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QR-submanifolds of a locally conformal quaternion Kaehler manifold

By BAYRAM ŞAHIN (Malatya) and RIFAT GÜNEŞ (Malatya)

Abstract. In this paper, we study QR-submanifolds of a locally conformal quaternion Kaehler manifold.We give the basic formulas for QR-submanifolds of a locally conformal quaternion Kaehler manifold and two examples of QR-submanifolds of a locally conformal quaternion Kaehler manifold. Necessary and sufficient conditions are given for a quaternion distribution on a QR-submanifold to be integrable. Also, a necessary and sufficient condition is given for a distribution $D^{\perp}$ on a QR-submanifold to be a totally geodesic foliation. Further, a theorem is obtained for a QR-submanifold to be mixed geodesic. Finally, totally umbilical QR-submanifolds are studied and some theorems are given.

1. Introduction

A locally conformal quaternion Kaehler manifold (shortly, l.c.q.K. manifold) is a quaternion Hermitian manifold whose metric is conformal to a quaternion Kaehler metric in some neighborhood of each point.

I. Vaisman reported on the locally conformal Kaehler structures in [10]. He also gave several results for locally conformal almost Kaehler manifolds to be Kaehler manifolds [10], [11], [12], [13], [14].

A. Bejancu introduced QR-submanifolds of a quaternion Kaehler manifold [1]. He obtained fundamental results about these submanifolds. It is known that a real hypersurface of a quaternion Kaehler manifold is a

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QR-submanifold [1]. The geometry of these submanifolds has been studied by many authors.

H. Pedersen, A. Swann and Y. S. Poon [8] introduced l.c.q.K. manifolds. They showed that a manifold is a quaternion Hermitian–Weyl manifold if and only if it is a l.c.q.K. manifold. L. Ornea and P. Piccini [6] showed that the Lee form of a compact l.c.q.K. manifold can be chosen as parallel form without any restrictions. It is known that this property is not guaranteed in the complex case [12], [13]. L. Ornea and P. Piccini [6] proved a theorem for a l.c.q.K. manifold to be a quaternion Kaehler manifold.

In this paper, we introduce QR-submanifolds of a l.c.q.K. manifold. We give some necessary and sufficient conditions for a quaternion distribution on a QR-submanifold of a l.c.q.K. manifold to be integrable. We also obtain a necessary and sufficient condition for a distribution $D^{\perp}$ of a QR-submanifold of a l.c.q.K. manifold to be a totally geodesic foliation. Moreover, we give a characterization for a QR-submanifold to be a mixed geodesic QR-submanifold. Finally, we give some results for totally umbilical QR-submanifolds.

2. Preliminaries

We denote a quaternion Hermitian manifold by $\left(\bar{M}, g, H\right)$, where H is a subbundle of $\mathrm{End}(T\bar{M})$ of rank 3 which is spanned by almost complex structures J_1, J_2 and J_3. We recall that a quaternion Hermitian metric g is said to be a quaternion Kaehler metric if its Levi–Civita connection $\bar{\nabla}$ satisfies $\bar{\nabla} H \subset H$.

A quaternion Hermitian manifold with metric g is a l.c.q.K. manifold if over neighborhoods $\{U_i\}$ covering M, $g|_{U_i} = e^{f_i} g'_i$ with g'_i a quaternion Kaehler metric on U_i. In this case, the Lee form ω is locally defined by $\omega|_{U_i} = df_i$ and satisfies

$$d\Theta = \omega \wedge \Theta, d\omega = 0 \tag{II.1}$$

where $\Theta = \sum_{\alpha=1}^{3} \Omega_\alpha \wedge \Omega_\alpha$ is the Kaehler 4-form. We note that property (II.1) is also a sufficient condition for a quaternion Hermitian metric to be a l.c.q.K. metric [6].

The Levi–Civita connections $\bar{D}^i$ of the local Kaehler metrics g_i' glue together on M to a connection $\bar{D}'$ related to the Levi–Civita connection $\bar{\nabla}$ of g by the formula

$$\bar{D}'_X Y = \bar{\nabla}_X Y - \frac{1}{2}\left\{\omega\left(X\right)Y + \omega\left(Y\right)X - g(X,Y)B\right\} \tag{II.2}$$

for any $X, Y \in \Gamma\left(T\bar{M}\right)$, where $B = \omega^{\#}$ is the Lee vector field [4].

Let $\bar{M}$ be a l.c.q.K. manifold and M a real submanifold of $\bar{M}$. Then M is called a QR-submanifold if there exists a vector subbundle ν of the normal bundle such that

$$J_a(\nu_x) = \nu_x \tag{II.3}$$

and

$$J_a(\nu_x^{\perp}) \subset T_M(x) \tag{II.4}$$

for $x \in M$ and $a = 1, 2, 3$, where $\nu^{\perp}$ is the orthogonal bundle complementary to ν in $TM^{\perp}$ [1]. Let M be a QR-submanifold of $\bar{M}$. Set $D_{ax} = J_a(\nu_x^{\perp})$. We consider $D_{1x} \oplus D_{2x} \oplus D_{3x} = D_x^{\perp}$. Then the $3s$-dimensional distribution $D^{\perp} : x \to D_x^{\perp}$ is globally defined on M, where $s = \dim \nu_x^{\perp}$. Also, we have for each $x \in M$

$$J_a(D_{ax}) = \nu_x^{\perp}, J_a(D_{bx}) = D_{cx} \tag{II.5}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. We denote the orthogonal distribution complementary to $D^{\perp}$ in TM by D. Then D is invariant with respect to the action of J_a, i.e. we have

$$J_a(D_x) = D_x \tag{II.6}$$

for any $x \in M$. D is called a quaternion distribution.

