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A class of contact Riemannian manifolds whose associated CR-structures are integable

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Abstract. We study the geometry of contact Riemannian manifolds whose associated CR-structures are integrable.

1. Introduction

There are two typical examples of contact manifolds; one is formed by the principal circle bundles over symplectic manifolds of integral class (including the odd-dimensional spheres) and the other is given by the unit tangent sphere bundles of Riemannian manifolds. These spaces have the standard Riemannian structures and their associated CR-structures. A contact structure η is a global differentiable 1-form on a smooth manifold M^{2n+1} such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is well-known that there exists an associated Riemannian metric structure g and $(1,1)$-type tensor ϕ, where (η, g) and (η, ϕ) are canonically related. We call the pair (η, g) a contact Riemannian structure and $M = (M; \eta, g)$ a contact Riemannian manifold. Sasaki and Hatakeyama [10] defined the normality of the contact Riemannian structure (see Section 2). A normal contact Riemannian manifold is said to be a Sasakian manifold. One the other

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hand, the associated CR-structure of a given contact Riemannian manifold $M = (M; \eta, g, \phi)$ is given by the holomorphic subbundle

$$\mathcal{H} = \{X - i\bar{\phi}X : X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM, where D is the subbundle of TM defined by the kernel of η and $\bar{\phi} = \phi \mid D$, the restriction of ϕ to D. Then we see that each fiber $\mathcal{H}_x$ $(x \in M)$ is of complex dimension n and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. We say that *the associated CR-structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$ we define the Levi form by

$$L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, \phi Y)$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is hermitian and positive definite, that is, the CR-structure is a *strongly pseudo-convex, pseudo-hermitian CR structure*. In [14] S. Tanno proved that for a given contact Riemannian manifold M the associated CR-structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q_1 = 0$ (see Section 2). Here, we note that the normality of a contact Riemannian structure implies the integrability of the associated CR-structure, but the converse does not always hold. The associated CR-structures of 3-dimensional contact Riemannian manifolds are always integrable (see [14]).

In this paper, we concentrate our attention on contact Riemannian manifolds whose associated CR-structures are integrable. We call such manifolds briefly, *contact Riemannian manifolds of class* $\mathfrak{Q}_1$. We introduce a class $\mathfrak{Q}$ of contact Riemannian manifolds whose associated CR-structures are integrable ($Q_1 = 0$) and satisfy the additional condition $Q_2 = 0$ (see Section 4). Here, we remark that these classes $\mathfrak{Q}_1$ and $\mathfrak{Q}$ are invariant under a D-homothetic deformation (see Sections 2, 4). Further, we can see that the class $\mathfrak{Q}$ contains the unit tangent sphere bundles of real space forms and the contact (k, μ)-spaces which appeared in [3], [6]. In Section 2, we prepare some fundamental facts about contact Riemannian manifolds and we review their associated CR-structures. In Section 3, we study the curvature tensor R of contact Riemannian manifolds of class $\mathfrak{Q}_1$. In Section 4, we obtain the nice form of the curvature tensor of a contact Riemannian manifold of class $\mathfrak{Q}$ and of constant ϕ-holomorphic sectional curvature. We call a simply connected and complete space belonging to $\mathfrak{Q}$ and of constant ϕ-holomorphic sectional curvature a *contact Riemannian space form*. In case of dimension three, the contact Riemannian space forms coincide with contact (k,μ)-spaces (see Remark 4.5). In higher dimensions, we can find (non-Sasakian) contact Riemannian space forms of constant ϕ-holomorphic sectional curvature $H > 0$, $H = 0$, or $H < 0$. From this observation, we see that the contact Riemannian space form is a proper extension of the Sasakian space form. Furthermore, in Section 4 we prove the equivalence theorem, the homogeneity and the ϕ-symmetry of a contact Riemannian space form.

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2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta\wedge(d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there exists an associated Riemannian metric g and a $(1,1)$-type tensor field ϕ such that

$$\eta(X) = g(X,\xi), \quad d\eta(X,Y) = g(X,\phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1) it follows that

$$\phi\xi = 0, \quad \eta\circ\phi = 0, \quad g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

Although g and ϕ are not uniquely determined for η, the two pairs (η, g) and (η,ϕ) are canonically related to each other by the equation $d\eta(X,Y) = g(X,\phi Y)$. A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a contact Riemannian manifold and is denoted by $M = (M;\eta, g)$. Given a contact Riemannian manifold M, we define a $(1,1)$-type tensor field h by $h = \frac{1}{2}L_\xi \phi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \tag{2.3}$$

$$(\nabla_X \eta)(Y) = g(X, \phi Y) - g(X, \phi h Y) \quad (\text{or } \nabla_X \xi = -\phi X - \phi h X), \tag{2.4}$$

$$\nabla_\xi h = \phi - \phi R_\xi - \phi h^2, \tag{2.5}$$

where ∇ is the Levi–Civita connection and $R_\xi = R(\cdot, \xi)\xi$. Here, R is the Riemannian curvature tensor of M defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z on M.

