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Steinhaus property for products of ideals

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Abstract. Let $\mathcal{M}$ and $\mathcal{N}$ stand for the ideals of meager sets and of null sets in $\mathbb{R}$, respectively. We prove that, for any Borel sets A, B in $\mathbb{R}^2$ which both are not in $\mathcal{M}\otimes\mathcal{N}$ (or $\mathcal{N}\otimes\mathcal{M}$), the set $A+B = \{a+b : a \in A,\ b \in B\}$ has the nonempty interior. Some general version of this theorem for $B = -A$ is also considered.

0. Introduction

Steinhaus [29] proved that, for each Lebesgue measurable set $A \subset \mathbb{R}$ of positive measure, the set $A - A$ of all differences $x - y$ with $x, y \in A$, contains a neighbourhood of 0. The analogous result for a linear set of the second category with the Baire property was obtained by Piccard [27]. The both results have been extended in various directions by several authors. (See e.g. [19].) The scheme given in the Steinhaus theorem can be formulated as the respective property of a pair consisting of an algebra and an ideal of sets in $\mathbb{R}$ (or, more generally, in a topological group). Other examples of pairs with the Steinhaus property and their applications to functional equations can be found in [6]. The Steinhaus property connected with invariant extensions of Lebesgue measure was investigated by Kharazishvili [15, pp. 123–132].

Let $\mathcal{M}$ and $\mathcal{N}$ stand, respectively, for the σ-ideals of meager sets and of null sets in $\mathbb{R}$. Products $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ (which will be defined in

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Section 1) form σ-ideals of sets in $\mathbb{R}^2$, which have been studied in several papers [21], [22], [11], [7], [8], [9], [10], [2], [4]. In [7], a weak version of the Steinhaus property for $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ assiociated with the σ-algebra of Borel sets in $\mathbb{R}^2$, was considered. Namely, the authors of [7] were interested in the case when there exists a countable set $W \subset \mathbb{R}^2$ such that $(A - A) \cap W \neq \emptyset$ for each Borel set $A \notin \mathcal{M} \otimes \mathcal{N}$ (or $A \notin \mathcal{N} \otimes \mathcal{M}$). From the theorems of Steinhaus and Piccard it easily follows that one can take as W the product $\mathbb{Q}^2$ of the rationals. The aim of our paper is to prove a general version of the Steinhaus property for $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$. Theorems 4 and 5 are our main results. In Section 1 we use a technique which turned out fruitful in [11], [9] and [2]. In Section 2 we follow some ideas of [16] and [4].

1. Very strong Steinhaus property

We use standard notation. Let $\mathbb{N} = \{1, 2, \dots\}$. By $\mathcal{P}(X)$ we denote the power set of X. Let $(\mathbb{G}, +, 0)$ be an Abelian topological group. For $A, B \subset \mathbb{G}$ and $x \in \mathbb{G}$, we denote

$$A \pm x = \{a \pm x : a \in A\}, \quad -A = \{-a : a \in A\},$$
$$A \pm B = \{a \pm b : a \in A,\ b \in B\}.$$

We say that $\mathcal{F} \subset \mathcal{P}(\mathbb{G})$ is *invariant* if $A + x \in \mathcal{F}$ for all $A \in \mathcal{F}$ and $x \in \mathbb{G}$. Let Σ and $\mathcal{I}$ be invariant families and let they form an algebra and an ideal of subsets of $\mathbb{G}$, respectively. We say that $(\Sigma, \mathcal{I})$ has the *Steinhaus property* (in short SP) if $A - A$ contains a neighbourhood of 0, for each $A \in \Sigma \setminus \mathcal{I}$. In the sequel, we shall use, as Σ, the algebra $\mathcal{B} = \mathcal{B}(\mathbb{G})$ of Borel sets in $\mathbb{G}$. Observe that, for $\mathbb{G} = \mathbb{R}$, the pair $(\mathcal{B}, \mathcal{N})$ has SP if and only if $(\Sigma, \mathcal{N})$ has SP where Σ stands for the algebra of Lebesgue measurable sets. The analogous statement holds in the category case. By that reason, we attribute the Steinhaus property to an ideal $\mathcal{I}$ regardless of an algebra, but this will mean that $(\mathcal{B}, \mathcal{I})$ has SP.

It is clear that for $A \subset \mathbb{G}$ we have

$$A - A = \{x \in \mathbb{G} : (A + x) \cap A \neq \emptyset\}.$$

Now, we shall graduate the strength of the Steinhaus-type properties for a given ideal. Denote by Nb(0) the family of all neighbourhoods of 0.

An ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{G})$ is called proper if $\{\emptyset\} \neq \mathcal{I} \neq \mathcal{P}(\mathbb{G})$. We say that an invariant proper ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{G})$ possesses:

(a) the *Steinhaus property*, if

$$(\forall\, A \in \mathcal{B} \setminus \mathcal{I})\; (\exists\, U \in \mathrm{Nb}(0))\;\; U \subset \{x \in \mathbb{G} : (A + x) \cap A \neq \emptyset\};$$

(b) the *strong Steinhaus property*, if

$$(\forall\, A \in \mathcal{B} \setminus \mathcal{I})\; (\exists\, U \in \mathrm{Nb}(0))\;\; U \subset \{x \in \mathbb{G} : (A + x) \cap A \notin \mathcal{I}\};$$

(c) the *very strong Steinhaus property*, if there is a countable family $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that $\mathcal{B} \setminus \mathcal{I} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ and

$$(\forall\, n \in \mathbb{N})\; (\exists\, U \in \mathrm{Nb}(0))\; (\forall\, A, B \in \mathcal{F}_n)\;\; U \subset \{x \in \mathbb{G} : (A + x) \cap B \notin \mathcal{I}\};$$

we then say that the very strong Steinhaus property for $\mathcal{I}$ *is realized by the family* $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.

The above properties will be written in short as SP, SSP and VSSP. Clearly VSSP $\implies$ SSP $\implies$ SP. The family of all countable subsets of $\mathbb{R}$ serves as a simple example of a σ-ideal without SP. Namely, it suffices to consider a nowhere dense perfect set $P \subset \mathbb{R}$ such that $P - P$ is nowhere dense. (See e.g. [30].) Several examples of ideals without SP can be derived from [3, Section 3].

Theorem 1.

(I) [25] *Assume that there exists a countable base $\{U_n\}_{n\in\mathbb{N}}$ of open neighbourhoods of 0 in $\mathbb{G}$. Then* SP $\iff$ SSP *for each invariant proper σ-ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{G})$.*

(II) [25] *There is an invariant proper ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{R})$ which witnesses that* SP $\not\Rightarrow$ SSP.

(III) *There is a Banach space in which the ideal of meager sets witnesses that* SSP $\not\Rightarrow$ VSSP.

Proof. I) It suffices to prove SP $\implies$ SSP. Suppose that $\mathcal{I}$ does not have SSP. So, there is an $A \in \mathcal{B} \setminus \mathcal{I}$ such that for each $n \in \mathbb{N}$ there is an $x_n \in U_n$ with $(A+x_n)\cap A \in \mathcal{I}$. Put $A_0 = A \setminus \bigcup_{n\in\mathbb{N}}(A+x_n)$. Thus $A_0 \in \mathcal{B}\setminus\mathcal{I}$

and $(A_0+x_n)\cap A_0=\emptyset$ for every n. Hence $U\not\subset\{x\in\mathbb{G}:(A+x)\cap A\neq\emptyset\}$ for each $U\in\mathrm{Nb}(0)$. This shows that $\mathcal{I}$ does not have SP.

(II) Let $\mathcal{I}$ denote the ideal of all sets of the form $A\cup B$ where $A\in\mathcal{N}$ and B is nowhere dense in $\mathbb{R}$. Then SP for $\mathcal{I}$ follows from SP for $\mathcal{N}$. Let $\{U_n\}_{n\in\mathbb{N}}$ be a fixed countable base of open sets in $\mathbb{R}$. Define nowhere dense perfect sets P_k, $k\in\mathbb{N}$, as follows. If $j\in\mathbb{N}$ is given and P_j, $i<j$, are chosen, pick a nowhere dense perfect set P_j of positive measure, with the diameter less than $1/(2j)$, and such that

$$P_j\subset U_j\setminus\bigcup_{i<j}\bigcup_{n<i+j}\left(P_i\pm\frac{1}{n}\right).$$

Then $B=\bigcup_{k\in\mathbb{N}}P_k\in\mathcal{B}\setminus\mathcal{I}$, and

$$\left(B+\frac{1}{n}\right)\cap B\subset\bigcup_{i+j\leq n}\left(P_i+\frac{1}{n}\right)\cap P_j\in\mathcal{I}\quad\text{for each } n\in\mathbb{N}.$$

Hence $U\not\subset\{x\in\mathbb{R}:(B+x)\cap B\in\mathcal{I}\}$ for each $U\in\mathrm{Nb}(0)$. This shows that $\mathcal{I}$ does not have SSP.