Let $\bar{M}$ be a l.c.q.K. manifold and $\bar{\nabla}$ be the connection of $\bar{M}$. Then the Weyl connection does not preserve the compatible almost complex structures individually but only their 3-dimensional bundle H. Indeed, Pedersen, Poon and Swann showed that

$$\bar{D}' J_a = \sum Q_{ab} \otimes J_b \tag{II.7}$$

for $a, b = 1, 2, 3$, and Q_{ab} is a skew-symmetric matrix of local forms [8]. Thus, from (II.1) and (II.2) we have

$$\bar{\nabla}_X J_a Y = J_a \bar{\nabla}_X Y + \frac{1}{2}\left\{\theta_o\left(Y\right)X - \omega\left(Y\right)J_a X - \Omega\left(X,Y\right)B + g(X,Y)J_a B\right\}$$

$$+ Q_{ab}(X)J_bY + Q_{ac}(X)J_cY \tag{II.8}$$

for any $X, Y \in \Gamma\left(T\bar{M}\right)$, where $\theta_o = \omega o J_a$.

We give the following

Theorem 2.1. [7] *Let* $(\bar{M}, \bar{g}, H)$ *be a compact quaternion Hermitian Weyl manifold, non-quaternion Kaehler, whose foliation* $\bar{D}$ *has compact leaves. Then the leaves space* $P = \bar{M}/\bar{D}$ *is a compact quaternion Kaehler orbifold with positive scalar curvature, the projection is a Riemannian, totally geodesic submersion and a fibre bundle map with fibres as described in Proposition* 4.10 *of* [7], *where* $\bar{D}$ *is locally generated by* $B, J_1B = B_1, B_2, B_3$.

If $\bar{D}$ is a regular foliation, then $P = \bar{M}/\bar{D}$ is a compact quaternion Kaehler manifold.

Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. Let P denote the projection morphism of TM to the quaternion distribution D and choose a local field of orthonormal frames $\{v_1, \ldots, v_s\}$ on the vector subbundle $\nu^\perp$ in $TM^\perp$. Then, on the distribution $D^\perp$, we have the local field of orthonormal frames

$$\{E_{11}, \ldots, E_{1s}, E_{21}, \ldots, E_{2s}, E_{31}, \ldots, E_{3s}\} \tag{II.9}$$

where $E_{ai} = J_a v_i$ and $i = 1, \ldots, s$. Thus any vector field Y tangent to M can be written locally as follows

$$Y = PY + \sum_{b=1}^{3} \sum_{i=1}^{s} W_{bi}(Y)E_{bi} \tag{II.10}$$

where the W_{bi} are 1-forms locally defined on M by

$$W_{bi}(Y) = g(Y, E_{bi}). \tag{II.11}$$

Applying J_a to (II.10) and taking account of (II.1) we have

$$J_aY = J_aPY + \sum_{i=1}^{s}\{W_{bi}(Y)E_{ci} - W_{ci}(Y)E_{bi}\} - W_{ai}(Y)v_i. \tag{II.12}$$

We can decompose J_aY as follows:

$$J_aY = \phi_aY + F_aY, a = 1, 2, 3, \tag{II.13}$$

for $Y \in \Gamma(TM)$, where $\phi_a Y$ and $F_a Y$ are the tangential and normal parts of $J_a Y$, respectively. Similarly, we get

$$J_a V = t_a V + f_a V. \tag{II.14}$$

Example 2.1. Let $\bar{M}$ be a l.c.q.K. manifold. Assume that the foliation $\bar{D}$ is regular. Then $P = \bar{M}/\bar{D}$ is a compact quaternion Kaehler manifold (cf. Theorem 2.1). We denote almost complex structures of $\bar{M}$ and P by J_a and J'_a, respectively. Now we consider the following commutative diagram:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\pi} & P = \bar{M}/\bar{D} \\ \uparrow i & & \uparrow j \\ N & \xrightarrow{\bar{\pi}} & \bar{N} \end{array}$$

where N and $\bar{N}$ are submanifolds of $\bar{M}$ and P, respectively. We denote the horizontal lift by *. Then we have

