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Some topological obstructions to symmetric curvature operators

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Dedicated to Professor Lajos Tamássy on his 70th birthday

0. Introduction

Let (M, J, g) be a Hermitian manifold and ∇ a complex connection on $M(\nabla J = 0)$. The problem what can be said about the topology of M has been considered in many papers. The attention was paid mainly to metric connections which curvature tensor has some additional symmetries. For example, in [CO] Kähler–Einstein manifolds are considered. Naturally, the corresponding question for complex vector bundles is also studied; in [Ko] for holomorphic vector bundles and in [GBNV] and [Bz] for formally holomorphic vector bundles. The case of non-metric connections is studied in [IKO].

Here, we consider some properties of the Chern characteristic classes $c_1(M)$ and $c_2(M)$ when the curvature operator of ∇ is a symmetric operator. We do not assume ∇ to be a metric connection. (If ∇ is a metric connection the curvature operator is anti-symmetric). A symmetric curvature operator has the skew-symmetric Ricci tensor. Connections with the skew-symmetric Ricci tensor appear naturally in the study of manifolds which admit absolute parallelizability of directions (see, for example, [No]). These connections are also studied in [AT, Section §7]. Examples of such connections are already constructed in [BB] and [BB1]. We express the Chern forms $\gamma_1^2(M)$ and $\gamma_2(M)$ in terms of the quadratic invariants of ∇ and, for example, we show that $c_1(M) = 0$. We give also some examples of these connections.

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1. Chern forms of Hermitian surface

Let M be a complex manifold, of complex dimension n, and let J $(J^2=-I)$ be the corresponding almost complex structure on TM. We denote by $\mathcal{X}_{\mathbb{C}}(M)$ and M_m the Lie algebra of C^∞ complex vector fields on M and the real tangent space to M at m. Let ∇ be an arbitrary complex, symmetric connection, i.e. a connection such that $\nabla J=0$ and

$$\nabla_X Y-\nabla_Y X=[X,Y]\,,$$

for $X,Y\in\mathcal{X}_{\mathbb{C}}(M)$. The curvature operator R of ∇ is defined by

$$R(X,Y)=\nabla_{[X,Y]}-[\nabla_X,\nabla_Y]\quad \text{for}\ \ X,Y\in\mathcal{X}_{\mathbb{C}}(M)\,,$$

and it satisfies

$$R(X,Y)=-R(Y,X)\,, \tag{1}$$

the first Bianchi identity

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0\,, \tag{2}$$

and the Kähler identity,

$$R(X,Y)\circ J=J\circ R(X,Y)\,, \tag{3}$$

for $X,Y\in\mathcal{X}_{\mathbb{C}}(M)$. Especially, ∇ is an affine Kähler connection if

$$R(X,Y)=R(JX,JY)$$

(see [NP]).

Let $E_1,JE_1,\dots,E_n$, JE_n, be a real orthonormal basis for the tangent space M_m and $\omega^1,\bar\omega^1,\dots,\omega^n$, $\bar\omega^n$ the corresponding dual base for M_m^*. Then we will write $E_{n+p}=JE_p=E_{\bar p}$ and similarly $\omega^{n+q}=\bar\omega^q$, $1\le p,\ q\le n$. In the next formulas, a pair of repeated indices will always indicate summation. Also, we use the following ranges for indices: $i,j,p,q=1,2\dots,n$, and $I,K,P,Q=1,2,\dots,2n$. We denote $JE_P=E_{\bar P}$ and

$$R(X,Y)E_P=R_{XYP}{}^{Q}E_Q\,.$$

For $X=E_I,\ Y=E_K$ we simplify our notation and write $R_{E_IE_KP}{}^{Q}=R_{IKP}{}^{Q}$ and $R_{XYP}{}^{Q}=R_{XYPQ}$. The fundamental 2–form is $\Phi=\sum\omega^i\wedge\bar\omega^i$.

It will be useful for our study of the Chern classes to introduce the following traces:

$$\tilde\varrho(X,Y)=\frac{1}{2}\,\mathrm{tr}\,\{V\to R(X,Y)V\}=R_{XY_i}{}^{i}\,,$$

(4) $$\bar{\varrho}(X,Y) = \frac{1}{2}\operatorname{tr}\{V \to J \circ R(X,JY)V\} = R_{XKY_i}{}^{i}\,,$$

and
$$\varrho(X,Y) = \operatorname{tr}\{V \to R(V,X)Y\} = R_{IXY}{}^{I}\,,$$

for $X,Y \in M_m \otimes \mathbb{C}$ and $V \in M_m$. We also consider the following tensor of the Ricci type, $\hat{\varrho}$, defined by

$$\hat{\varrho}_I^K = \frac{1}{2}R_{P\bar{P}I}{}^{K} = R_{p\bar{p}I}{}^{K}\,.$$

Notice that $\tilde{\varrho}, \bar{\varrho}$ and ϱ do not depend on the choice of the metric g.