(III) Let $\mathcal{I}$ stand for the ideal of meager sets in the Banach space X of all bounded functions on $[0,1]$, endowed with the supremum norm. Fix an uncountable family $\mathcal{F}$ of pairwise disjoint balls in X. Then $\mathcal{F}\subset\mathcal{B}\setminus\mathcal{I}$. Suppose that $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ fulfils the statement of VSSP. Thus we can find an ucountable $\mathcal{F}_n$. This yields a contradiction since $A\cap B\neq\emptyset$ for any $A,B\in\mathcal{F}_n$, and $A\cap B=\emptyset$ for any distinct $A,B\in\mathcal{F}$. It is not hard to check that $\mathcal{I}$ possesses SSP. (See e.g. [28, Theorem 3.5.12].) □

Immediatelly from the definitions we obtain the following:

Proposition 1. *If proper invariant ideals $\mathcal{I},\mathcal{J}\subset\mathcal{P}(\mathbb{G})$ possess SP (respectively, SSP, VSSP) then $\mathcal{I}\cap\mathcal{J}$ possesses SP (respectively, SSP, VSSP). Moreover, if VSSP for $\mathcal{I}$ and $\mathcal{J}$ is realized by $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$, then VSSP for $\mathcal{I}\cap\mathcal{J}$ is realized by $\{\mathcal{F}_n\}_{n\in\mathbb{N}}\cup\{\mathcal{G}_n\}_{n\in\mathbb{N}}$.*

Now, we are going to show that $\mathcal{M}$ and $\mathcal{N}$ have VSSP. Then we shall obtain a general result which implies that $\mathcal{M}\otimes\mathcal{N}$ and $\mathcal{N}\otimes\mathcal{M}$ have VSSP and consequently, they have SP.

Lebesgue measure on $\mathbb{R}$ will be denoted by μ. Let $\{I_n\}_{n\in\mathbb{N}}$ stand for the family of all bounded open intervals with rational endpoints.

Theorem 2. *The ideal $\mathcal{M}$ has VSSP realized by the family $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ where*

$$\mathcal{F}_n = \{A \in \mathcal{B} : I_n \setminus A \in \mathcal{M}\} \quad \textit{for } n \in \mathbb{N}.$$

Proof. Clearly $\mathcal{B}\setminus\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Fix an $n \in \mathbb{N}$ and put $U = I_n - I_n$. Then U is an open interval and

$$U = \{x \in \mathbb{R} : (I_n + x) \cap I_n \neq \emptyset\} = \{x \in \mathbb{R} : (I_n + x) \cap I_n \notin \mathcal{M}\}.$$

Assume that $A, B \in \mathcal{F}_n$ and $x \in U$. Thus

$$\begin{aligned}(A + x) \cap B &\supset ((I_n \cap A) + x) \cap (I_n \cap B)\\ &\supset (((I_n \setminus (I_n \setminus A)) + x) \cap (I_n \setminus (I_n \setminus B))\\ &= (I_n + x) \cap I_n \setminus (((I_n \setminus A) + x) \cup (I_n \setminus B)).\end{aligned}$$

Since $(I_n + x) \cap I_n \notin \mathcal{M}$ and $I_n \setminus A$, $I_n \setminus B \in \mathcal{M}$, we have $(A + x) \cap B \notin \mathcal{M}$ as desired. □

Theorem 3. *The ideal $\mathcal{N}$ has VSSP realized by the family $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ where*

$$\mathcal{G}_n = \left\{A \in \mathcal{B} : \mu(A \cap I_n) > \frac{2}{3}\mu(I_n)\right\} \quad \textit{for } n \in \mathbb{N}.$$

Proof. Let us show that $\mathcal{B} \setminus \mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$. Inclusion "$\supset$" is obvious. To prove inclusion "$\subset$" consider an $A \in \mathcal{B} \setminus \mathcal{N}$. Thus there exists an $h > 0$ such that $\mu(A \cap K)/\mu(K) > 5/6$ where $K = (a - h, a + h)$. Pick an $I_n \subset K$ such that $\mu(K \setminus I_n) < \mu(K)/6$. We have

$$\begin{aligned}\mu(A \cap I_n)/\mu(I_n) &= (\mu(A \cap K) - \mu(A \cap (K \setminus I_n)))/\mu(I_n)\\ &\geq (\mu(A \cap K) - \mu(K \setminus I_n))/\mu(K) > \frac{5}{6} - \frac{1}{6} = \frac{2}{3}.\end{aligned}$$

Hence $A \in \mathcal{G}_n$.

Now, fix an $n \in \mathbb{N}$ and put $U = (-\mu(I_n)/4, \mu(I_n)/4)$. It easily follows that

$$U = \{x \in \mathbb{R} : \mu(I_n \cap (I_n + x)) > 3\mu(I_n)/4\}.$$

Assume that $A, B \in \mathcal{G}_n$ and $x \in U$. Thus

$$\begin{aligned}\mu((A+x)\cap B) &\geq \mu(((I_n \cap A)+x)\cap (I_n \cap B)) \\ &= \mu((I_n + x)\cap I_n \setminus (((I_n \setminus A)+x)\cup (I_n \setminus B))) \\ &\geq \mu((I_n + x)\cap I_n) - \mu(I_n \setminus A) - \mu(I_n \setminus B) \\ &> 3\mu(I_n)/4 - \mu(I_n)/3 - \mu(I_n)/3 > 0. \qquad \square\end{aligned}$$