A contact Riemannian manifold for which ξ is Killing, is called a K-contact Riemannian manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A Sasakian manifold is characterized by a condition

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.6}$$

for all vector fields X and Y on the manifold.

It is well-known that a contact Riemannian manifold M is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

for all vector fields X and Y. For more details about contact Riemannian manifolds, we refer to [5], [10], [11].

For a contact Riemannian manifold $M = (M; \eta, \phi)$, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$(direct sum), where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. The $2n$-dimensional distribution D is called the *contact distribution*. We see that the restriction $\bar{\phi} = \phi|D$ of ϕ to D defines an almost complex structure on D. Then the associated CR-structure of a given contact Riemannian manifold M is given by the holomorphic subbundle

$$\mathcal{H} = \{X - i\bar{\phi}X : X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM. Then we see that each fiber $\mathcal{H}_x$ $(x \in M)$ is of complex dimension n and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. We say that *the associated CR-structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$, we define the Levi form, by

$$L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, \phi Y)$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is hermitian and positive definite, that is, the CR-structure is a *strongly pseudo-convex, pseudo-hermitian CR-structure*.

Since $d\eta(\phi X, \phi Y) = d\eta(X, Y)$, we see that $[\bar{\phi}X, \bar{\phi}Y] - [X, Y] \in D$ and $[\phi X, Y] + [X, \phi Y] \in D$ for $X, Y \in D$. Thus, the associated CR-structure is integrable ($[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$) if and only if

$$[\bar{\phi}, \bar{\phi}](X, Y) = 0$$

for $X, Y \in D$, where $[\bar{\phi}, \bar{\phi}]$ is the Nijenhuis torsion of $\bar{\phi}$. It was obtained ([14, Proposition 2.1]) that for a given contact Riemannian manifold M the associated CR-structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q_1 = 0$, where Q_1 is a $(1, 2)$-type tensor field on M defined by

$$Q_1(X, Y) = (\nabla_X \phi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX)$$

for all vector fields X, Y on M. We remark here that the class of contact strongly pseudo-convex integrable CR-manifolds is invariant under D-homothetic deformations ([12])

$$\tilde{\eta} = a\eta, \quad \tilde{\xi} = \frac{1}{a}\xi, \quad \tilde{\phi} = \phi, \quad \tilde{g} = ag + a(a - 1)\eta \otimes \eta,$$

where a is a positive constant. In fact, by direct computations, we have $\tilde{h} = \frac{1}{a}h$ and

$$(\tilde{\nabla}_X \tilde{\phi})Y = (\nabla_X \phi)Y + (a-1)\eta(Y)\phi^2 X - \frac{a-1}{a} g(X, hY)\xi.$$

From this, we easily see that $Q_1 = 0$ implies $\tilde{Q}_1 = 0$. Taking account of (2.6) we see that for a Sasakian manifold the associated CR-structure is strongly pseudo-convex integrable ($h = 0$). For 3-dimensional contact Riemannian manifolds their associated CR structures are always integrable (see [14]). One other class of contact Riemannian manifolds whose associated-CR structure is integrable is the class which is determined by the condition (see [6])

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

$k, \mu \in \mathbb{R}$. A space belonging to this class is called a contact (k, μ)-space ([3], [6]). In [6] the authors proved that $k \leq 1$. If $k = 1$, then $h = 0$ and the structure is Sasakian. It was also proved in [6] that the standard contact Riemannian structure of the unit tangent sphere bundle is a (k, μ)-space if and only if the base manifold is of constant curvature c with $k = c(2 - c)$ and $\mu = -2c$.

Now, we give

Definition 2.1. The class $\mathfrak{Q}_1$ is formed by the contact Riemannian manifolds whose associated CR-structures are integrable, that is,

$$\mathfrak{Q}_1 = \{(M, \eta, g) : Q_1 = 0\}.$$

3. A class $\mathfrak{Q}_1$ of contact Riemannian manifolds

Let M be a contact Riemannian manifold belonging to $\mathfrak{Q}_1$. Then we have

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{3.1}$$

for all vector fields X and Y. Comparing with (2.6), we see that a contact Riemannian manifold $\in \mathfrak{Q}_1$ is normal (or Sasakian) if and only if $h = 0$. We put $P(X, Y) = (\nabla_X h)Y - (\nabla_Y h)X$. Then we see that P is a $(1, 2)$-type tensor field on M. We have the following

Proposition 3.1. *For all vector fields X, Y, Z on M*

$$R(X,Y)\xi = \eta(Y)(X+hX) - \eta(X)(Y+hY) + \phi P(Y,X), \tag{3.2}$$

$$\begin{aligned} g(R(\xi,X)Y,Z) = {} & \eta(Z)g(Y+hY,Z) - \eta(Y)g(Z+hZ,X) \\ & + g(\phi P(Z,Y),X), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} R(X,Y)\phi Z = {} & \phi R(X,Y)Z - g(Y+hY,Z)(\phi X + \phi hX) \\ & + g(X+hX,Z)(\phi Y + \phi hY) + g(\phi X + \phi hX, Z)(Y+hY) \\ & - g(\phi Y + \phi hY, Z)(X+hX) + g(P(X,Y),Z)\xi \\ & - \eta(Z)P(X,Y). \end{aligned} \tag{3.4}$$