Because of the first Bianchi identity we have the following relations

(5) $$2\bar{\varrho}(X,Y) = \varrho(X,Y) + \varrho(JY,JX)\,,$$

(6) $$2\tilde{\varrho}(X,Y) = \varrho(Y,X) - \varrho(X,Y)\,.$$

From now on, we will assume for our complex symmetric connection ∇ to have a symmetric curvature operator, i.e., $R(X,Y)$ satisfies the relation

$$g(R(X,Y)Z,V) = g(R(X,Y)V,Z)\,.$$

(We do not assume that ∇ is a metric connection.) For example, we have the relations

(7) $$R(JX,JY) = R(X,Y)\,,$$

(8) $$\varrho(X,Y) = -\varrho(Y,X), \quad \varrho(JX,JY) = \varrho(X,Y)\,,$$

(for proofs see [Ni]). It means that a complex, symmetric connection with the symmetric curvature operator is an affine Kähler connection. Hence, $\bar{\varrho} = 0$, $\tilde{\varrho} = -\varrho$ and

$$\hat{\varrho}_I^K = \frac{1}{2}R_{P\bar{P}I}{}^{K} = -\frac{1}{2}(R_{IP\bar{P}}{}^{K} + R_{\bar{P}IP}{}^{K}) = \varrho_{IK}\,.$$

For the scalar curvatures

$$\tau = \sum \varrho_{PP} = 0, \quad \tau^* = \sum \varrho_{P\bar{P}}\,.$$

We use the following quadratic invariants for the curvature tensor R

$$\|R\|^2 = \sum R_{PQIK}R_{PQIK}, \quad \|\varrho\|^2 = \sum \varrho_{PQ}\varrho_{PQ}\,.$$

Quadratic invariants of the curvature tensor for a complex connection are considered in [MN].

We put

$$\Omega_I^K(X,Y) = R_{XYI}{}^{K}, \quad \text{i.e.} \quad \Omega_I^K = R_{PQI}{}^{K}\omega^P \wedge \omega^Q\,,$$

and

$$\Theta_i^j(X,Y) = -(\Omega_i^j(X,Y) - \sqrt{-1}\,\Omega_{\bar{i}}^j(X,Y))\,,$$

for $X, Y \in M_m \otimes \mathbb{C}$. Then (Θ_p^q) is a matrix of complex 2-forms and

$$\det\left(\delta_p^q - \frac{1}{2\pi\sqrt{-1}}\Theta_p^q\right) = 1 + \gamma_1 + \cdots + \gamma_n$$

is a globally defined closed form which represents the total Chern class of M via de Rham's theorem (see [KN, p.307]). Chern classes determined by γ_1, γ_2 are denoted by c_1, c_2 respectively. The corresponding Chern numbers for a compact manifold M are defined by $c_1^2[M] = \int_M \gamma_1^2$ and $c_2[M] = \int_M \gamma_2$.

In particular, the first two Chern forms are given by

$$\gamma_1 = \frac{\sqrt{-1}}{2\pi}\sum \Theta_i^i = \frac{\sqrt{-1}}{2\pi}\left(\Omega_i^i - \sqrt{-1}\,\Omega_{\bar{i}}^i\right)\,, \tag{9}$$

$$\gamma_2 = -\frac{1}{4\pi^2}\sum_{1\le i<j\le 2}\left\{\Theta_i^i \wedge \Theta_j^j - \Theta_i^j \wedge \Theta_j^i\right\}\,. \tag{10}$$

Since the metric tensor g is not parallel with respect to ∇, in general, it is interesting to compute γ_2 and γ_1^2 for a complex connection on a Hermitian surface. More precisely, we prove the following theorem.