Now, from Proposition 1 and Theorems 2, 3 we deduce

Corollary 1. *The ideal $\mathcal{M}\cap\mathcal{N}$ has VSSP.*

Assume that $\mathbb{G}_1$ and $\mathbb{G}_2$ are topological groups, and let $\mathcal{I}$ and $\mathcal{J}$ be invariant proper ideals of sets in $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. For an $A \subset \mathbb{G}_1 \times \mathbb{G}_2$ we put

$$A(\mathcal{J}) = \{x \in \mathbb{G}_1 : A_x \notin \mathcal{J}\}$$

where $A_x = \{y \in \mathbb{G}_2 : (x,y) \in A\}$, $x \in \mathbb{G}_1$. We define

$$\mathcal{I} \otimes \mathcal{J} = \{A \subset \mathbb{G}_1 \times \mathbb{G}_2 : A(\mathcal{J}) \in \mathcal{I}\}.$$

It is easy to check that $\mathcal{I}\otimes\mathcal{J}$ is an invariant proper ideal of sets in the group $\mathbb{G}_1 \times \mathbb{G}_2$. Moreover, if $\mathcal{I}$ and $\mathcal{J}$ are σ-ideals, so is $\mathcal{I}\otimes\mathcal{J}$.

Now, we are ready to formulate our main result:

Theorem 4. *Assume that $\mathcal{I}$ and $\mathcal{J}$ are proper invariant ideals of sets in $\mathbb{R}$, and $\mathcal{I}$ is moreover a σ-ideal. Assume also the following conditions:*

(1) *$\mathcal{I}$ has VSSP realized by a family $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$,*

(2) *$\mathcal{J}$ has VSSP realized by a family $\{\mathcal{G}_m\}_{m\in\mathbb{N}}$,*

(3) $(\forall\, A \in \mathcal{B}(\mathbb{R}^2))(\forall\, m \in \mathbb{N})\{x \in \mathbb{R} : A_x \in \mathcal{G}_m\} \in \mathcal{B}(\mathbb{R})$.

Then $\mathcal{I}\otimes\mathcal{J}$ has VSSP realized by the family $\{\mathcal{H}_{mn}\}_{m,n\in\mathbb{N}}$ where

$$\mathcal{H}_{mn} = \{A \in \mathcal{B}(\mathbb{R}^2) : \{x \in \mathbb{R} : A_x \in \mathcal{G}_m\} \in \mathcal{F}_n\}$$

for $m, n \in \mathbb{N}$.

Proof. For brevity we write $\mathcal{B}(\mathbb{R}) = \mathcal{B}$ and $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}^2$. First, we shall prove that

$$\mathcal{B}^2 \setminus (\mathcal{I}\otimes\mathcal{J}) = \bigcup_{m,n\in\mathbb{N}} \mathcal{H}_{m,n}. \tag{4}$$

So, let $A \in \mathcal{B}^2 \setminus (\mathcal{I} \otimes \mathcal{J})$. Since $A \in \mathcal{B}^2$, we have $A_x \in \mathcal{B}$ for each $x \in \mathbb{R}$. (See e.g. [28, 3.1.20].) Hence $A(\mathcal{J}) = \{x \in \mathbb{R} : A_x \in \mathcal{B} \setminus \mathcal{J}\}$. Thus by (2) we have

$$A(\mathcal{J}) = \bigcup_{m \in \mathbb{N}} \{x \in \mathbb{R} : A_x \in \mathcal{G}_m\}. \tag{5}$$

From $A \notin \mathcal{I} \otimes \mathcal{J}$ it follows that $A(\mathcal{J}) \notin \mathcal{I}$. Since $\mathcal{I}$ is a σ-ideal, by (5) there exists an $m \in \mathbb{N}$ such that $\{x \in \mathbb{R} : A_x \in \mathcal{G}_m\} \notin \mathcal{I}$. Now, by (3) and (1) we can find an $n \in \mathbb{N}$ such that $\{x \in \mathbb{R} : A_x \in \mathcal{G}_m\} \in \mathcal{F}_n$. Consequently, $A \in \mathcal{H}_{mn}$.

Now, let $A \in \mathcal{H}_{mn}$ for some $m, n \in \mathbb{N}$. Hence

$$A(\mathcal{J}) \supset \{x \in \mathbb{R} : A_x \in \mathcal{G}_m\} \in \mathcal{F}_n \subset \mathcal{B} \setminus \mathcal{I}$$

and thus $A \notin \mathcal{I} \otimes \mathcal{J}$. So (4) has been proved.