Proof. From the definition of the curvature tensor R, by using (2.4), (3.1) and the fundamental symmetries of the curvature tensor, we obtain (3.2) and (3.3). The Ricci identity for ϕ is given as

$$R(X,Y)\phi Z - \phi R(X,Y)Z = (\nabla^2_{X,Y}\phi)Z - (\nabla^2_{Y,X}\phi)Z, \tag{3.5}$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$. From (3.1) we have

$$\begin{aligned} (\nabla^2_{X,Y}\phi)Z = {} & -g(Y+hY,Z)(\phi X + \phi hX) + g(\phi X + \phi hX, Z)(Y+hY) \\ & + g((\nabla_X h)Y,Z)\xi - \eta(Z)(\nabla_X h)Y, \end{aligned} \tag{3.6}$$

and thus (3.4) follows easily from this and (3.5). □

Now, we prove

Proposition 3.2. *Let M be a space $\in \mathfrak{Q}_1$. Then the necessary and sufficient condition for M to have pointwise constant ϕ-holomorphic sectional curvature H is*

$$\begin{aligned} & g(R(X,Y)Z,W) \\ & = \frac{1}{4}\big\{(H+3)\big[(g(Y,Z) - \eta(Y)\eta(Z))(g(X,W) - \eta(X)\eta(W)) \\ & - (g(X,Z) - \eta(X)\eta(Z))(g(Y,W) - \eta(Y)\eta(W))\big] \end{aligned}$$

$$+ (H-1)\big[g(\phi Y,Z)g(\phi X,W)-g(\phi X,Z)g(\phi Y,W)-2g(\phi X,Y)g(\phi Z,W)\big]$$
$$+ 2\big[2g(hY,Z)(g(X,W)-\eta(X)\eta(W))-2g(hX,Z)(g(Y,W)-\eta(Y)\eta(W))$$
$$+ 2g(hX,W)(g(Y,Z)-\eta(Y)\eta(Z)) - 2g(hY,W)(g(X,Z)-\eta(X)\eta(Z))$$
$$+ g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) - g(hY,\phi Z)g(hX,\phi W)$$
$$+ g(hX,\phi Z)g(hY,\phi W)\big]\big\}$$
$$+ \eta(X)\big[g(\phi P(W,Z) - \eta(W)\phi P(\xi,Z) - \eta(Z)\phi P(W,\xi),Y)\big]$$
$$- \eta(Y)\big[g(\phi P(W,Z) - \eta(W)\phi P(\xi,Z) - \eta(Z)\phi P(W,\xi),X)\big]$$
$$+ \eta(Z)\big[g(\phi P(Y,X) - \eta(Y)\phi P(\xi,X) - \eta(X)\phi P(Y,\xi),W)\big]$$
$$- \eta(W)\big[g(\phi P(Y,X) - \eta(Y)\phi P(\xi,X) - \eta(X)\phi P(Y,\xi),Z)\big]$$
$$- \eta(X)\eta(Z)\big[g(Y+hY,W) - \eta(Y)\eta(W) + g(\phi P(\xi,Y),W)\big]$$
$$+ \eta(X)\eta(W)\big[g(Y+hY,Z) - \eta(Y)\eta(Z) + g(\phi P(\xi,Y),Z)\big]$$
$$+ \eta(Y)\eta(Z)\big[g(X+hX,W) - \eta(X)\eta(W) + g(\phi P(\xi,X),W)\big]$$
$$- \eta(Y)\eta(W)\big[g(X+hX,Z) - \eta(X)\eta(Z) + g(\phi P(\xi,X),Z)\big] \tag{3.7}$$

for all vector fields X, Y, Z, W in M.

Proof. For $X, Y \in D$, using the first Bianchi identity and the fundamental properties of the curvature tensor, (2.1), (2.2) and (2.3), we obtain from (3.4)

$$g(R(X,\phi X)Y,\phi Y) = g(R(X,\phi Y)Y,\phi X) + g(R(X,Y)\phi X,\phi Y) \tag{3.8}$$

and

$$\begin{aligned} g(R(X,Y)\phi X,\phi Y) &= g(R(X,Y)X,Y) - g(X,Y)^2 - g(hX,Y)^2 \\ &\quad - 2g(X,Y)g(hX,Y) + g(X,X)g(Y,Y) + g(X,X)g(hY,Y) \\ &\quad + g(Y,Y)g(hX,X) + g(hX,X)g(hY,Y) - g(\phi X,Y)^2 \\ &\quad + g(\phi hX,Y)^2 - g(\phi hX,X)g(\phi hY,Y). \end{aligned} \tag{3.9}$$

