}{2^k}\right)\right\| \leq \frac{2^n+1}{n!}\delta\left(\frac{1}{2^{n-1}} - \frac{1}{2^{nk}}\right), \quad x \in X, \ k \in \mathbb{N}. \tag{19}$$

For $k = 1$ inequality (19) is (18).

Assume that (19) holds true for some $k \in \mathbb{N}$. Applying (18) and the induction hypothesis we get for every $x \in X$:

$$\begin{aligned}
\left\|\frac{1}{2^{n(k+1)}} f(x) - f\left(\frac{x}{2^{k+1}}\right)\right\| &= \left\|\frac{1}{2^{nk}}\left(\frac{1}{2^n} f(x) - 2^{nk} f\left(\frac{x}{2^{k+1}}\right)\right)\right\| \\
&= \left\|\frac{1}{2^{nk}}\left(\frac{1}{2^n} f(x) - f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right) - 2^{nk} f\left(\frac{x}{2^{k+1}}\right)\right)\right\| \\
&\leq \frac{1}{2^{nk}}\left\|\frac{1}{2^n} f(x) - f\left(\frac{x}{2}\right)\right\| + \left\|\frac{1}{2^{nk}} f\left(\frac{x}{2}\right) - f\left(\frac{x}{2^{k+1}}\right)\right\| \\
&\leq \frac{1}{2^{nk}}\,\frac{2^n+1}{n!}\,\frac{\delta}{2^n} + \frac{2^n+1}{n!}\delta\left(\frac{1}{2^{n-1}} - \frac{1}{2^{nk}}\right)
\end{aligned}$$

$$= \frac{2^n+1}{n!}\delta\left(\frac{1}{2^{n-1}} + \frac{1}{2^{n(k+1)}} - \frac{1}{2^{nk}}\right) = \frac{2^n+1}{n!}\delta\left(\frac{1}{2^{n-1}} + \frac{1-2^n}{2^{n(k+1)}}\right)$$
$$\leq \frac{2^n+1}{n!}\delta\left(\frac{1}{2^{n-1}} - \frac{1}{2^{n(k+1)}}\right)$$

which ends the inductive proof of (19).

Let us define the functions $g_k : X \to Y$ by

$$g_k(x) = \frac{f(2^k x)}{2^{nk}}, \qquad x \in X,\ k \in \mathbb{N}. \tag{20}$$

Let us note that in view of (1) we easily get

$$\frac{1}{2^{kn}}\Delta^n_{2^k y} f(2^k x) = \Delta^n_y g_k(x), \quad x, y \in X,\ k \in \mathbb{N}. \tag{21}$$

We prove that $(g_k(x))_{k\in\mathbb{N}}$ is a Cauchy sequence for every $x \in X$. Let $x \in X$ be any fixed and choose an $m \in \mathbb{N}$ arbitrarily. By substituting $2^{k+m}x$ for x in (19) and dividing this inequality by 2^{mn}, we obtain:

$$\begin{aligned}&\left\|\frac{1}{2^{n(k+m)}} f\left(2^{k+m}x\right) - \frac{1}{2^{nm}} f\left(2^m x\right)\right\| \\ &\qquad \leq \frac{1}{2^{nm}}\frac{2^n+1}{n!}\delta\left(\frac{1}{2^{n-1}} - \frac{1}{2^{nk}}\right), \qquad x \in X,\ k, m \in \mathbb{N}\end{aligned}$$

or, taking (20) into account,

$$\begin{gathered}\|g_{k+m}(x) - g_m(x)\| \leq \frac{1}{2^{nm}}\frac{2^n+1}{n!}\delta\left(\frac{1}{2^{n-1}} - \frac{1}{2^{nk}}\right), \\ x \in X,\ k, m \in \mathbb{N}.\end{gathered} \tag{22}$$

Since the right-hand side of the inequality (22) tends to 0 as m tends to ∞, the sequence $(g_k(x))_{k\in\mathbb{N}}$ is a Cauchy sequence for every $x \in X$.