The proof will be finished, if we show the condition

$$\begin{aligned}&(\forall\, m, n \in \mathbb{N})(\exists\, W \in \mathrm{Nb}(0,0))(\forall\, A, B \in \mathcal{H}_{mn})\\ &W \subset \{(x,y) \in \mathbb{R}^2 : (A + (x,y)) \cap B \notin \mathcal{I} \otimes \mathcal{J}\}.\end{aligned} \tag{6}$$

Fix any $m, n \in \mathbb{N}$. By (1) and (2) we deduce the existence of sets $U, V \in \mathrm{Nb}(0)$ such that

$$(\forall\, C, C' \in \mathcal{F}_n)\ U \subset \{x \in \mathbb{R} : (C + x) \cap C' \notin \mathcal{I}\}, \tag{7}$$

$$(\forall\, D, D' \in \mathcal{G}_m)\ V \subset \{x \in \mathbb{R} : (D + x) \cap D' \notin \mathcal{J}\}. \tag{8}$$

Define $W = U \times V$. Let $A, B \in \mathcal{H}_{mn}$. Then $\widetilde{A}, \widetilde{B}$ given by

$$\widetilde{A} = \{x \in \mathbb{R} : A_x \in \mathcal{G}_m\}, \quad \widetilde{B} = \{x \in \mathbb{R} : B_x \in \mathcal{G}_m\}$$

are both in $\mathcal{F}_n$. Let $(x, y) \in W$, that is $x \in U$ and $y \in V$. Observe that

$$(\widetilde{A} + x) \cap \widetilde{B} \subset \{s \in \mathbb{R} : (A_{s-x} + y) \cap B_s \notin \mathcal{J}\}. \tag{9}$$

Indeed, let $s \in (\widetilde{A} + x) \cap \widetilde{B}$. Then $s - x \in \widetilde{A}$ and $s \in \widetilde{B}$. Hence $A_{s-x}, B_s \in \mathcal{G}_m$. Now from $y \in V$ and (8) we obtain $(A_{s-x} + y) \cap B_s \notin \mathcal{J}$.

We know that $\widetilde{A}, \widetilde{B} \in \mathcal{F}_n$ and $x \in U$, so from (7) it follows that $(\widetilde{A} + x) \cap \widetilde{B} \notin \mathcal{I}$. Thus by (9) we have

$$\{s \in \mathbb{R} : (A_{s-x} + y) \cap B_s \notin \mathcal{J}\} \notin \mathcal{I}. \tag{10}$$

To finish the proof of (6) we have to show that $((A+(x,y))\cap B)(\mathcal{J})\notin \mathcal{I}$. Observe that

$$((A+(x,y))\cap B)(\mathcal{J})=\{s\in\mathbb{R}:(A_{s-x}+y)\cap B_s\notin\mathcal{J}\}.$$

Thus the assertion follows from (10). □

Remark 1. If condition (1) in Theorem 4 is replaced by "$\mathcal{I}$ has SSP" and the remaining assumptions are unchanged then the assertion will be "$\mathcal{I}\otimes\mathcal{J}$ has SSP". Let us sketch the proof. Let $A\in\mathcal{B}^2\setminus(\mathcal{I}\otimes\mathcal{J})$. We can find an $n\in\mathbb{N}$ such that $B:=\{x\in\mathbb{R}:A_x\in\mathcal{G}_m\}\notin\mathcal{I}$. Pick $U,V\in \mathrm{Nb}(0)$ such that $U\subset\{x\in\mathbb{R}:(B+x)\cap B\notin\mathcal{I}\}$ and $V\subset\{y\in\mathbb{R}:(A_x+y)\cap A_{x'}\notin\mathcal{J}\}$ for all $x,x'\in B$ (note that $A_x,A_{x'}\in\mathcal{G}_m$). Then

$$U\times V\subset\{(x,y)\in\mathbb{R}^2:(A+(x,y))\cap A\notin\mathcal{I}\otimes\mathcal{J}\}.$$

Indeed, if $(x,y)\in U\times V$, we have

$$(B+x)\cap B\subset\{s\in\mathbb{R}:(A_{s-x}+y)\cap A_s\notin\mathcal{J}\}=((A+(x,y))\cap A)(\mathcal{J}).$$

Since $(B+x)\cap B\notin\mathcal{I}$, the proof is finished.

Another version of Theorem 4 with the phrases "$\mathcal{I}$ has SP" and "$\mathcal{I}\otimes\mathcal{J}$ has SP" also works, with a similar demonstration.

Remark 2. Theorem 4 and its versions given in Remark 1 remain valid if $\mathbb{R}$ is replaced, respectively, by Abelian topological groups $\mathbb{G}_1$ and $\mathbb{G}_2$.