$$(J'_a X)^* = J_a X^*. \tag{II.15}$$

We note that the projection π is a totally geodesic Riemannian submersion and a fibre bundle map. Hence $\bar{\pi}$ is also a Riemannian submersion. We denote the vertical distribution of the Riemannian submersion π by υ, i.e. $\ker \pi_* = \upsilon$. Let $\bar{H}$ be the horizontal distribution of π. Then we have $T\bar{M} = \bar{H} \oplus \upsilon$. We denote the horizontal distribution of $\bar{\pi}$ by H_0. We will investigate the relation between normal spaces of N and $\bar{N}$. We denote the Riemannian metrics of $\bar{M}$ and P by g and g', respectively. Let V^* be the horizontal lift of $V \in \Gamma(T\bar{N}^\perp)$. Then we get

$$g(V^*, X) = g((\pi_*)^* V, X) = g'(\pi_* X, V) = 0,$$

for any $X \in H_0$. Thus, $(T\bar{N}^\perp)^*$ is orthogonal to H_0. Note that the normal space is always horizontal. Hence $(T\bar{N}^\perp)^*$ is orthogonal to υ. Consequently, we have $(T\bar{N}^\perp)^* \subseteq TN^\perp$. Since π is a Riemannian submersion we get

$$\left(T\bar{N}^\perp\right)^* = TN^\perp. \tag{II.16}$$

Now, let t_a and f_a be the operators on $\bar{N}$ appearing in (II.14). We denote the operators in N corresponding to t_a and f_a by t'_a and f'_a, respectively. From (II.15) and (II.16) we obtain

$$(t_a V)^* = t'_a V^* \tag{II.17}$$

and

$$(f_a V)^* = f'_a V^*. \tag{II.18}$$

So, from (II.17) and (II.18) we see that N is a QR-submanifold of $\bar{M}$ if and only if $\bar{N}$ is a QR-submanifold of P.

Example 2.2. Let $\bar{M}$ be a l.c.q.K. manifold. We assume that the distribution $\bar{D}$ is regular. Then $P = \bar{M}/\bar{D}$ is a quaternion Kaehler manifold. It is known that a real hypersurface of a quaternion Kaehler manifold is a QR-submanifold [1]. From the previous example, a real hypersurface of a l.c.q.K. manifold is a QR-submanifold. Let M be a real hypersurface of a l.c.q.K. manifold $\bar{M}$. We denote the normal space of M by $TM^{\perp}$. Set $TM^{\perp} = Sp\{N\}$. Since $\dim(T_x M^{\perp}) = 1$ and $g(J_a N, N) = 0$, we obtain $J_a(TM^{\perp}) \subset TM$. Thus, $\nu_x = \{0\}$ and $\nu_x^{\perp} = T_x M^{\perp}$ for $x \in M$.

Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. The formulae of Gauss and Weingarten are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{II.19}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \tag{II.20}$$

for vector fields X, Y tangent to M and any vector field V normal to M, where ∇ is the induced Riemann connection in M, h is the second fundamental form, A_V is the fundamental tensor field of Weingarten with respect to the normal section V and $\nabla^{\perp}$ is the normal connection. Moreover, we have the relation

$$g(h(X, Y), V) = g(A_V X, Y). \tag{II.21}$$

3. QR-submanifolds of a l.c.q.K. manifold

Lemma 3.1. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Then we have*

$$h(X, E_{ai}) = W_{ai}(A_{v_i} X) v_i + f_a \nabla_X^{\perp} v_i \tag{III.1}$$

and

$$g(\nabla_X Y, E_{ai}) = \frac{1}{2}\omega(v_i)g(J_aX, Y) - \frac{1}{2}\theta_o(v_i)g(X, Y) + g(J_aPA_{v_i}X, Y) \tag{III.2}$$

for any $X, Y \in \Gamma(D)$ *and* $v_i \in \Gamma\left(\nu^{\perp}\right)$.

Proof. From (II.8), (II.19) and (II.20), we obtain

$$\begin{aligned} h(X, E_{ai}) &= \bar{\nabla}_X E_{ai} - \nabla_X E_{ai} \\ &= \frac{1}{2}\left\{\theta_o(v_i)X - \omega\left(v_i\right)J_aX\right\} - Q_{ab}(X)E_{ci} + Q_{ac}(X)E_{bi} \\ &\quad - \nabla_X E_{ai} - J_aPA_{v_i}X - \sum_{i=1}^{s}\{W_{bi}(A_{v_i}X)E_{ci} - W_{ci}(A_{v_i}X)E_{bi} \\ &\quad - W_{ai}(A_{v_i}X)v_i\} + t_a\nabla^{\perp}_X v_i + f_a\nabla^{\perp}_X v_i. \end{aligned}$$

Considering the tangential and normal parts of the last equation we get

$$h(X, E_{ai}) = \sum_{i=1}^{s} W_{ai}(A_{v_i}X)v_i + f_a\nabla^{\perp}_X v_i,$$

and

$$\begin{aligned} 0 = \frac{1}{2}&\left\{\theta_o(v_i)X - \omega\left(v_i\right)J_aX\right\} - Q_{ab}(X)E_{ci} + Q_{ac}(X)E_{bi} - \nabla_X E_{ai} \\ &- J_aPA_{v_i}X - \sum_{i=1}^{s}\{W_{bi}(A_{v_i}X)E_{ci} - W_{ci}(A_{v_i}X)E_{bi}\} + t_a\nabla^{\perp}_X v_i. \end{aligned}$$

The proof of the lemma is complete. □

As a result of the lemma we have the following

Corollary 3.1. *Let* $\bar{M}$ *be a l.c.q.K. manifold and* M *be a QR-submanifold of* $\bar{M}$. *If the Lee vector field is tangent to* D *and* $A_{v_i}X \in \Gamma\left(D^{\perp}\right)$ *then* D *defines a totally geodesic foliation.*

Definition 3.1. A QR-submanifold is called mixed geodesic if $h(X, Y) = 0$ for any $X \in \Gamma\left(D\right)$ and $Y \in \Gamma\left(D^{\perp}\right)$ [2].

