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A continuity result on t-Wright-convex functions

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Abstract. Gy. Maksa, K. Nikodem and Zs. Páles have found an example of a noncontinuous t-Wright-convex function bounded above on the real line. On the other hand, J. Matkowski and M. Wróbel have proved that every lower semicontinuous t-Wright-convex function has to be continuous everywhere. We prove that every t-Wright-convex function continuous at a point is continuous at each point.

A function $f : (a, b) \to \mathbb{R}$ is called Wright-convex if the following condition

$$f(tx + (1 - t)y) + f((1 - t)x + ty) \leq f(x) + f(y), \qquad x, y \in (a, b), \quad (1)$$

is fulfilled for every $t \in (0, 1)$. If (1) is satisfied for some $t \in (0, 1)$ then f is called t-Wright-convex on the interval (a, b). C. T. Ng [6] characterizes Wright-convex functions in the following way: A function f is Wright-convex iff it is of the form $f = a + F$, where a is additive and F is convex in the usual sense (cf. also [2]). Addressing Matkowski's problem, Gy. Maksa, K. Nikodem and Zs. Páles [4] have constructed a discontinuous t-Wright-convex function defined on the whole real line $\mathbb{R}$ bounded above on $\mathbb{R}$ and Jensen-concave. This shows that the assumption of the upper boundedneity of a t-Wright-convex function does not imply its continuity. On the other hand, J. Matkowski and M. Wróbel [5] proved that every lower semicontinuous t-Wright-convex function has to be continuous. The task is to find another set of sufficient conditions of the continuity of t-Wright-convex functions. In this note we show that one

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of these is a condition of the continuity at a point. Our proof is based upon a remark proven by Gy. Maksa, K. Nikodem and Zs. Páles [4] and a lemma.

Remark. Let $f : (a,b) \to \mathbb{R}$ be a t-Wright-convex function. Then the set

$$W_f := \{\lambda \in (0,1);\ f \text{ is } \lambda\text{-Wright-convex}\}$$

is dense in the interval $(0,1)$.

Lemma. *Let $f : (a,b) \to \mathbb{R}$ be a t-Wright-convex function and assume that f has a limit at a point $x_0 \in (a,b)$. Then*

(i) $\forall_{x<x_0} \quad (-\infty < \limsup_{u\to x+} f(u) \le f(x) \le \liminf_{u\to x-} f(u) < \infty)$;

(ii) $\forall_{x>x_0} \quad (-\infty < \limsup_{u\to x-} f(u) \le f(x) \le \liminf_{u\to x+} f(u) < \infty)$.

Moreover, f is continuous at x_0.

Proof. (i). Let us fix an $x < x_0$ and let $(u_n)_{n\in\mathbb{N}}$ be an arbitrary sequence tending to x from the right. Based on the Remark we can choose a $t_n \in W_f$ such that

$$\frac{u_n - x}{x_0 - x} < t_n < \frac{u_n - x}{x_0 - u_n}. \tag{2}$$

Putting

$$v_n := \frac{1}{t_n}u_n - \frac{1-t_n}{t_n}x, \qquad x_n := t_n x + (1-t_n)v_n,$$

we observe that $u_n = t_n v_n + (1-t_n)x$. According to (2) one can easily check that

$$0 < x_0 - v_n < u_n - x,$$

whereas the condition $u_n \to x+$ implies that

$$t_n \to 0, \quad x_n \to x_0-, \quad v_n \to x_0- .$$

By virtue of (1) we obtain

$$f(u_n) + f(x_n) \le f(x) + f(v_n).$$

Thus

$$\limsup_{n\to\infty} [f(u_n) + f(x_n)] \le f(x) + \limsup_{n\to\infty} f(v_n),$$

and hence

$$\limsup_{n\to\infty} f(u_n) \le f(x).$$

Due to the arbitrariness of $(u_n)_{n\in\mathbb{N}}$ we get

$$\limsup_{u\to x+} f(u) \le f(x).$$

If u_n tends to x from the left then we choose a $t_n \in W_f$ fulfilling the condition

$$\frac{x-u_n}{x-u_n+x_0-u_n} < t_n < \frac{x-u_n}{x_0-u_n} \tag{3}$$

and we put

$$x_n := \frac{1}{t_n}x - \frac{1-t_n}{t_n}u_n, \qquad v_n := t_n u_n + (1-t_n)x_n.$$