Remark 3. Montgomery [24] proved that, for any Borel set $A\subset\mathbb{R}^2$ and $r>0$, the sets $\{x\in\mathbb{R}:A_x\notin\mathcal{M}\}$ and $\{x\in\mathbb{R}:\mu(A_x)>r\}$ are Borel. Consequently, if I is an interval, then the set

$$\{x\in\mathbb{R}:I\setminus A_x\in\mathcal{M}\}=\mathbb{R}\setminus\{x\in\mathbb{R}:((\mathbb{R}\times I)\setminus A)_x\notin\mathcal{M}\}$$

is Borel. Similarly, the set

$$\{x\in\mathbb{R}:\mu(I\cap A_x)>r\}=\{x\in\mathbb{R}:\mu((\mathbb{R}\times I)\setminus A)_x>r\}$$

is Borel. (See also [14, 16.1, 22.22, 22.25].) Hence, for any member of the families $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ from Theorems 2 and 3, condition (3) in Theorem 4 is fulfilled.

Now, using Theorems 2, 3, 4 together with Remark 3 and Proposition 1 we conclude

Corollary 2. $\mathcal{M} \otimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{M}$ *and* $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ *have VSSP.*

Remark 4. Note that the ideals $\mathcal{M} \otimes \mathcal{M}$ and $\mathcal{N} \otimes \mathcal{N}$ are greater than the ideals of meager sets and of null sets in $\mathbb{R}^2$, respectively. (See [26].) We shall obtain the respective equalities, if we reduce $\mathcal{M} \otimes \mathcal{M}$ and $\mathcal{N} \otimes \mathcal{N}$ to $\mathcal{M} \mathbin{\widetilde{\otimes}} \mathcal{M}$ and $\mathcal{N} \mathbin{\widetilde{\otimes}} \mathcal{N}$ where

$$\mathcal{I} \widetilde{\otimes} \mathcal{J} = \{A \subset \mathbb{R}^2 : (\exists\, B \in \mathcal{B}(\mathbb{R}^2) \cap (\mathcal{I} \otimes \mathcal{J}))\ A \subset B\}.$$

Analogously, we can consider $\mathcal{M} \widetilde{\otimes} \mathcal{N}$ and $\mathcal{N} \widetilde{\otimes} \mathcal{M}$ instead of $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$. The reduced products seem more natural in some contexts. However, since

$$\mathcal{B}(\mathbb{R}^2) \setminus (\mathcal{I} \otimes \mathcal{J}) = \mathcal{B}(\mathbb{R}^2) \setminus (\mathcal{I} \mathbin{\widetilde{\otimes}} \mathcal{J}),$$

there is no difference which kind of products we use to investigate the Steinhaus-type properties. Sometimes it is convenient to associate with $\mathcal{I} \mathbin{\widetilde{\otimes}} \mathcal{J}$ the smallest σ-algebra Σ containing $\mathcal{B}(\mathbb{R}^2) \cup (\mathcal{I} \mathbin{\widetilde{\otimes}} \mathcal{J})$. (See [2].) Clearly, each set from $\Sigma \setminus (\mathcal{I} \mathbin{\widetilde{\otimes}} \mathcal{J})$ contains a set from $\mathcal{B}(\mathbb{R}^2) \setminus (\mathcal{I} \otimes \mathcal{J})$.

The Steinhaus property has important applications in functional equations theory. For instance, it leads to a simple proof of the fact that an additive function bounded on a measurable set of positive measure is continuous (the Ostrowski theorem; [20, p. 210]). A similar fact holds in the Baire category case [20, p. 210]. Moreover, there is a general theorem [20, Theoremm 2, p. 240] from which, together with SP for $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$, we conclude the following

Corollary 3. *Let* $\mathcal{I} = \mathcal{M} \otimes \mathcal{N}$ *or* $\mathcal{I} = \mathcal{N} \otimes \mathcal{M}$, *and let* $T \in \mathcal{B}(\mathbb{R}^2) \setminus \mathcal{I}$. *Then every additive function* $f \colon \mathbb{R}^2 \to \mathbb{R}$ *bounded on* T *is continuous.*

In turn, from Corollary 3 and Remark 4 we derive the next result, by the use of an argument similar to that in [20, p. 218] or [16, p. 146].