Similarly, from (3.4) we get

$$g(R(X,\phi Y)X,\phi Y) = g(R(X,\phi Y)Y,\phi X) + g(X,Y)^2 - g(hX,Y)^2$$

$$
\begin{aligned}
&- g(\phi hX,X)g(\phi hY,Y) - g(X,X)g(Y,Y) - g(Y,Y)g(hX,X)\\
&+ g(X,X)g(hY,Y) + g(hX,X)g(hY,Y) + g(\phi X,Y)^2\\
&+ g(\phi hX,Y)^2 + 2g(\phi X,Y)g(\phi hX,Y) \qquad (3.10)
\end{aligned}
$$

and

$$
\begin{aligned}
g(R(Y,\phi X)Y,\phi X) &= g(R(X,\phi Y)Y,\phi X) + g(X,Y)^2 - g(hX,Y)^2\\
&- g(\phi hX,X)g(\phi hY,Y) - g(X,X)g(Y,Y) + g(Y,Y)g(hX,X)\\
&- g(X,X)g(hY,Y) + g(hX,X)g(hY,Y) + g(\phi X,Y)^2\\
&+ g(\phi hX,Y)^2 - 2g(\phi X,Y)g(\phi hX,Y). \qquad (3.11)
\end{aligned}
$$

We now suppose that the ϕ-holomorphic sectional curvature is constant, i.e., $K(X,\phi X) = H$ for any $X \in D$. Then we have

$$
g(R(X,\phi X)\phi X,X) = Hg(X,X)^2, \qquad (3.12)
$$

for any $X \in D$. Replacing X by $X+Y$ and $X-Y$ for $X,Y \in D$ in (3.12) respectively, and summing them, we get

$$
\begin{aligned}
&2g(R(X,\phi X)\phi Y,Y) + g(R(X,\phi Y)\phi Y,X) + 2g(R(X,\phi Y)\phi X,Y)\\
&\quad + g(R(Y,\phi X)\phi X,Y) = 2H\{2g(X,Y)^2 + g(X,X)g(Y,Y)\}. \qquad (3.13)
\end{aligned}
$$

From (3.8), (3.9), (3.10), (3.11) and (3.13), we get

$$
\begin{aligned}
&3g(R(X,\phi Y)\phi X,Y) + g(R(X,Y)Y,X) + 2g(hX,Y)^2\\
&\quad + 2g(X,Y)g(hX,Y) - g(X,X)g(hY,Y) - g(Y,Y)g(hX,X)\\
&\quad - 2g(hX,X)g(hY,Y) - 2g(\phi hX,Y)^2 + 2g(\phi hX,X)g(\phi hY,Y)\\
&= H\{2g(X,Y)^2 + g(X,X)g(Y,Y)\}. \qquad (3.14)
\end{aligned}
$$

Replacing Y by ϕY in (3.14) and using (2.1), (2.2) and (2.3), we have

$$
\begin{aligned}
&3g(R(X,Y)\phi Y,\phi X) - g(R(X,\phi Y)X,\phi Y) + 2g(\phi hX,Y)^2\\
&\quad - 2g(X,\phi Y)g(hX,\phi Y) + g(X,X)g(hY,Y) - g(Y,Y)g(hX,X)\\
&\quad + 2g(hX,X)g(hY,Y) - 2g(hX,Y)^2 - 2g(\phi hX,X)g(\phi hY,Y)
\end{aligned}
$$

$$= H\{2g(X, \phi Y)^2 + g(X, X)g(Y, Y)\}. \tag{3.15}$$

From (3.15), together with (3.9) and (3.10), we get

$$\begin{aligned}
&- 3g(R(X, Y)Y, X) - g(R(X, \phi Y)\phi X, Y) + 2g(X, Y)^2 + 2g(hX, Y)^2 \\
&\quad + 6g(X, Y)g(hX, Y) - 2g(X, X)g(Y, Y) - 3g(X, X)g(hY, Y) \\
&\quad - 3g(Y, Y)g(hX, X) - 2g(hX, X)g(hY, Y) \\
&\quad + 2g(X, \phi Y)^2 - 2g(\phi hX, Y)^2 + 2g(\phi hX, Y)g(\phi hY, Y) \\
&= H\{2g(X, \phi Y)^2 + g(X, X)g(Y, Y)\}.
\end{aligned} \tag{3.16}$$