It follows from (3) that

$$0 < x_n - x_0 < x - u_n,$$

so that the condition $u_n \to x$ implies that $t_n \to 0$, $x_n \to x_0+$ and $v_n \to x_0-$. By virtue of (1)

$$f(x) + f(v_n) \le f(x_n) + f(u_n)$$

and, consequently,

$$f(x) + \liminf_{n\to\infty} f(v_n) \le \liminf_{n\to\infty} [f(x_n) + f(u_n)],$$

$$f(x) \le \liminf_{n\to\infty} f(u_n),$$

and

$$f(x) \le \liminf_{u\to x-} f(u).$$

The proof of the relevant part in (ii) runs in a similar manner.

Assume that $\rho := f(x_0) - \lim_{u\to x_0} f(u) > 0$. Then, there exists a $\delta > 0$ such that for each $v, 0 < |x_0 - v| < \delta$ we have

$$|\lim_{u\to x_0} f(u) - f(v)| < \frac{1}{3}\rho.$$

Take $z \in (x_0 - \delta, x_0)$, $r, v \in (x_0, x_0 + \delta)$ and $t \in W_f$ such that

$$x_0 = tz + (1-t)v, \quad r = (1-t)z + tv.$$

Then

$$f(x_0) + \lim_{u \to x_0} f(u) - \frac{1}{3}\rho < f(x_0) + f(r) \le f(z) + f(v) < 2 \lim_{u \to x_0} f(u) + \frac{2}{3}\rho,$$

which is impossible. If $\rho := \lim_{u \to x_0} f(u) - f(x_0) > 0$ then we choose $z, r, v \in (x_0, x_0 + \delta)$ and $t \in W_f$ such that $z = tx_0 + (1-t)v$, and $r = (1-t)x_0 + tv$. Now

$$2 \lim_{u \to x_0} f(u) - \frac{2}{3}\rho < f(z) + f(r) \le f(x_0) + f(v) < f(x_0) + \lim_{u \to x_0} f(u) + \frac{1}{3}\rho,$$

and, consequently,

$$\rho = \lim_{u \to x_0} f(u) - f(x_0) < \rho,$$

which is a contradiction. This ends the proof of the continuity of f at the point x_0.

Now we shall show that if $x < x_0$ then $\limsup_{u \to x+} f(u) > -\infty$. Let m, M and $\delta > 0$ be chosen so that

$$u \in (x_0 - \delta, x_0 + \delta) \quad \Longrightarrow \quad m < f(u) < M.$$

Take a $u \in (x, x + \delta)$ and choose a $u_0 \in (x, u)$ such that

$$f(u_0) < f(u) - (M - m).$$

It follows from the density of W_f in $(0,1)$ that there exist $t \in W_f$, $v, v_0 \in (x_0 - \delta, x_0 + \delta)$ such that $u = tu_0 + (1-t)v_0$ and $v = (1-t)u_0 + tv_0$.

By virtue of (1) we get

$$f(u) + f(v) \le f(u_0) + f(v_0).$$

Consequently,

$$f(u) + m < f(u) - (M - m) + M,$$

which is a contradiction. Therefore

$$-\infty < \limsup_{u \to x+} f(u).$$

In a similar way one can prove that

$$x < x_0 \Longrightarrow \liminf_{u \to x-} f(u) < \infty,$$

as well as

$$x > x_0 \Longrightarrow \big(\limsup_{u \to x-} f(u) > -\infty \text{ and } \liminf_{u \to x+} f(u) < \infty\big).$$

Thus, the proof of our lemma is finished. □

Now, we are in a position to prove our main theorem.