Theorem 3.1. *Let* $\bar{M}$ *be a l.c.q.K. manifold and* M *be a QR-submanifold of* $\bar{M}$. *Then* M *is mixed geodesic if and only if*

$$A_{v_i}X \in \Gamma(D) \tag{III.3}$$

and

$$\nabla^{\perp}_{X} v_i \in \Gamma\left(\nu^{\perp}\right) \tag{III.4}$$

for any $X \in \Gamma(D)$.

Proof. (⇒) Let M be a mixed geodesic QR-submanifold. From (III.1) we get

$$0 = W_{ai}(A_{v_i} X)v_i + f_a \nabla^{\perp}_{X} v_i$$

or

$$W_{ai}(A_{v_i} X)v_i = 0, f_a \nabla^{\perp}_{X} v_i = 0.$$

Thus we have $A_{v_i} X \in \Gamma(D)$ and $\nabla^{\perp}_{X} v_i \in \Gamma\left(\nu^{\perp}\right)$.

(⇐) We suppose that (III.3) and (III.4) are satisfied. From (III.1) we have $h(X, E_{ai}) = 0$ for any $X \in \Gamma(D)$. □

Let $\bar{M}$ be a l.c.q.K. manifold and M a QR-submanifold of $\bar{M}$. From (II.8), (II.13), (II.14), (II.19) and (II.20) we get

$$\begin{aligned} h(X, J_a PY) = {} & f_a h(X,Y) - W_{ai}(\nabla_X Y)v_i + \frac{1}{2}\omega(Y)W_{ai}(X)v_i \\ & - \frac{1}{2}\Omega(X,Y)B^{\perp} + \frac{1}{2}g(X,Y)B^{\perp}_{0} - Q_{ab}(X)\omega_{bi}(Y)v_i \\ & - Q_{ac}(X)W_{ci}(Y)v_i - W_{bi}(Y)h(X,E_{ci}) + W_{ci}(Y)h(X,E_{bi}) \\ & + X(W_{ai}(Y))v_i + W_{ai}(Y)\nabla^{\perp}_{X} v_i \end{aligned} \tag{III.5}$$

for any $X, Y \in \Gamma(TM)$, where $B_o = J_a B, B^{\perp} = NorB, B^T = TanB$.

Lemma 3.2. *Let* $\bar{M}$ *be a l.c.q.K. manifold and* M *be a QR-submanifold of* $\bar{M}$. *Then we have*

$$\begin{aligned} h(X, J_a Y) = {} & f_a h(X,Y) - W_{ai}(\nabla_X Y)v_i - \frac{1}{2}\Omega(X,Y)B^{\perp} \\ & + \frac{1}{2}g(X,Y)B^{\perp}_{0} \end{aligned} \tag{III.6}$$

for any $X, Y \in \Gamma(D)$.

Proof. It can easily be seen from (III.5). □

From Lemma 3.4 we have the following

Corollary 3.2. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. If D is integrable and $h(X, J_aY) = h(J_aX, Y)$ for any $X, Y \in \Gamma(D)$, $a = 1, 2, 3$, then the Lee vector field is tangent to M.*

Definition 3.2. Let M be a QR-submanifold of a l.c.q.K. manifold. Then M is called D-geodesic if $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.

Theorem 3.2. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Assume that the Lee vector field is tangent to M. Then the following assertions are equivalent:*

1) *$h(X, J_aY) = h(J_aX, Y)$ for any $X, Y \in \Gamma(D)$.*

2) *M is D-geodesic.*

3) *The quaternion distribution is integrable.*

Proof. (1) $\Longrightarrow$ (2): Since $\bar{M}$ is a quaternion Hermitian manifold, we have $J_c o J_b = -J_b o J_c = J_a$. Thus we get

$$h(X, J_aY) = h(J_aX, Y) = h((J_c o J_b)\, X, Y)$$
$$= h(J_bX, J_cY) = h(X, (J_b o J_c)Y) = -h(X, J_aY).$$

Hence we have $h(X, J_aY) = 0$.

(2) $\Longrightarrow$ (3): By using (III.6) we get

$$-W_{ai}(\nabla_X Y)v_i + \frac{1}{2}g(X, Y)B_0^{\perp} = 0.$$

Thus, interchanging X and Y in the last equation, we have

$$-W_{ai}(\nabla_Y X)v_i + \frac{1}{2}g(Y, X)B_0^{\perp} = 0.$$

Hence we obtain $[Y, X] \in \Gamma\,(D)$.