From (3.14) and (3.16), we have

$$\begin{aligned}
4g(R(X, Y)Y, X) &= (H + 3)\{g(X, X)g(Y, Y) - g(X, Y)^2\} \\
&\quad + 3(H - 1)g(X, \phi Y)^2 - 2\{g(hX, Y)^2 + 4g(X, Y)g(hX, Y) \\
&\quad - 2g(X, X)g(hY, Y) - 2g(Y, Y)g(hX, X) \\
&\quad - g(hX, X)g(hY, Y) - g(\phi hX, Y)^2 + g(\phi hX, X)g(\phi hY, Y)\}
\end{aligned} \tag{3.17}$$

for any $X, Y \in D$. Replacing $X = X + Z$ in (3.17), we obtain

$$\begin{aligned}
4g(R(X, Y)Y, Z) &= (H + 3)\{g(X, Z)g(Y, Y) - g(X, Y)g(Y, Z)\} \\
&\quad + 3(H - 1)g(X, \phi Y)g(Z, \phi Y) - 2\{g(hX, Y)g(hY, Z) \\
&\quad + 2g(X, Y)g(hY, Z) + 2g(Y, Z)g(hX, Y) \\
&\quad - 2g(X, Z)g(hY, Y) - 2g(Y, Y)g(hX, Z) - g(hX, Z)g(hY, Y) \\
&\quad - g(\phi hX, Y)g(\phi hZ, Y) + g(\phi hX, Z)g(\phi hY, Y)\}.
\end{aligned} \tag{3.18}$$

If we replace $Y = Y + W$ in (3.18) again and use (2.3), then we obtain

$$\begin{aligned}
&4\{g(R(X, Y)W, Z) + g(R(X, W)Y, Z)\} \\
&= (H + 3)\{2g(X, Z)g(Y, W) - g(X, Y)g(W, Z) - g(X, W)g(Y, Z)\} \\
&\quad + 3(H - 1)\{g(X, \phi Y)g(Z, \phi W) + g(X, \phi W)g(Z, \phi Y)\} \\
&\quad - 2\{g(hX, Y)g(hZ, W) + g(hX, W)g(hZ, Y) + 2g(X, Y)g(hZ, W) \\
&\quad + 2g(X, W)g(hZ, Y) + 2g(Z, Y)g(hX, W) + 2g(Z, W)g(hX, Y)
\end{aligned}$$

$$- 4g(X,Z)g(hY,W) - 4g(Y,W)g(hX,Z) - 2g(hX,Z)g(hY,W)$$
$$- g(\phi hX,Y)g(\phi hZ,W) - g(\phi hX,W)g(\phi hZ,Y)$$
$$+ 2g(\phi hX,Z)g(\phi hY,W)\} \tag{3.19}$$

and we have

$$\begin{aligned}
&4\{g(R(X,Z)W,Y) + g(R(X,W)Z,Y)\} \\
&\quad = (H+3)\{2g(X,Y)g(Z,W) - g(X,Z)g(W,Y) - g(X,W)g(Z,Y)\} \\
&\qquad + 3(H-1)\{g(X,\phi Z)g(Y,\phi W) + g(X,\phi W)g(Y,\phi Z)\} \\
&\qquad - 2\{g(hX,Z)g(hY,W) + g(hX,W)g(hY,Z) \\
&\qquad + 2g(X,Z)g(hY,W) + 2g(X,W)g(hY,Z) + 2g(Y,Z)g(hX,W) \\
&\qquad + 2g(Y,W)g(hX,Z) - 4g(X,Y)g(hZ,W) - 4g(Z,W)g(hX,Y) \\
&\qquad - 2g(hX,Y)g(hZ,W) - g(\phi hX,Z)g(\phi hY,W) - g(\phi hX,W)g(\phi hY,Z) \\
&\qquad + 2g(\phi hX,Y)g(\phi hZ,W)\}.
\end{aligned} \tag{3.20}$$

We subtract (3.20) from (3.19). Then by using the first Bianchi identity and (2.3), we get

$$\begin{aligned}
&4g(R(X,Y)Z,W) = (H+3)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} + (H-1) \\
&\quad \times \{g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\} \\
&\qquad + 2\{2g(hY,Z)g(X,W) - 2g(hX,Z)g(Y,W) + 2g(Y,Z)g(hX,W) \\
&\qquad - 2g(X,Z)g(hY,W) + g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) \\
&\qquad - g(hY,\phi Z)g(hX,\phi W) + g(hX,\phi Z)g(hY,\phi W)\},
\end{aligned} \tag{3.21}$$

where $X,Y,Z,W \in D$. We now let X be an arbitrary vector field on M. Then we may write

$$X = X^T + \eta(X)\xi,$$

where X^T denotes *the horizontal part* of X. Then we have for all vector fields X,Y,Z,W in M:

$$g(R(X,Y)Z,W) = g(R(X^T,Y^T)Z^T,W^T) + \eta(X)g(R(\xi,Y^T)Z^T,W^T)$$

$$
\begin{aligned}
&+ \eta(Y)g(R(X^T,\xi)Z^T,W^T) + \eta(Z)g(R(X^T,Y^T)\xi,W^T)\\
&+ \eta(W)g(R(X^T,Y^T)Z^T,\xi) + \eta(X)\eta(Z)g(R(\xi,Y^T)\xi,W^T)\\
&+ \eta(X)\eta(W)g(R(\xi,Y^T)Z^T,\xi) + \eta(Y)\eta(Z)g(R(X^T,\xi)\xi,W^T)\\
&+ \eta(Y)\eta(W)g(R(X^T,\xi)Z^T,\xi). \qquad (3.22)
\end{aligned}
$$