Corollary 4. *Let* $\mathcal{I} = \mathcal{M} \mathbin{\widetilde{\otimes}} \mathcal{N}$ *or* $\mathcal{I} = \mathcal{N} \mathbin{\widetilde{\otimes}} \mathcal{M}$, *and let* Σ *denote the smallest* σ*-algebra containing* $\mathcal{B}(\mathbb{R}^2) \cup \mathcal{I}$. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be an additive function such that* $f|P$ *is* Σ*-measurable for some* $P \in \Sigma \setminus \mathcal{I}$. *Then* f *is continuous.*

2. Extended Steinhaus property

Fix an Abelian topological group $\mathbb{G}$ and an invariant proper ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{G})$. Denote by $\mathrm{int}(A)$ the interior of a set $A \subset \mathbb{G}$. Note that if VSSP for $\mathcal{I}$ is realized by $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, then $\mathrm{int}(B-A) \neq \emptyset$ for all $A, B \in \mathcal{F}_n$ and for each $n \in \mathbb{N}$. It is natural to ask whether $\mathrm{int}(B-A) \neq \emptyset$ for all $A, B \in \mathcal{B} \setminus \mathcal{I}$. The answer is affirmative for $\mathcal{I} = \mathcal{M}$ and $\mathcal{I} = \mathcal{N}$. The respective results are well known and their various generalizations were studied in several papers. (See [5], [18], [19], [23], [12], [13].) We are going to establish this kind of Steinhaus property for $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$. We shall follow the method used in [16, Theoremm 2, p. 115]. First let us connect our investigations with the results of the previous section.

We say that an invariant proper ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{G})$ possesses:

(a) the *extended Steinhaus property*, if

$$(\forall A, B \in \mathcal{B} \setminus \mathcal{I})\ \mathrm{int}(\{x \in \mathbb{G} : (A+x) \cap B \neq \emptyset\}) \neq \emptyset;$$

(b) the *extended strong Steinhaus property*, if

$$(\forall A, B \in \mathcal{B} \setminus \mathcal{I})\ \mathrm{int}(\{x \in \mathbb{G} : (A+x) \cap B \notin \mathcal{I}\}) \neq \emptyset.$$

Condition (a) states exactly that $\mathrm{int}(B-A) \neq \emptyset$ for all $A, B \in \mathcal{B} \setminus \mathcal{I}$. The properties given in (a) and (b) will be written in short as ESP and ESSP. Clearly ESP $\implies$ SP, ESSP $\implies$ SSP and ESSP $\implies$ ESP. The following proposition shows how to obtain ESP or ESSP when SP or SSP holds.

Proposition 2. *For an invariant proper ideal* $\mathcal{I} \subset \mathcal{P}(\mathbb{G})$ *satisfying the condition:*

$$(\forall\, A, B \in \mathcal{B} \setminus \mathcal{I})\, (\exists\, z \in \mathbb{G}) \quad (A+z) \cap B \notin \mathcal{I}, \tag{11}$$

we have SP $\iff$ ESP *and* SSP $\iff$ ESSP.

Proof. We shall prove SSP $\implies$ ESSP; the argument for SP $\implies$ ESP is analogous. Let $A, B \in \mathcal{B} \setminus \mathcal{I}$. Pick a $z \in \mathbb{G}$ as in (11) and put $Z = (A+z) \cap B$. By SSP we have

$$U := \mathrm{int}(\{x \in \mathbb{G} : (Z+x) \cap Z \notin \mathcal{I}\}) \neq \emptyset.$$

Observe that

$$U + z \subset \{y \in \mathbb{G} : (Z + y - z) \cap Z \notin \mathcal{I}\} \subset \{y \in \mathbb{G} : (A + y) \cap B \notin \mathcal{I}\}.$$

Hence $\operatorname{int}(\{y \in \mathbb{G} : (A + y) \cap B \notin \mathcal{I}\}) \neq \emptyset$. □

Remark 5. T. Natkaniec [25] observed that the following version of Theorem 1(I) holds. If $\mathbb{G}$ has a countable base of open sets then ESP $\iff$ ESSP for each proper invariant σ-ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{G})$. Indeed, suppose that $\mathcal{I}$ does not have ESSP. Thus there are sets $A, B \in \mathcal{B} \setminus \mathcal{I}$ such that $\operatorname{int}(\{x \in \mathbb{G} : (A + x) \cap B \notin \mathcal{I}\}) = \emptyset$. If $\{U_n\}_{n\in\mathbb{N}}$ is a base of open sets in $\mathbb{G}$, then for each $n \in \mathbb{N}$, pick an $x_n \in U_n$ with $(A + x_n) \cap B \in \mathcal{I}$. Thus $B_0 = B \setminus \bigcup_{n\in\mathbb{N}}(A + x_n) \in \mathcal{B} \setminus \mathcal{I}$ and $(A + x_n) \cap B_0 = \emptyset$ for every n. Hence $\operatorname{int}(\{x \in \mathbb{G} : (A + x) \cap B_0 \neq \emptyset\}) = \emptyset$ which shows that $\mathcal{I}$ does not have ESP.