(3) $\Longrightarrow$ (1): We suppose that D is integrable. From (III.5) we obtain

$$-W_{ai}(\nabla_X Y)v_i - \frac{1}{2}\Omega\,(X, Y)\,B^{\perp} + \frac{1}{2}g(X, Y)B_0^{\perp} = 0,$$

or

$$-W_{ai}(\nabla_Y X)v_i - \frac{1}{2}\Omega\,(Y, X)\,B^{\perp} + \frac{1}{2}g(Y, X)B_0^{\perp} = 0,$$

for any $X, Y \in \Gamma(D)$. Hence we get

$$h(X, J_aY) - h(J_aX, Y) = \Omega(Y, X)B^{\perp}.$$

Since B is tangent to M we have $h(X, J_aY) = h(J_aX, Y)$. □

Corollary 3.3. *Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. Then $D^{\perp}$ defines a totally geodesic foliation if and only if*

$$A_{v_i}V \in \Gamma\left(D^{\perp}\right)$$

for any $V \in \Gamma\left(D^{\perp}\right)$.

Proof. From (II.8) we have

$$\begin{aligned}\bar{\nabla}_{E_{bj}}v_i &= -\bar{\nabla}_{E_{bj}}J_aE_{ai} = -\left(\bar{\nabla}_{E_{bj}}J_a\right)E_{ai} - J_a\bar{\nabla}_{E_{bj}}E_{ai}\\ &= -\frac{1}{2}\{\theta_0(E_{ai})E_{bj} - \omega(E_{ai})J_aE_{bi} - \Omega\left(E_{bj}, E_{ai}\right)B + g\left(E_{bj}, E_{ai}\right)J_aB\}\\ &\quad + Q_{ab}(E_{bj})J_bE_{ai} + Q_{ac}(E_{bj})J_cE_{ai} - J_a(\nabla_{E_{bj}}E_{ai} + h(E_{bj}, E_{ai}).\end{aligned}$$

Considering (II.20) we have

$$\begin{aligned}&PA_{v_i}E_{bj} + \sum_{i=1}^{s}W_{bi}(A_{v_i}E_{bj})E_{ci} - W_{ci}(A_{v_i}E_{bj})E_{bi} + t_a\nabla_{E_{bj}}v_i\\ &\quad = -\frac{1}{2}\{\theta_o(E_{ai})E_{cj} + \omega\left(E_{ai}\right)E_{bi}\}\\ &\qquad + Q_{ab}(E_{bj})E_{bi} + Q_{ac}(E_{bj})E_{ci} + \nabla_{E_{bj}}E_{ai}.\end{aligned} \tag{III.7}$$

If $D^{\perp}$defines a totally geodesic foliation then we have $PA_{v_i}E_{bj} = 0$. Hence $A_{v_i}E_{bj} \in \Gamma\left(D^{\perp}\right)$. Conversely, if $A_{v_i}E_{bj} \in \Gamma\left(D^{\perp}\right)$ then $D^{\perp}$ defines a totally geodesic foliation. □

Lemma 3.3. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Then, we have*

$$A_jE_{ai} = A_iE_{aj} + \frac{1}{2}\omega\left(v_i\right)E_{aj} - \frac{1}{2}\omega\left(v_j\right)E_{ai} \tag{III.8}$$

for any $v_i, v_j \in \Gamma\left(\nu^{\perp}\right)$.

Proof. From (II.8), (II.19) and (II.20) we have

$$\nabla_XE_{ai} + h(X, E_{ai}) = -J_aA_{v_i}X + J_a\nabla^{\perp}_Xv_i$$

$$+\frac{1}{2}\{\theta_o(v_i)X-\omega\left(v_i\right)J_aX-g(X,E_{ai})B\}$$
$$+Q_{ab}(X)E_{bi}+Q_{ac}(X)E_{ci} \tag{III.9}$$

or

$$g(h(X,E_{ai}),v_j)=-g(J_aA_{v_i}X,v_j)+g(J_a\nabla^{\perp}_{X}v_i,v_j)$$
$$+\frac{1}{2}\omega\left(v_i\right)g(X,J_av_j)-\frac{1}{2}\omega\left(v_j\right)g(E_{ai},X)$$

$$g(A_jE_{ai},X)=g(A_{v_i}X,E_{aj})+\frac{1}{2}\omega\left(v_i\right)g(X,E_{aj})-\frac{1}{2}\omega\left(v_j\right)g(E_{ai},X)$$

for any $X\in\Gamma\left(TM\right)$. Hence we get

$$A_jE_{ai}=A_iE_{aj}+\frac{1}{2}\omega\left(v_i\right)E_{aj}-\frac{1}{2}\omega\left(v_j\right)E_{ai}. \qquad \square$$

Lemma 3.4. *Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. Then, we have*

$$B_{aij}(X)=-\frac{1}{2}\delta_{ij}\omega\left(X\right)+g(A_jE_{ai},J_aX) \tag{III.10}$$

for any $X\in\Gamma(D)$, where $B_{aij}(X)=g(\nabla_{E_{ai}}E_{aj},X)$.