Furthermore, from (3.22), by using (3.2), (3.3), (3.21) and straightforward calculations, we obtain (3.7). □

From (3.6), by using (2.4) and (2.5), we find for the Ricci tensors

$$
\begin{aligned}
\rho(X,Y) = \frac{1}{4}\big\{&(2(n+1)H+6n-2)(g(X,Y)-\eta(X)\eta(Y))\\
&+4(2n-1)g(hX,Y)\big\}\\
&-\eta(X)\sum_i g(\phi P(Y,e_i),e_i) + \eta(Y)\sum_i g(\phi P(X,e_i),e_i)\\
&+g(\phi P(\xi,X),Y) + \eta(X)\eta(Y)(2n-\operatorname{tr} h^2)
\end{aligned}
\qquad (3.23)
$$

for all vector fields X and Y in M, where $\{e_i\}$ ($i = 1, 2, \ldots, 2n+1$) is an arbitrary local orthonormal frame field on M. Since the trace of h vanishes, from (3.23), we have for the scalar curvature

$$
\tau = \frac{1}{2}\cdot n\big(2(n+1)H+6n-2\big) + 2n - 2(\operatorname{tr} h^2).
$$

4. A class of contact Riemannian manifolds $\mathfrak{Q}$

There are two typical examples of contact manifolds; one is formed by the principal circle bundles over symplectic manifolds of integral class (including the odd-dimensional spheres) and the other is given by the unit tangent sphere bundles. The former admit a Riemannian metric which is Sasakian. Concerning the latter, in [15], it was proved that the associated CR-structure of a unit tangent sphere bundle T_1M with standard contact Riemannian structure is integrable if and only if the base manifold is of constant curvature. Here, we note that the unit tangent sphere bundle of a space of constant curvature satisfies ([6])

$$
g((\nabla_{X^T}h)Y^T, Z^T) = 0, \qquad (1)
$$

that is, h is *η-parallel* and at the same time it also satisfies

$$\nabla_\xi h = \mu h\phi \tag{2}$$

where μ is a constant. Now, we consider a contact Riemannian manifolds whose structure tensor h satisfies (1) and (2) simultaneously. Then

$$\begin{aligned}
0 &= g((\nabla_{X^T} h)Y^T, Z^T) = g((\nabla_{X-\eta(X)\xi} h)(Y - \eta(Y)\xi, Z - \eta(Z)\xi) \\
&= g((\nabla_X h)Y, Z) - \eta(X)g((\nabla_\xi h)Y, Z) - \eta(Y)g((\nabla_X h)\xi, Z) \\
&\quad - \eta(Z)g((\nabla_X h)Y, \xi) + \eta(X)\eta(Y)g((\nabla_\xi h)\xi, Z) + \eta(Y)\eta(Z)g((\nabla_X h)\xi, \xi) \\
&\quad + \eta(Z)\eta(X)g((\nabla_\xi h)Y, \xi) - \eta(X)\eta(Y)\eta(Z)g((\nabla_\xi h)\xi, \xi).
\end{aligned}$$

From the above equation, by using (2.3), (2.4) and $\nabla_\xi h = \mu h\phi$, we have

$$(\nabla_X h)Y = g((h - h^2)\phi X, Y)\xi + \eta(Y)(h - h^2)\phi X + \mu\eta(X)h\phi Y \tag{4.1}$$

for any vector fields X and Y. Now, we define a $(1, 2)$-tensor field $Q_2(X, Y)$ by

$$\begin{aligned}
Q_2(X, Y) = (\nabla_X h)Y &- g((h - h^2)\phi X, Y)\xi \\
&- \eta(Y)(h - h^2)\phi X - \mu\eta(X)h\phi Y.
\end{aligned}$$

Definition 4.1. The class $\mathfrak{Q}$ is given by the spaces belonging to $\mathfrak{Q}_1$ and satisfying $Q_2 = 0$, that is,

$$\mathfrak{Q} = \{(M, \eta, g) \in \mathfrak{Q}_1 : Q_2 = 0\}.$$

Then, as mentioned before, we can see that this class $\mathfrak{Q}$ is invariant under D-homothetic deformations. More precisely, for a D-homothetic deformation, we have

$$(\tilde{\nabla}_X h)Y = \tilde{g}((\tilde{h} - \tilde{h}^2)\tilde{\phi} X, Y)\tilde{\xi} + \tilde{\eta}(Y)(\tilde{h} - \tilde{h}^2)\tilde{\phi} X + \tilde{\mu}\tilde{\eta}(X)h\tilde{\phi} Y,$$

where $\tilde{\mu} = (2(a-1)+\mu)/a$. Furthermore, assume that $hY = \lambda Y$ for $Y \in D$ and $\|Y\| = 1$. Then from (4.1), we easily get $g((\nabla_X h)Y, Y) = 0$ for any vector X, from which we have

Lemma 4.2. *The eigenvalues of h are constant.*

Further, from (4.1), we have

$$P(X,Y) = -g((\phi h^2 + h^2\phi)X,Y)\xi + \eta(X)((\mu-1)h\phi Y + h^2\phi Y) \quad (4.2)$$
$$- \eta(Y)((\mu-1)h\phi X + h^2\phi X),$$

$$\phi P(X,Y) = \eta(X)((\mu-1)hY - h^2 Y) - \eta(Y)((\mu-1)hX - h^2 X). \quad (4.3)$$

Now, we prove a Schur-type theorem for the class $\mathfrak{Q}$.