Because of the completeness of Y there exists

$$g(x) := \lim_{k\to\infty} g_k(x), \qquad x \in X. \tag{23}$$

Note what in particular we have (cf. (20))

$$g(0) = \lim_{k\to\infty}\frac{f(0)}{2^{nk}} = 0. \tag{24}$$

Now, if we replace x by $2^k x$ and y by $2^k y$ in the condition (15), we get:

$$\left\|\Delta^n_{2^k y} f(2^k x) - n! f(2^k y)\right\| \le \delta, \quad x, y \in X,\ k \in \mathbb{N},\ \|2^k x\| + \|2^k y\| \ge d,$$

or equivalently

$$\left\|\Delta^n_{2^k y} f(2^k x) - n! f(2^k y)\right\| \le \delta, \quad x, y \in X,\ k \in \mathbb{N},\ \|x\| + \|y\| \ge \frac{d}{2^k}. \quad (25)$$

Let us fix now $x, y \in X$ so that $\|x\| + \|y\| > 0$, and let $p \in \mathbb{N}$ be such that $\|x\| + \|y\| \ge \frac{d}{2^p}$. Then for every $k \ge p$ we have by (25)

$$\left\|\frac{1}{2^{nk}} \Delta^n_{2^k y} f(2^k x) - n! \frac{f(2^k y)}{2^{kn}}\right\| \le \frac{\delta}{2^{kn}},$$

or in view of (21)

$$\left\|\Delta^n_y g_k(x) - n! g_k(y)\right\| \le \frac{\delta}{2^{nk}}.$$

Whence letting k tend to infinity we obtain

$$\left\|\Delta^n_y g(x) - n! g(y)\right\| \le 0$$

or

$$\Delta^n_y g(x) = n! g(y). \quad (26)$$

The inequality (26) holds true for all $x, y \in X$. Indeed, if $x = y = 0$, then we get (26) immediately, taking into account (24). Thus g is a monomial of degree n.

Now, we will prove that $f - g$ is bounded on X. Substituting $2^k x$ for x in (19), we get:

$$\left\|\frac{1}{2^{nk}} f(2^k x) - f(x)\right\| \le \frac{2^n + 1}{n!} \delta \left(\frac{1}{2^{n-1}} - \frac{1}{2^{nk}}\right), \quad x \in X,\ k \in \mathbb{N}.$$

Letting k go to infinity in the inequality above and using (20), we obtain:

$$\|g(x) - f(x)\| \le \frac{2^n + 1}{2^{n-1} n!} \delta, \quad x \in X. \quad (27)$$

It remains to prove the uniqueness of function $g : X \to Y$, which satisfies (27) and (26). Suppose that there exists a monomial function $h : X \to Y$

of degree n which is different from g and satisfies

$$\|f(x) - h(x)\| \le \frac{2^n + 1}{2^{n-1} n!}\delta, \quad x \in X.$$

Substituting $2^k x$ for x, we get

$$\left\| f(2^k x) - h(2^k x) \right\| \le \frac{2^n + 1}{2^{n-1} n!}\delta, \quad x \in X.$$

Since h is a monomial function of degree n, we can write the above inequality in the form

$$\left\| f(2^k x) - 2^{nk} h(x) \right\| \le \frac{2^n + 1}{2^{n-1} n!}\delta, \quad x \in X.$$

Now, dividing the above by 2^{nk} and letting k tend to infinity, we obtain:

$$\|g(x) - h(x)\| \le 0, \qquad x \in X$$

that is,

$$g(x) = h(x), \qquad x \in X.$$

This ends the proof. □

Remark 1. If $n = 1$ in Theorem 2, we get the stability of additive functions on a restricted domain. This problem was solved by F. Skof [6] in the case of functions from $\mathbf{R}$ into a Banach space. Using another method of proof, Skof showed that the difference $f - g$ is bounded by 9δ. In this case, our theorem yields the estimation constant equal to 3δ.

Remark 2. Monomials of degree 2 can be also characterized as solutions of the equation

$$f(u + v) + f(u - v) = 2f(u) + 2f(v), \qquad u, v \in X \tag{28}$$

The asymptotic stability of (28) was investigated by Soon-Mo Jung in [5]. In particular S. M. Jung proved the following (cf. [5, Theorem 2]):

Let $d > 0$ and $\delta \ge 0$ be given. Suppose that $f : X \to Y$ satisfies

$$\begin{gathered} \|f(u + v) + f(u - v) - 2f(u) - 2f(v)\| \le \delta, \\ u, v \in X, \ \|u\| + \|v\| \ge d \end{gathered} \tag{29}$$

Then there exists a unique quadratic mapping Q (i.e. a solution of (28)) such that

$$\|f(u) - Q(u)\| \leq \frac{7}{2}\delta, \qquad u \in X. \tag{30}$$

We show that Jung's result follows from our Theorem 2, moreover, in (30) we get a better estimation. Indeed, suppose that $f : X \to Y$ satisfies (29) for some $\delta \geq 0$ and $d > 0$. Let $x, y \in X$ be such that $\|x\| + \|y\| \geq 2d$. Then

$$\begin{aligned}\|x+y\| + \|y\| &\geq \frac{1}{2}\|x+y\| + \|y\| = \frac{1}{2}\left(\|x+y\| + \|-y\|\right) + \frac{1}{2}\|y\| \\ &\geq \frac{1}{2}\left(\|x\| + \|y\|\right) \geq d,\end{aligned}$$

and hence by (29)

$$\begin{aligned}&\left\|\Delta_y^2 f(x) - 2f(y)\right\| = \|f(x+2y) - 2f(x+y) + f(x) - 2f(y)\| \\ &= \|f((x+y)+y) + f((x+y)-y) - 2f(x+y) - 2f(y)\| \leq \delta.\end{aligned}$$