Proof. From (III.9) we get

$$g\left(\nabla_{E_{ai}}E_{aj},X\right)=-g\left(J_aA_{v_j}E_{ai},X\right)-\frac{1}{2}g(E_{ai},E_{aj})g(B,X)$$
$$=g\left(A_{v_j}E_{ai},J_aX\right)-\frac{1}{2}\delta_{ij}\omega(X). \qquad \square$$

Lemma 3.5. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Then we have*

$$g\left(\nabla_{E_{ai}}E_{bj},X\right)=-B_{aji}(J_cX)-\frac{1}{2}\delta_{ij}g(B,J_cX) \tag{III.11}$$

for any $X\in\Gamma(D)$.

Proof. From (II.20) we obtain

$$g\left(\nabla_{E_{ai}}E_{bj},X\right)=g\left(\bar{\nabla}_{E_{ai}}E_{bj},X\right)=g\left(J_c\bar{\nabla}_{E_{ai}}E_{bj},J_cX\right)$$

$$= g\left(\bar{\nabla}_{E_{ai}} J_c E_{bj} - \left(\bar{\nabla}_{E_{ai}} J_c\right) E_{bj}, J_c X\right).$$

Since $J_c E_{bj} = -E_{aj}$, we get

$$g\left(\nabla_{E_{ai}} E_{bj}, X\right) = -g\left(\bar{\nabla}_{E_{ai}} E_{aj}, J_c X\right) - g(\left(\bar{\nabla}_{E_{ai}} J_c\right) E_{bj}, J_c X).$$

By using (II.8) we get

$$g\left(\nabla_{E_{ai}} E_{bj}, X\right) = -B_{aij}(J_c X) - \frac{1}{2}\delta_{ij} g(B, J_c X). \qquad \square$$

From Lemma 3.4, Lemma 3.5 and Corollary 3.3, we have the following corollaries:

Corollary 3.4. *Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. Then $D^{\perp}$ defines a totally geodesic foliation if and only if B is normal to D and $B_{aij}(X) = 0, X \in \Gamma(D)$.*

Corollary 3.5. *Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. If the distribution $D^{\perp}$ is integrable and $B_{aij}(X) = 0$, $X \in \Gamma(D)$ for all $i, j = 1, \ldots, s$, then B is normal to D.*

Proof. From (III.8) and (III.10) we get

$$\begin{aligned} B_{aij}(X) &= -\frac{1}{2}\delta_{ij}\omega(X) + g(A_i E_{aj} + \frac{1}{2}\omega\left(v_i\right) E_{aj} - \frac{1}{2}\omega\left(v_j\right) E_{ai}, J_a X) \\ &= -\frac{1}{2}\delta_{ij}\omega(X) + g(A_j E_{ai}, J_a X) = B_{aji}(X). \end{aligned} \tag{III.12}$$

On the other hand we have

$$g\left(\nabla_{E_{bj}} E_{ai}, X\right) = -B_{bji}(J_c X) - \frac{1}{2}\delta_{ij} g(B, J_c X). \tag{III.13}$$

Thus, from (III.11) and (III.13) we get

$$g(\left[E_{ai}, E_{bj}\right], X) = -B_{aij}(J_c X) - B_{bji}(J_c X) + \delta_{ij} g(B, J_c X). \tag{III.14}$$

From (III.14) and (III.12) the proof is results. $\square$

The rest of this section is devoted to the study of totally umbilical QR-submanifolds of a l.c.q.K. manifold.

We recall that any submanifold is called totally umbilical in a Riemann manifold if

$$h(X, Y) = g(X, Y) H \tag{III.15}$$

for any $X, Y \in \Gamma(TM)$, where H is the mean curvature vector.

Corollary 3.6. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a totally umbilical QR-submanifold of $\bar{M}$. Then $D^{\perp}$ defines a totally geodesic foliation if and only if B is normal to the quaternion distribution.*

Proof. From (III.10) and (III.15) we have

$$B_{aij}(X) = -\frac{1}{2}\delta_{ij}\omega(X) + g(A_j E_{ai}, J_a X) = -\frac{1}{2}\delta_{ij} g(X, B) + g(h\left(E_{ai}, J_a X\right), v_j) = -\frac{1}{2}\delta_{ij} g(X, B), \tag{III.16}$$

for any $X \in \Gamma(D)$. By using (III.11) we get

$$g\left(\nabla_{E_{ai}} E_{bj}, X\right) = -B_{aij}(J_c X) - \frac{1}{2}\delta_{ij} g(B, J_c X). \tag{III.17}$$

Thus from (III.16) and (III.17) we have the assertion of the corollary. □

Theorem 3.3. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a totally umbilical QR-submanifold of $\bar{M}$. Assume that the Lee vector field is tangent to M. If $\dim \nu_x^{\perp} > 1$ for $x \in M$, then the QR-submanifold is totally geodesic.*