Theorem 4.3. *Let $M = (M^{2n+1};\eta,g)$ $(n>1)$ be a contact Riemannian manifold belonging to the class $\mathfrak{Q}$. If the ϕ-holomorphic sectional curvature at any point of M is independent of the choice of ϕ-holomorphic section, then it is constant on M and the curvature tensor is given by*

$$\begin{aligned}
g(R(X,Y)Z,W) &= \frac{1}{4}\big\{(c+3)\big[(g(Y,Z)-\eta(Y)\eta(Z))(g(X,W)\\
&- \eta(X)\eta(W)) - (g(X,Z)-\eta(X)\eta(Z))(g(Y,W)-\eta(Y)\eta(W))\big]\\
&+ (c-1)\big[g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\big]\\
&+ 2\big[2g(hY,Z)(g(X,W)-\eta(X)\eta(W)) - 2g(hX,Z)(g(Y,W)-\eta(Y)\eta(W))\\
&+ 2g(hX,W)(g(Y,Z)-\eta(Y)\eta(Z)) - 2g(hY,W)(g(X,Z)-\eta(X)\eta(Z))\\
&+ g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) - g(hY,\phi Z)g(hX,\phi W)\\
&+ g(hX,\phi Z)g(hY,\phi W)\big]\big\}\\
&- \eta(X)\eta(Z)g(Y+\mu hY - h^2Y,W) + \eta(X)\eta(W)g(Y+\mu hY-h^2Y,Z)\\
&+ \eta(Y)\eta(Z)g(X+\mu hX - h^2X,W)\\
&- \eta(Y)\eta(W)g(X+\mu hX-h^2X,Z) \qquad (4.4)
\end{aligned}$$

for all vector fields X, Y, Z, W in M.

Proof. Suppose that M has pointwise constant ϕ-holomorphic sectional curvature H. Then, taking account of (4.1), (4.2) and (4.3), from (3.23) we obtain

$$\begin{aligned}
\rho(X,Y) = \frac{1}{4}\big\{&(2(n+1)H + 6n - 2)(g(X,Y)-\eta(X)\eta(Y))\\
&+ 4(2n-2+\mu)g(hX,Y) - 4g(h^2X,Y)\big\}
\end{aligned}$$

$$+\eta(X)\eta(Y)(2n-\operatorname{tr}h^2), \tag{4.5}$$

$$\tau=\frac{1}{2}\cdot n\Big(2(n+1)H+6n-2\Big)+2n-2(\operatorname{tr}h^2). \tag{4.6}$$

From (4.1) and by using (2.4) and Lemma 4.2, we have

$$\begin{aligned}(\nabla_X\rho)(Y,Z)=\frac{1}{4}\big\{&(2(n+1)(XH))(g(Y,Z)-\eta(Y)\eta(Z))\\
&+(2(n+1)H+6n-2)(\eta(Z)g(\phi X+\phi hX,Y)\\
&+\eta(Y)g(\phi X+\phi hX,Z))\\
&+4(2n-2+\mu)g(g((h-h^2)\phi X,Y)\eta(Z)\\
&+\eta(Y)g((h-h^2)\phi X,Z)+\mu\eta(X)g(h\phi Y,Z))\\
&-4(g((h-h^2)\phi X,hY)\eta(Z)+g((h-h^2)\phi X,hZ)\eta(Y))\big\}\\
&+(\eta(Z)g(-\phi X-\phi hX,Y)\\
&+\eta(Y)g(-\phi X-\phi hX,Z))(2n-\operatorname{tr}h^2),\end{aligned}$$

which yields

$$\sum_i(\nabla_{e_i}\rho)(X,e_i)=\frac{1}{2}(n+1)\{(XH)-(\xi H)\eta(X)\}. \tag{4.7}$$

By the well-known formula

$$\nabla_X\tau=2\sum_i(\nabla_{e_i}\rho)(X,e_i)$$

for any local orthonormal frame field $\{e_i\}$ $(i=1,2,\dots,2n+1)$ and by using (4.6), (4.7) and Lemma 4.2, we have

$$(n+1)\{XH-(\xi H)\eta(X)\}=n(n+1)XH.$$

This says that $\xi H=0$ and $(n-1)XH=0$. Since $n>1$, we see that H is constant, say c. By applying (4.1), (4.2) and (4.3) in Proposition 3.2, we obtain (4.4). $\square$

Definition 4.4. A complete and simply connected contact Riemannian manifold of class $\mathfrak{Q}$ with constant ϕ-holomorphic sectional curvature is said to be a *contact Riemannian space form*.