In other words, f satisfies (15) (for $n = 2$) with δ and $2d$ instead of d. From Theorem 2 it follows that there exists a unique monomial g of degree 2 satisfying

$$\|f(x) - g(x)\| \leq \frac{5}{4}\delta, \qquad x \in X.$$

Since g is a monomial of degree 2 if and only if it satisfies (28), putting $Q = g$ we obtain the assertion of Jung's theorem (with $\frac{5}{4}\delta$ instead of $\frac{7}{2}\delta$).

Let us note that a asymptotic stability of slightly more general equation, i.e.

$$f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) = 0$$

was been studied by F. Skof in [9, Theorem 3] for functions mapping $\mathbb{R}$ into a Banach space.

We conclude with a result generalizing theorems of F. Skof and S. M. Jung to the case of monomials arbitrary degree (cf. [6, Teorema 4 and Corollario] and [5, Corollary 4]).

Theorem 3. *Let X be a real normed space, Y be a Banach space and let n be a positive integer. A mapping $f : X \to Y$ is a monomial function* *of degree n if and only if the asymptotic condition*

$$\left\|\Delta_y^n f(x) - n!f(y)\right\| \to 0 \quad \text{as} \quad \|x\| + \|y\| \to \infty \tag{31}$$

holds true.

Proof. Assume that $f : X \to Y$ satisfies the asymptotic condition (31). Then for any sequence $(\delta_k)_{k\in\mathbb{N}}$ monotonically decreasing to 0 there exists a sequence $(d_k)_{k\in\mathbb{N}}$ of real positive numbers such that

$$\left\|\Delta_y^n f(x) - n!f(y)\right\| \le \delta_k, \quad x, y \in X, \ \|x\| + \|y\| \ge d_k. \tag{32}$$

According to (32) and Theorem 2, there exists a unique monomial function $g_k : X \to Y$ of degree n such that

$$\|f(x) - g_k(x)\| \le \frac{2^n + 1}{2^{n-1}n!}\delta_k, \qquad x \in X. \tag{33}$$

Obviously, for any $k, m \in \mathbb{N}$, $k \neq m$ we obtain $g_k = g_m = g$ (since $g_k - g_m$ is bounded monomial function, hence $g_k - g_m = 0$). $\square$

Letting k tend to infinity in (33), we obtain:

$$\|f(x) - g(x)\| \le 0, \qquad x \in X,$$

that is,

$$f(x) = g(x), \qquad x \in X.$$

Hence, we conclude that $f : X \to Y$ is a monomial function of degree n.

The converse is trivial.

Remark 3. We would like to call attention to the paper by A. Gilányi [3] in which he proves that a function f from $\mathbb{R}$ into Banach space satisfying

$$\lim \frac{\|\Delta_y^n f(x) - n!f(y)\|}{|y|^\alpha} = 0$$

for a positive integer n, a real number $\alpha < n$ and as (x, y) tends to $(-\infty, \infty)$, $(\infty, -\infty)$, (∞, ∞) or $(-\infty, -\infty)$ is approximated by a monomial function of degree n. For the first reading, one might think the results in the present paper are special cases of Gilányi's ones. However, this is not the case. If $\alpha = 0$ in Gilányi's [3]; Corollary 1 we get the following:

Let X be a Banach space and $n \in \mathbb{N}$. If a function $f : \mathbb{R} \to X$ satisfies

$$\lim_{(x,y)\to(-\infty,\infty)} \|\Delta_y^n f(x) - n! f(y)\| = 0$$

then there exists a uniquely determined monomial function $g : \mathbb{R} \to X$ of degree n for which

$$\lim_{|y|\to\infty} \|f(y) - g(y)\| = 0.$$

In the above theorem we consider a function f on the set of reals, while in the Theorem 3 of this paper f is a mapping defined in a real normed space. On the other hand we assume a stronger convergence requiring that $\|x\| + \|y\| \to \infty$. As a reward we get that (31) forces f to be a monomial. In Gilányi's corollary f is asymptotically equal to a monomial.

Acknowledgement. The author is grateful to Professor Maciej Sablik for his precious suggestion in this paper. She is indebted to the referees for their valuable comments.

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(Received February 27, 2002, revised September 26, 2002)