Proof. From (III.8) we have

$$A_j E_{ai} = A_i E_{aj} + \frac{1}{2}\omega\left(v_i\right) E_{aj} - \frac{1}{2}\omega\left(v_j\right) E_{ai}$$

for $X, Y \in \Gamma\left(D_{ax}\right)$, hence

$$A_{J_a X} Y = A_{J_a Y} X + \frac{1}{2}\omega\left(v_i\right) X - \frac{1}{2}\omega\left(v_j\right) Y,$$

where $J_a v_i = Y, J_a v_j = X$. Since $t_a H \in \Gamma\left(D_{ax}\right)$ at each $x \in M$, we have

$$A_{J_a X} t_a H = A_{J_a t_a H} X + \frac{1}{2}\omega\left(t_a H\right) X - \frac{1}{2}\omega\left(v_j\right) t_a H.$$

Now we derive

$$g\left(A_{J_a X} t_a H, X\right) = g\left(A_{J_a t_a H} X, X\right) + \frac{1}{2}\omega\left(J_a t_a H\right) g(X, X) - \frac{1}{2}\omega\left(v_j\right) g(t_a H, X)$$

$$g(h(t_aH,X),J_aX)=g(h(X,X),J_at_aH)+\frac{1}{2}\omega(J_at_aH)g(X,X)$$
$$-\frac{1}{2}\omega(v_j)g(t_aH,X).$$

Since M is totally umbilical, we have

$$\begin{aligned}g\left(t_aH,X\right)g\left(H,J_aX\right)&=g(X,X)g\left(H,J_at_aH\right)+\frac{1}{2}\omega\left(J_at_aH\right)g(X,X)\\&\quad-\frac{1}{2}\omega\left(v_j\right)g(t_aH,X)\\&=-g(X,X)g\left(t_aH,t_aH\right)+\frac{1}{2}\omega\left(J_at_aH\right)g(X,X)-\frac{1}{2}\omega\left(v_j\right)g(t_aH,X).\end{aligned}$$

By the hypothesis of the theorem, we can choose $X\in\Gamma(TM)$ such that $X\neq 0$ and X is orthogonal to B_aH. Since the Lee vector field is tangent to M, we obtain

$$0=-g\left(X,X\right)g\left(t_aH,t_aH\right),$$

that is

$$t_aH=0. \tag{III.18}$$

On the other hand, by using (II.12), (II.14), (II.19) and (II.20) in (II.8) and taking the tangential parts we obtain

$$\begin{aligned}\nabla_Yt_aV-A_{f_aV}Y=&-J_aPA_VY-W_{bk}\left(A_VY\right)E_{ck}+W_{ck}\left(A_VY\right)E_{bk}\\&+t_a\nabla^{\perp}_YV+\frac{1}{2}\theta_o(V)Y-\frac{1}{2}\omega(V)\phi_aY\\&-\frac{1}{2}\Omega(Y,V)B^T+Q_{ab}(Y)t_bV+Q_{ac}(Y)t_cV\end{aligned}$$

or

$$\begin{aligned}P\nabla_Yt_aV-PA_{f_aV}Y=&-J_aPA_VY+\frac{1}{2}\theta_o(V)PY-\frac{1}{2}\omega(V)P\phi_aY\\&-\frac{1}{2}\Omega\left(Y,V\right)PB^T\end{aligned} \tag{III.19}$$

for any $Y\in\Gamma(TM)$ and $V\in\Gamma\left(TM^{\perp}\right)$. From (III.18) and (III.19) we get

$$-PA_{f_aH}Y=-J_aPA_HY+\frac{1}{2}\theta_o(H)PY-\frac{1}{2}\Omega(Y,H)PB^T.$$

For $Z \in \Gamma(D)$, we have

$$g\left(PA_{f_aH}Y, Z\right) = -g\left(A_{f_aH}Y, Z\right) = -g\left(h(Y,Z), J_aH\right)$$
$$= -g(Y,Z)g(H, J_aH) = 0.$$

Hence we obtain

$$g\left(J_aPA_HY, Z\right) + \frac{1}{2}\theta_o(H)g(PY,Z) - \frac{1}{2}\Omega(Y,H)g\left(PB^T, Z\right) = 0$$

$$g\left(PA_HY, J_aZ\right) + \frac{1}{2}\theta_o(H)g(PY,Z) - \frac{1}{2}g\left(Y, J_aH\right)g\left(PB^T, Z\right) = 0$$

$$g\left(A_HY, J_aZ\right) + \frac{1}{2}\theta_o(H)g(PY,Z) - \frac{1}{2}g\left(Y, t_aH\right)g\left(PB^T, Z\right) = 0$$

$$g\left(h\left(Y, J_aZ\right), H\right) + \frac{1}{2}\theta_o(H)g(PY,Z) - \frac{1}{2}g\left(Y, t_aH\right)g\left(PB^T, Z\right) = 0.$$

Since the Lee vector field is tangent to M and $t_aH = 0$, we get

$$g\left(h\left(Y, J_aZ\right), H\right) = 0$$
$$g\left(Y, J_aZ\right)g\left(H, H\right) = 0.$$

Thus, we obtain $H = 0$ for $Y = J_aZ$. □

Let $\bar{M}$ be a compact l.c.q.K. manifold. Then we can choose the metric g such that

i) The fixed metric g makes ω parallel:

$$\bar{\nabla}\omega = 0, \tag{III.20}$$

ii)

$$\|\omega\| = 1 \tag{III.21}$$

[6]. From now on we will denote a compact l.c.q.K. manifold by $\bar{M}$.