So, from the proof for Proposition 3.2 and Theorem 4.3, we have

Theorem 4.5. *Let M be a complete and simply connected space belonging to the class $\mathfrak{Q}$. Then M is a contact Riemannian space form if and only if the curvature tensor R is given by* (4.4)*.*

Remark 4.6. From (3.2) and (4.2), it follows that a manifold $\in \mathfrak{Q}$ satisfies

$$R(X,Y)\xi = \eta(Y)(X - h^2X) - \eta(X)(Y - h^2Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

It is easily seen that the class $\mathfrak{Q}$ coincides with the class of contact (k,μ)-space if and only if $h^2 = (k-1)\phi^2$, $k \in \mathbb{R}$. Thus, taking account of Lemma 4.2, we see that for dimension 3, the class $\mathfrak{Q}$ coincides with the class of contact (k,μ)-spaces. But, in higher dimensions, we do not know yet of an example in $\mathfrak{Q}$ which is not a contact (k,μ)-space.

Examples of (non-Sasakian) contact Riemannian space forms

(1) All 3-dimensional non-Sasakian contact (k,μ)-spaces have constant ϕ-holomorphic sectional curvatures $H = -(k+\mu)$. That is, $SU(2)$, the universal covering space of $SL(2,\mathbb{R})$, the universal covering space of the group $E(2)$ of rigid motions of Euclidean 2-space and the group space $E(1,1)$ of rigid motions of Minkowski 2-space, respectively with a special left-invariant metric, are contact Riemannian space forms (see [6]).

(2) Tangent sphere bundles $T_1M(c)$ (with standard contact Riemannian structure) of n-dimensional spaces of constant curvature $c = 2 \pm \sqrt{5}$ have constant ϕ-holomorphic sectional $H = c^2$. We know that $T_1M(c)$ is simply connected when $n > 2$.

(3) By D-homothetic deformations, we can construct more examples with $H > 0$, $H < 0$, or $H = 0$. More explicitly, for the unit tangent sphere bundle of a space of constant curvature c,

$$\begin{cases} H > 0, & \text{if } (-11 + 4\sqrt{6}\,)/5 < c, \\ H = 0, & \text{if } c = (-11 + 4\sqrt{6}\,)/5, \\ H < 0, & \text{if } -1 < c < (-11 + 4\sqrt{6}\,)/5. \end{cases}$$

For the above (2) and (3), we refer to [9].

We close this section showing the equivalence theorem, the homogeneity and the ϕ-symmetry of the contact Riemannian space forms. First,

Theorem 4.7 (Equivalence theorem)**.** *Let* $(M^{2n+1};\eta,g)$ *and* $(M^{2n+1};\eta',g')$ *be two contact Riemannian space forms with the same* $\mu\in\mathbb{R}$*. Suppose that the eigenvalues and the dimensions of their eigenspaces of* h *and* h' *are equal to each other. Then they are isometric as contact Riemannian spaces.*

Proof. The theorem follows from the expression (4.4) for the Riemannian curvature tensor and the fomulas (2.4), (3.1) and (4.1), using similar arguments as in [3] or [13]. □

Now, we prove the homogeneity. We define a $(1,2)$-tensor field T by

$$\begin{aligned} T(X,Y) = &-g(\phi X + \phi hX, Y)\xi + \eta(Y)(\phi X + \phi hX) \\ &+ (\mu/2)\eta(X)\phi Y \end{aligned} \tag{4.8}$$

for vector fields X, Y ([2]). Let $\bar\nabla$ be the connection determined by $\bar\nabla = \nabla + T$. Then we easily get

$$\bar\nabla g = 0, \quad \bar\nabla\xi = 0, \quad \bar\nabla\eta = 0, \quad \bar\nabla\phi = 0. \tag{4.9}$$

Also, we obtain from (4.1)

$$\bar\nabla h = 0. \tag{4.10}$$

Thus, in view of the form (4.4) and (4.8), using (4.9) and (4.10), it follows easily that

$$\bar\nabla T = 0 \tag{4.11}$$

and

$$\bar\nabla R = 0. \tag{4.12}$$

At last, together with (4.9), (4.11) and (4.12), by Kiričenko's generalzation ([8]) of the Ambrose–Singer theorem ([1] or [16]), we have

Theorem 4.8 (Homogeneity)**.** *A contact Riemannian space form is a locally homogeneous contact Riemannian manifold.*

We remark that a Sasakian space form is a naturally reductive homogeneous space ([7]). Furthermore, together with (2.4), (3.1), (4.1) and (4.4), and in a similar way as in the proof of Lemma 7 and Theorem 1 in [2], we obtain

Theorem 4.9 (ϕ-symmetry)**.** *A contact Riemannian space form is locally ϕ-symmetric in the strong sense, that is, the characteristic reflections are local isometries.*

For more details about the ϕ-symmetry in the weak or strong sense, we refer to [2], [4].

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