Theorem 1.1. *Let (M, J, g) be a Hermitian surface and ∇ a complex symmetric connection on M, with the symmetric curvature operator. Then c_1^2 and c_2 are given by the following 4-forms:*

$$\gamma_1^2 = \frac{-1}{8\pi^2}(\tau^{*2} - 2\|\varrho\|^2)\Phi^2\,, \tag{11}$$

and

$$\gamma_2 = \frac{1}{16\pi^2}\{4\|\varrho\|^2 - \|R\|^2 - \tau^{*2}\}\Phi^2\,. \tag{12}$$

Proof. For the formula (11) we have

$$\begin{aligned} 4\pi^2\gamma_1^2 &= -\sum \Theta_i^i \wedge \Theta_j^j \\ &= -\sum\left[\Omega_i^i \wedge \Omega_j{}^j - \Omega_{\bar{i}}{}^i \wedge \Omega_{\bar{j}}{}^j - 2\sqrt{-1}\,\Omega_i^i \wedge \Omega_{\bar{j}}{}^j\right]\,. \end{aligned}$$

Then by (4),

$$\Omega_i^i = \tilde{\varrho}_{PQ}\omega^P \wedge \omega^Q, \qquad \Omega_{\bar{i}}^i = 0$$

which implies (11).

In a similar way, it follows from (10)

$$\gamma_2 = -\frac{1}{4\pi^2} \sum_{1\leq i<j\leq 2} \Big\{ \Big(\Omega_i^i \wedge \Omega_j^j - \Omega_{\bar{i}}^{\bar{i}} \wedge \Omega_{\bar{j}}^{\bar{j}} - \Omega_i^j \wedge \Omega_j^i + \Omega_{\bar{i}}^{\bar{j}} \wedge \Omega_{\bar{j}}^{\bar{i}} \Big) \\ -\sqrt{-1} \Big(\Omega_i^i \wedge \Omega_{\bar{j}}^{\bar{j}} + \Omega_{\bar{i}}^{\bar{i}} \wedge \Omega_j^j - \Omega_i^j \wedge \Omega_{\bar{j}}^{\bar{i}} - \Omega_{\bar{i}}^{\bar{j}} \wedge \Omega_j^i \Big) \Big\} \,.$$

By the straightforward and long computation, using the symmetries of the curvature tensor R, formulas (4) and (7), we obtain (12).

Remark. The special classes of metric connections whose curvature tensor satisfies some additional symmetries have been already studied. For example, the formulas (11) and (12) generalize the corresponding relations obtained in [CO] for Kähler manifolds, in [Ko] for holomorphic vector bundles and in [Bz] for complex vector bundles with a formally holomorphic connection.

2. Some inequalities for quadratic invariants of symmetric curvature operators

We recall now some basic facts concerning the decomposition of the curvature tensor of a complex, symmetric connection with the symmetric curvature operator. For more details see [Ni].

Let $\mathcal{R}(T_pM)$ be the vector space of all curvature tensors which satisfy (1), (2) and (3) with the symmetric curvature operator defined on the tangent space T_pM in an arbitrary point p of a Hermitian surface M. $\mathcal{R}(T_pM)$ splits into the direct sum

$$\mathcal{R}(T_p, M) = \mathcal{R}_1(T_pM) \oplus \mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM)\,,$$

where

$$\mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM) = \{R \in \mathcal{R}(T_pM) \mid \tau^*(R) = 0\}\,,$$
$$\mathcal{R}_3(T_pM) = \{R \in \mathcal{R}(T_pM) \mid \varrho(R) = 0\}\,,$$

$$\mathcal{R}_2(T_pM) = \text{orthogonal complement of } \mathcal{R}_3(T_pM) \text{ in } \mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM)\,,$$
$$\mathcal{R}_1(T_pM) = \text{orthogonal complement of } \mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM) \text{ in } \mathcal{R}(T_pM)\,.$$

Moreover, for an arbitrary $R \in \mathcal{R}(T_pM)$ we have

$$R = R_1 + R_2 + R_3\,,$$

where

$$R_1(X,Y)Z = \frac{-\tau^*}{24}\Big[g(JX,Z)Y - g(JY,Z)X + 2g(JX,Y)Z - g(X,Z)JY + g(Y,Z)JX\Big], \tag{13}$$

$$\begin{aligned} R_2(X,Y,Z,V) = -\frac{1}{8}\Big[&S(X,Z)g(Y,V) - S(Y,X)g(X,V) \\ &+ S(X,V)g(Y,Z) - S(Y,V)g(X,Z) \\ &+ 2S(X,Y)g(Z,V) - S(X,JZ)g(JY,V) \\ &+ S(Y,JZ)g(JX,V) - S(X,JV)g(JY,Z) \\ &+ S(Y,JV)g(JX,Z) + 2S(Z,JV)g(JX,Y)\Big], \end{aligned} \tag{14}$$

and

$$S(X,Y) = \varrho(X,Y) - \frac{\tau^*}{4}g(JX,Y)\,. \tag{15}$$

In the following lemma we state some inequalities which will be used later.