Theorem 5. *$\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ have ESSP.*

Proof. Let $\mathcal{J} = \mathcal{M} \otimes \mathcal{N}$ (the case of $\mathcal{N} \otimes \mathcal{M}$ is analogous). By virtue of Corollary 2 it suffices to check condition (11) in Proposition 2 for $\mathcal{J}$. To this aim we use the notion of a $\mathcal{J}$-density point considered in [4]. Let $\varphi(E)$ denote the set of $\mathcal{J}$-density points of a set E from the σ-algebra generated by $\mathcal{B}(\mathbb{R}^2) \cup (\mathcal{M} \widetilde{\otimes} \mathcal{N})$. In [4], it is proved that φ has usual properties of the lower density operator (cf. [26, Chap. 22]). Let A, B be Borel sets in $\mathbb{R}^2$ that are not in $\mathcal{J}$. Pick $a \in \varphi(A), b \in \varphi(B)$ and put $z = b - a$. Now, $a \in \varphi(A)$ implies $b \in \varphi(A) + z = \varphi(A + z)$, and thus $b \in \varphi(A + z) \cap \varphi(B) = \varphi((A + z) \cap B)$. Hence $(A + z) \cap B \notin \mathcal{J}$. □

An ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{G})$ is called *symmetric* if $-A \in \mathcal{I}$ whenever $A \in \mathcal{I}$. Obviously, if $\mathcal{I}$ is symmetric, then ESP for $\mathcal{I}$ is equivalent to

$$(\forall A, B \in \mathcal{B} \setminus \mathcal{I}) \ \operatorname{int}(A + B) \neq \emptyset.$$

Observe that if ideals $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(\mathbb{G})$ are symmetric, so is $\mathcal{I} \otimes \mathcal{J}$. Thus from Theorem 5 it immediately results the following corollary.

Corollary 5. *For arbitrary Borel sets A, B in $\mathbb{R}^2$ which both are not in $\mathcal{M} \otimes \mathcal{N}$ (or $\mathcal{N} \otimes \mathcal{M}$), the set $A + B$ has nonempty interior.*

Let us finish with the observation that $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ does not possess the extended Steinhaus property. This will follow from the known general scheme.

We say that ideals $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(\mathbb{G})$ are *Borel orthogonal* if there is a Borel set $A \in \mathcal{I}$ such that $\mathbb{G} \setminus A \in \mathcal{J}$. For $D \subset \mathbb{G}$, we say that an ideal $\mathcal{I}$ is *D-additive* if $A + D \in \mathcal{I}$ whenever $A \in \mathcal{I}$. Clearly, if $\mathcal{I}$ is an invariant σ-ideal then $\mathcal{I}$ is D-additive for each countable set $D \subset \mathbb{G}$.

Proposition 3 (cf. [1], [17])**.** *Assume that $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(\mathbb{G})$ are Borel orthogonal proper ideals. Let additionally, $\mathcal{I}$ be invariant, symmetric and D-additive for some countable dense subgroup D of $\mathbb{G}$. Then there are Borel sets $A, B \notin \mathcal{I} \cap \mathcal{J}$ such that $A+B = A-B = B-A$ and* $\operatorname{int}(A+B) = \emptyset$.

Proof. Let $C \subset \mathbb{G}$ be a Borel set such that $C \in \mathcal{I}$ and $\mathbb{G} \setminus C \in \mathcal{J}$. Put $B = (D - C) \cup (D + C)$. Then $B \in \mathcal{I}$, $-B = B$, and for $A = \mathbb{G} \setminus B$ we have $A \in \mathcal{J}$, $-A = A$. Hence $A + B = A - B = B - A$. Since $A \cup B = \mathbb{G}$, we infer that $A, B \notin \mathcal{I} \cap \mathcal{J}$. Moreover, $A - B \subset \mathbb{G} \setminus D$ by the definition of B. Since D is dense, we have $\operatorname{int}(A - B) = \emptyset$. □

Proposition 3 applies to $\mathcal{M}$ and $\mathcal{N}$ and to $\mathbb{Q}$, the additive group of rationals in $\mathbb{R}$. (See [1], [17]). Observe that it also applies to $\mathcal{M} \otimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{M}$ and $\mathbb{Q}^2$ in $\mathbb{R}^2$. Namely, if C, E are disjoint Borel sets such that $C \in \mathcal{M}$, $E \in \mathcal{N}$ and $C \cup E = \mathbb{R}$ (cf. [26]), then $C \times \mathbb{R} \in \mathcal{M} \otimes \mathcal{N}$ and $E \times \mathbb{R} \in \mathcal{N} \otimes \mathcal{M}$. Thus we may formulate the following result which contrasts with Corollary 2.

Corollary 6. *There are Borel sets $A, B \subset \mathbb{R}^2$ which both are not in $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$, and such that $A+B = A-B = B-A$,* $\operatorname{int}(A+B) = \emptyset$.

As we have seen, the ideals $\mathcal{M} \cap \mathcal{N}$ and $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ witness that ESP can be false even while VSSP is true. Consequently, condition (11) in Proposition 2 is not fulfilled by these ideals.

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