Lemma 3.6. *Let K_0 be the curvature tensor field of the Weyl connection $\bar{D}'$ of the l.c.q.K. manifold $\bar{M}$ and $\bar{R}$ the curvature tensor field of the Levi–Civita connection ∇ of the l.c.q.K. manifold $\bar{M}$. Then we have*

$$\begin{aligned} K_0(X,Y)Z &= \bar{R}(X,Y)Z + \frac{1}{4}\left\{\omega(Z)\omega(Y)X - \omega(Z)\omega(X)Y\right\} \\ &+ \frac{1}{4}\left\{-\omega(Y)g(X,Z) + \omega(X)g(Y,Z)\right\}B - \frac{1}{4}(X \wedge Y)Z, \end{aligned} \tag{III.22}$$

for any $X, Y, Z \in \Gamma\left(T\bar{M}\right)$.

Proof. From (II.2) we have

$$K_0(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2}\{L_0(X,Z)Y - L_0(Y,Z)X + g(X,Z)L_0(Y,\,.\,)^{\#} - g(Y,Z)L_0(X,\,.\,)^{\#}\} - \frac{1}{4}\|\omega\|^2(X \wedge Y)Z,$$

for any $X, Y, Z \in \Gamma\left(T\bar{M}\right)$, where $L_0 = \bar{\nabla}\omega + \frac{1}{2}\omega \otimes \omega$. Since $\bar{\nabla}\omega = 0$ we have

$$\bar{\nabla}B = 0. \tag{III.23}$$

Thus, from (III.20), (III.21) and (III.23) we have the assertion of the lemma. □

Let $\bar{D}'$ be the Weyl connection of $\bar{M}$. Then we have

$$K_0(X,Y)J_1Z - J_1K_0(X,Y)Z = \alpha\,(X,Y)\,J_2Z - \beta\,(X,Y)\,J_3Z, \tag{III.24}$$

where

$$\alpha = dQ_{12} + Q_{32} \wedge Q_{13}$$

and

$$\beta = dQ_{13} + Q_{23} \wedge Q_{12}.$$

Theorem 3.4. *There exist no proper totally umbilical QR-submanifolds in negatively curved L.c.q.K. manifolds with* $B^T = 0$.

Proof. Considering the definition of a QR-submanifold, from (III.22) and (III.24) we have

$$-\bar{R}(X, E_{ai}, J_aX, v_i) - \bar{R}(X, E_{ai}, X, E_{ai}) = -\frac{1}{4}\omega(X)\omega(X) - \frac{1}{4}\omega\,(E_{ai})\,\omega\,(E_{ai}) + \frac{1}{4}$$

for any orthonormal vector field $X \in \Gamma(D)$ and $E_{ai} \in \Gamma(D^{\perp})$. Thus, if B is normal to M we get

$$-\bar{R}(X, E_{ai}, J_aX, v_i) + \bar{R}(X, E_{ai}, E_{ai}, X) = \frac{1}{4}. \tag{III.25}$$

Now suppose that M is a proper totally umbilical QR-submanifold of $\bar{M}$ with $K_{\bar{M}} < 0$.

Then from the equation of Codazzi we have

$$g(\bar{R}(X,Y)Z,W) = g(Y,Z)g\left(\nabla^{\perp}_{X}H,W\right) - g(X,Z)g\left(\nabla^{\perp}_{Y}H,W\right),$$

for any X, Y, Z tangent to M and V normal to M. Thus, if we take $X \in \Gamma(D), Z = J_1X, Y = E_{1i}$ and $W = v_i$ we obtain

$$\bar{R}(X, E_{ai}, J_aX, v_i) = 0. \tag{III.26}$$

Using (III.25) and (III.26) we get $K_{\bar{M}}(X, E_{1i}) = \frac{1}{4}$ which is a contradiction. □

From the Gauss equation for totally umbilical submanifolds we have

$$K_M(X,Y) = K_{\bar{M}}(X,Y) + \|H\|^2,$$

for any X, Y tangent to M [3]. Now we take $X \in \Gamma(D)$ and $E_{ai} = Y$ in this equation and taking account of $K_{\bar{M}}(X, E_{ai}) = 0$ we obtain

$$K_M(X, E_{ai}) = \|H\|^2 + \frac{1}{4}. \tag{III.27}$$

Thus, from (III.27) and (III.26) we have the following.

Corollary 3.7. *There exist no proper totally umbilical negatively curved QR-submanifolds of a l.c.q.K. manifold with* $B^T = 0$.

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B. ṢAHIN and R. GÜNEṢ
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ART
INONU UNIVERSITY
44100-MALATYA
TURKEY

E-mail: bsahin@inonu.edu.tr

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