Lemma 2.1. *Let M be a complex Hermitian surface with $R \in \mathcal{R}(T_pM)$. Then*

$$\|R\|^2 - \frac{1}{3}\tau^{*2} \geq 0\,, \tag{16}$$

$$\|\varrho\|^2 \geq \frac{\tau^{*2}}{4}\,. \tag{17}$$

The equality holds in (16) *if $R \in \mathcal{R}_1(T_pM)$. The equality holds in* (17) *if $R \in \mathcal{R}_1(T_pM) \oplus \mathcal{R}_3(T_pM)$.*

Proof. The inequalities

$$\sum\Big[R_{PQIK} + \frac{\tau^*}{24}\big(g_{\bar{P}I}g_{QK} - g_{\bar{Q}I}g_{PK} + 2g_{\bar{P}Q}g_{IK} - g_{PI}g_{\bar{Q}K} + g_{QI}g_{\bar{P}K}\big)\Big]^2 \geq 0$$

and

$$\sum\Big[\varrho_{IK} - \frac{\tau^*}{4}g_{\bar{I}K}\Big]^2 \geq 0$$

imply by direct computations (16) and (17).

3. Chern numbers of Hermitian surface with symmetric curvature operator

Now we shall study some properties of the Chern numbers for a Hermitian surface admiting a complex connection with the symmetric curvature operator. We do not assume that this is a metric connection.

Proposition 3.1. *Assume that a Hermitian surface M admits a complex symmetric connection ∇ with the symmetric curvature operator. Then $c_1(M) = 0$.*

Proof. For γ_1 we have

$$\gamma_1(X,Y) = \frac{\sqrt{-1}}{2\pi}\left(\tilde{\varrho}(X,Y) + \sqrt{-1}\,\bar{\varrho}(X,JY)\right) .$$

Then

$$\bar{\varrho}(X,Y) = R_{XY\bar{i}}{}^{i} = R_{XYi}{}^{\bar{i}} = -R_{XYi}{}^{\bar{i}} = 0$$

implies

$$\gamma_1 = \frac{\sqrt{-1}}{2\pi}\tilde{\varrho} = \frac{-\sqrt{-1}}{2\pi}\varrho\,.$$

Since $c_1(M)$ is a real cohomology class, $\tilde{\varrho}$ is an exact 2-form and $c_1(M) = [\gamma_1] = 0$, where $[\gamma]$ denotes for a closed form γ the corresponding de Rham cohomology class. That is $\gamma_1 = d\eta$ for some global 1-form η on M. Hence, $c_1^2(M) = 0$.

Moreover, from (11),

$$\gamma_1^2 = -\frac{1}{8\pi^2}\left(\tau^{*2} - 2\|\varrho\|^2\right)\Phi^2 = d(\eta \wedge d\eta)\,, \tag{19}$$

and γ_1^2 is an exact form.

Proposition 3.2. *Assume that a Hermitian surface M admits a complex symmetric connection ∇ with the symmetric curvature operator. Then*

$$c_2(M) = [\tilde{\gamma}_2] = [\hat{\gamma}_2]$$

where

$$\tilde{\gamma}_2 = \frac{1}{16\pi^2}\left(2\|\varrho\|^2 - \|R\|^2\right)\Phi^2\,, \tag{20}$$

and

$$\hat{\gamma}_2 = \frac{1}{16\pi^2}\left(\tau^{*2} - \|R\|^2\right)\Phi^2\,. \tag{21}$$

Proof. By (12), we have

$$\gamma_2 = \frac{1}{16\pi^2} \sum \left\{ 4\|\varrho\|^2 - \|R\|^2 - \tau^{*2} + 2\varrho_{\bar{I}J}\varrho_{IJ} \right\} \Phi^2 ,$$

and then (19) implies $[\gamma_2] = [\tilde{\gamma}_2]$, $[\gamma_2] = [\hat{\gamma}_2]$. This completes the proof.