Theorem. *Let $f : (a, b) \to \mathbb{R}$ be a t-Wright-convex function and assume that f has a limit at a point. Then, f is continuous and convex.*

Proof. By our lemma, f has a continuity point x_0. We show that f is Jensen-convex in the interval (a, x_0). For that, take arbitrary $x, y \in (a, x_0)$, $x < y$, and put $z := \frac{x+y}{2}$.

Let $(u_n)_{n\in\mathbb{N}}$ be an arbitrary sequence tending to z from the right. For $n \in \mathbb{N}$ we choose a $t_n \in W_f$ such that

$$\frac{u_n - x}{u_n - x + \frac{y-x}{2}} < t_n < \frac{u_n - x}{y - x}. \tag{4}$$

Define points y_n and v_n in the following manner:

$$y_n := \frac{1}{t_n} u_n - \frac{1 - t_n}{t_n} x, \qquad v_n := t_n x + (1 - t_n) y_n.$$

Then $u_n = t_n y_n + (1 - t_n)x$. It follows from (4) that

$$0 < y_n - y < u_n - z \quad \text{and} \qquad v_n < z,$$

so that the condition $u_n \to z+$ implies that $t_n \to \frac{1}{2}$, $y_n \to y+$, $v_n \to z-$. By virtue of (1) we obtain

$$f(u_n) + f(v_n) \le f(x) + f(y_n),$$

and, by our lemma (condition (i)),

$$\limsup_{n\to\infty} [f(u_n) + f(v_n)] \le f(x) + \limsup_{n\to\infty} f(y_n) \le$$

$$\le f(x) + \limsup_{u \to y+} f(u) \le f(x) + f(y).$$

Therefore,

$$\limsup_{n \to \infty} f(u_n) + \liminf_{v \to z-} f(v) \le f(x) + f(y).$$

Due to the arbitrariness of the sequence $(u_n)_{n \in \mathbb{N}}$, we obtain

$$\limsup_{u \to z+} f(u) + \liminf_{v \to z-} f(v) \le f(x) + f(y). \tag{5}$$

We shall show that

$$2f(z) \le \limsup_{u \to z+} f(u) + \liminf_{v \to z-} f(v). \tag{6}$$

For indirect proof of (6), we assume that

$$f(z) - \limsup_{u \to z+} f(u) > \liminf_{v \to z-} f(v) - f(z) := \rho_2.$$

For that, take a $\rho_1 \in (\rho_2, f(z) - \limsup_{u \to z+} f(u))$. It follows from our Lemma that

$$\rho_2 \ge 0. \tag{7}$$

Let us put $\varepsilon := \frac{1}{3}(\rho_1 - \rho_2) > 0$. There exists a $\delta > 0$ such that

$$\forall_{u \in (z, z+\delta)} f(u) < f(z) - \rho_1 \quad \text{and} \quad \forall_{v \in (z-\delta, z)} f(v) > f(z) + \rho_2 - \varepsilon. \tag{8}$$

Take a $v \in (z - \delta, z)$ sufficiently close to z such that $f(v) < f(z) + \rho_2 + \varepsilon$. Then, there exist s, r, u and $t \in W_f$ fulfilling the following conditions

$$v < s < r < z < u, \; s = tu + (1-t)v \quad \text{and} \quad r = (1-t)u + ts.$$

It follows from (1) that

$$f(s) + f(r) \le f(u) + f(v).$$

Hence and by (8)

$$2f(z) + 2\rho_2 - 2\varepsilon < 2f(z) + \rho_2 - \rho_1 + \varepsilon,$$

or, equivalently,

$$\rho_2 < 0,$$

which contradicts (7). This proves (6). From (6) and (5) it follows that f is Jensen-convex in the interval (a, x_0).

Quite similarly one can show Jensen-convexity of f in the interval (x_0, b). Since f is continuous at x_0, it is bounded above in a neighbourhood of x_0 and by Bernstein–Doetsch theorem ([1], cf also [3]) f is continuous at each point of (a, b). Moreover, f being t-Wright-convex and continuous is convex, too. This completes the proof of the theorem. □

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