Corollary 3.3. *Let ∇ be a symmetric complex connection on a compact Hermitian surface with $R \in \mathcal{R}_1(T_pM)$. Then ∇ is a flat connection.*

Proof. By Proposition 3.1, $c_1(M) = 0$, so (19) implies

$$\int_M \left(\tau^{*2} - 2\|\varrho\|^2 \right) \Phi^2 = 0 .$$

Hence, because of Lemma 2.1, $\int_M \tau^{*2}\Phi^2 = 0$ and $\tau^* \equiv 0$ on M. Moreover $R = 0$, i.e., ∇ is flat.

Corollary 3.4. *Suppose that a symmetric complex connection ∇ exists on a compact Hermitian surface M with $R \in \mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM)$. Then $c_2[M] \le 0$. The equality holds if and only if ∇ is a flat connection.*

Proof. For $R \in \mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM)$ on M, $\tau^* = 0$. Hence, from (21) it follows

$$c_2[M] = \int_M \gamma_2 = \frac{-1}{16\pi^2} \int_M \|R\|^2 \Phi^2 \le 0 .$$

Clearly, the equality holds if and only if $R = 0$.

Corollary 3.5. *Let (M, J) be a compact Hermitian surface which admits a Kähler–Einstein metric. Then every complex symmetric connection with $R \in \mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM)$ on M is flat.*

Proof. Kähler–Einstein surfaces satisfy the Miyaoka inequality

$$c_1^2[M] = 3c_2[M] \le 0 ,$$

where the equality holds if and only if M is a complex space form (see [CO]). If M admits a complex symmetric connection ∇ with $R \in \mathcal{R}_2(T_pM)$ $\oplus \mathcal{R}_3(T_pM)$, then Proposition 3.1 implies $c_1^2(M) = 0$ and Corrolary 3.4 gives $c_2[M] \le 0$. The proof now follows by the Miyaoka inequality.

Corollary 3.6. *Let (M, J) be a compact Hermitian surface which admits a Kähler–Einstein metric. Then every complex symmetric connection with $R \in \mathcal{R}_1(T_pM) \oplus \mathcal{R}_3(T_pM)$ on M is flat.*

Proof. Let ∇ be a symmetric complex connection defined on (M, J) with $R \in \mathcal{R}_1(T_pM) \oplus \mathcal{R}_3(T_pM)$. We use Lemma 2.1 to see $4\|\varrho\|^2 = \tau^{*2}$ for this curvature tensor. Moreover, by (19) $\tau^* = \varrho = 0$. Now (20) gives

$$c_2[M] = \int_M \gamma_2 = \frac{-1}{16\pi^2} \int_M \|R\|^2 \Phi^2 \leq 0\,.$$

By Miyaoka inequality and Corollary 3.3 we get $c_2(M) = 0$ and ∇ is flat.

Remark. We have similar results for a compact surface of a general type since Miyaoka inequality holds in that case.

4. Examples

The main purpose of this section is to construct some symmetric complex connections on reducible Hermitian surfaces M with the generic $R \in \mathcal{R}(T_pM)$ or with R belonging to some vector subspaces of $\mathcal{R}(T_pM)$. Example 2 shows that the compactness of M is an essential assumption in the Corollaries 3.4, 3.5 and 3.6. These examples can be seen as modifications of the examples constructed on complex curves in [BB] and [BB1].

Let M be a reducible Hermitian surface. It means $M = M' \times M''$, where M', M'' are complex curves, endowed with symmetric connections ∇', ∇'', respectively, such that the corresponding curvature operators R' and R'' are symmetric.

Example 1. Firstly, we construct a symmetric connection $\nabla = \nabla' \times \nabla''$ on a torus $T^4 = {T'}^2 \times {T''}^2$. For this purpose we recall some results from [BB]. We consider the standard embeding of the torus into the Euclidean space $\mathbb{R}^4$ defined by

$$x_1 = \cos\alpha, \quad x_2 = \sin\alpha, \quad x_3 = \cos\beta, \quad x_4 = \sin\beta\,,$$

$0 \leq \alpha \leq 2\pi,\ 0 \leq \beta \leq 2\pi$. The metric tensor g is given by $g_{11}{=}g_{22}{=}1$, $g_{12} = 0$. Then the components of the Levi–Civita connection vanish, i.e.

$${}^{LC}\Gamma^k_{ij} = 0, \quad i, j, k = 1, 2\,.$$

If E_1, E_2 is a basis for the tangent space at an arbitrary point $p \in T^2$ we define an almost complex structure J by

$$JE_1 = E_2, \quad JE_2 = -E_1\,.$$

Let Γ_{ij}^k $(i,j,k=1,2)$ be the components of a complex symmetric connection ∇. They have to satisfy the following conditions

$$\Gamma_{11}^1=\Gamma_{12}^2=\Gamma_{21}^2=-\Gamma_{22}^1, \quad \Gamma_{12}^1=\Gamma_{21}^1=\Gamma_{22}^2=-\Gamma_{11}^2\,.$$

It follows that the components Γ_{11}^1 and Γ_{12}^1 can be arbitrary smooth functions which are periodical with respect to α and β and all other components depend on these two. To satisfy

$$\varrho_{11}=\varrho_{22}=0, \quad \varrho_{12}=-\varrho_{21}$$

we find

$$\Gamma_{11}^1=-\cos\alpha\sin\beta, \quad \Gamma_{12}^1=\sin\alpha\cos\beta\,.$$

Therefore

$$\varrho_{12}=2\cos\alpha\cos\beta \tag{22}$$

and

$$\tau^*=4\cos\alpha\cos\beta\,. \tag{23}$$

The components of our symmetric complex connection are defined globally on the torus (T^2,g).

Now we easily obtain the components of the tensor S on the torus T^4. We use (15) to see

$$S_{12}=\varrho_{12}-\frac{\tau^*}{4}g_{22}\,, \tag{24}$$

where

$$\tau^*=\tau'^*+\tau''^* \tag{25}$$

and

$$\varrho_{12}=\varrho'_{12}=\frac{\tau'^*}{2}g'_{22}, \quad g'_{22}=g_{22}\,.$$

Using (22) and (23) we get

$$S_{12}=\cos\alpha\cos\beta-\cos\gamma\cos\delta=-S_{21}\,. \tag{26}$$

Similarly

$$S_{34}=\varrho_{34}-\frac{\tau^*}{4}g_{44}, \quad \varrho_{34}=\varrho''_{12}=\frac{\tau''^*}{2}g''_{22}, \quad g''_{22}=g_{44}\,,$$

i.e.

$$S_{34}=\cos\gamma\cos\delta-\cos\alpha\cos\beta=-S_{43}\,; \tag{27}$$

(γ and δ are the parameters for the standard embeding of the torus T''^2 into $\mathbb{R}^4$). All other S_{ij} vanish.

Now by substitution (23), (25), (26), (27) into (13) and (14) we obtain that the curvature tensor R belongs to $\mathcal{R}(T_pM)$ with all components R_i different from zero.

Example 2. A nonflat complex symmetric connection ∇ with $R \in \mathcal{R}_1(T_pM) \oplus \mathcal{R}_3(T_pM)$ or $R \in \mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM)$ can be obtained using [BB1]. There we have constructed, on a complex line $\mathbb{C}$, a nonflat complex symmetric connection ∇ with the components

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = -Bx + (A-\alpha)y + E\,,$$
$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{11} = Ax + By + C\,,$$

where A, B, C, E are constants and $z = x + \sqrt{-1}\,y$. ∇ has also the constant scalar curvature $\tau^* = 2\alpha$. The Ricci tensor ϱ for ∇ has the components

$$\varrho_{11} = \varrho_{22} = 0, \quad \varrho_{12} = \varrho_{21} = \alpha\,.$$

Of course, the Christoffel symbols Γ^k_{ij} are zero for the Levi–Civita connection on $\mathbb{C}$.

Now, if we have two complex lines with complex symmetric connections ∇' and ∇'' and constant scalar curvatures $\tau'^* = 2\alpha$ and $\tau''^* = 2\beta$, then for the curvature tensor R of the complex plane $(\mathbb{C}^2, \nabla' \times \nabla'')$ we get

(i) $R \in \mathcal{R}_1(T_pM) \oplus \mathcal{R}_3(T_pM)$ if $\alpha = \beta$ (as $S = 0$, $\tau^* = 2(\alpha+\beta)$ and therefore $R_2 = 0$, see (14));

(ii) $R \in \mathcal{R}_2(T_pM) \oplus \mathcal{R}_3(T_pM)$ if $\alpha = -\beta$ (as $\tau^* = 0$, $S \neq 0$, $S_{12} = (\alpha-\beta)/2$, $S_{34} = (\beta-\alpha)/2$ and therefore $R_1 = 0$, see (13)).

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