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On a characterization of infinite cyclic groups

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Abstract. In a paper on representation theory of algebras a characterization of infinite cyclic group was given. We provide a simpler proof of this characterization and some related results.

In [2] the following result was proved:

Theorem 1. *Let G be a torsion-free finitely generated group. G is cyclic if and only if G satisfies the following two conditions:*

(a) *for any nontrivial subgroup H of G the index $|N_G(H) : H|$ is finite,*

(b) *for any two cyclic subgroups H_1, H_2 of G the intersection $H_1 \cap H_2$ is a nontrivial subgroup of G.*

This theorem was crucial for the proof of the main result of [2], concerning representation theory of algebras. The proof given on pages 138–141 of [2], depends on cohomological arguments. Here we give two proofs obtained by elementary methods. The first one is based on the well known classical result due to I. Schur.

Lemma 1. *If the center $Z(G)$ of a group G has finite index in G, then its commutator subgroup G' is finite.*

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An easy combinatorial proof of the lemma can be found in [7] (Lemma 2.1, p. 4). Other proofs are available for instance in [4] (Theorem 1.4, p. 49) or [8] (Theorem 10.1.4, p. 287).

The second proof of Theorem 1 is also based on elementary group theory. We will use the following well known lemma due to Miller and Moreno.

Lemma 2. *If every proper subgroup of a finite group G is abelian, then G is solvable.*

The proof of this lemma one can find, for example, either in the introductory part of [6] (Theorem 6.3, p. 62) or in [5] (Exercise 6, p. 14).

So both proofs are essentially different than that given in [2] (we prove only the part $\Leftarrow$ because the part $\Rightarrow$ is obvious). Our notation and terminology is standard, as for example in [3], [8].

In both proofs we shall apply the following lemma, almost proved on page 141 in [2].

Lemma 3. *If G satisfies the conditions* (a) *and* (b) *of Theorem* 1, then the center $Z(G)$ of G has finite index in G.

Proof. Let $G = \langle x_1, \dots, x_n \rangle$. By (b) the intersection $\langle x_1 \rangle \cap \langle x_2 \rangle \cap \dots \cap \langle x_n \rangle$ is a nontrivial cyclic subgroup. Let c be its generator. Since c is a power of each x_i, it commutes with each x_i and then it commutes with all elements of G that is c is a central element. Now by (a) $|G : \langle c \rangle|$ is finite because $G = N_G(\langle c \rangle)$. Since $|G : Z(G)| \leqslant |G : \langle c \rangle|$, the lemma follows. □

The first proof of the theorem. By Lemma 3 we know that $[G : Z(G)] < \infty$. Thus, by Lemma 1 the commutator subgroup G' of G is finite. But G is torsion-free, so G' is trivial and then G is abelian. Hence by the structure theorem for finitely generated abelian groups G is cyclic. □

The second proof of the theorem. It follows from Lemma 3 that $|G : C| < \infty$ where $C \subseteq Z(G)$ is a cyclic subgroup. We proceed by induction on $|G : C|$. If $|G : C|$ is prime, then obviously G is abelian and consequently it is cyclic. So by the induction hypothesis we may assume that every proper subgroup H of G such that $C \leqslant H$ is cyclic. Hence in the finite group $\overline{G} = G/Z(G)$ all proper subgroups are cyclic. Thus by Lemma 2 the group $\overline{G}$ is solvable. The group $\overline{G}$, being solvable, contains a maximal normal subgroup $\overline{M}$ of prime index, say q. Its preimage M is, by induction assumption, cyclic and normal in G.

Let $M = \langle m \rangle$, and let $k \neq 0$ be such that $m^k = z \in Z(G)$. If $g \in G \setminus M$ then obviously, as on page 138 of [2], $g^{-1}mg = m^s$ for some integer s, since $M \trianglelefteq G$. Thus $z = g^{-1}zg = g^{-1}(m^k)g = (g^{-1}mg)^k = m^{sk} = z^s$, which implies $s = 1$ and so $mg = gm$. Therefore $M \subseteq Z(G)$ and G is abelian because the factor group $G/Z(G)$ is cyclic. Hence G is cyclic. □

The above considerations give an opportunity to discuss some modifications of the assumptions of Theorem 1. In particular the following version of condition (a) can be considered:

(a') for any nontrivial cyclic subgroup C of G the index $|N_G(C) : C|$ is finite.

Notice that, if $C \subseteq G$ is a cyclic subgroup then $C \subseteq Z_G(C) \subseteq N_G(C)$ and $|N_G(C) : Z_G(C)| < \infty$, because $\mathrm{Aut}(C)$ is a finite group. Hence, in the condition (a') one can replace $N_G(C)$ by $Z_G(C)$. We also have the following immediate observation:

Lemma 4. *If a group G satisfies the condition* (a) *or* (b) *of Theorem 1 or the condition* (a') *then every subgroup of G satisfies the same condition.*

Now we have the following slightly stronger version of Theorem 1

Theorem 2. *Let G be a torsion-free group. G is cyclic if and only if G satisfies the conditions* (a') *and* (b)*.*

Proof. $\Leftarrow$ Let $H \subseteq G$ be a finitely generated subgroup. Then in view of Lemma 4 H satisfies the same condition. Hence, by the proof of Theorem 1 we know, that H is cyclic. It means that G is locally cyclic and hence abelian. Now if $1 \neq C$ is a cyclic subgroup of G then C is of finite index in G. Hence G is cyclic, by previous theorem. □

If G is torsion-free abelian, then we already used that the Theorem 1 follows immediately from (a') or (b) and from the structure theorem for finitely generated abelian groups. The following generalization is true:

Proposition 1. *Let G be a torsion-free finitely generated solvable group. If G satisfies the condition either* (a') *or* (b) *then G is cyclic.*

Proof. Let A be a maximal abelian normal subgroup of G.

If G satisfies (a') then, by Lemma 4, every cyclic subgroup of A has finite index in A and hence, since A is torsion-free, it follows that A must be cyclic. Moreover, again by (a') $|G : A| < \infty$, so G satisfies the assumption (b) which gives G cyclic by Theorem 1.

Suppose now that G satisfies the assumption (b) only. Thus A must be locally cyclic. Let $g \in G \setminus A$ be an arbitrary element. Since $\langle g \rangle \cap A \neq \{e\}$, g induces an automorphism of finite order on A. But it is known ([3]) that the only nontrivial automorphism of finite order of a locally cyclic torsion-free group is of the form $a \longrightarrow a^{-1}$. Therefore using the same arguments as in the end of the second proof of Theorem 1 we obtain that $G = A$. Hence G is cyclic because it is finitely generated. □

Using Lemma 4 we immediately obtain

Corollary 1. *Let G be a torsion-free locally solvable group. If G satisfies the condition* (a') *then G is cyclic. If G satisfies the condition* (b) *then G is locally cyclic.*

In the proof of the above proposition we have not used the assumption that G is solvable in full generality. In fact we showed that

Corollary 2. *Let G be a torsion-free group and let A be a maximal abelian subgroup of G. If G satisfies the condition* (a') *then A is cyclic; If G satisfies the condition* (b) *then A is locally cyclic. In both cases $N_G(A) = A$.*

Remark 1. In [1] there was constructed a nonabelian finitely generated non-torsion group G, which is torsion-by-cyclic. Obviously this group satisfies the condition (b), hence it is torsion-free, and does not satisfy the condition (a) and even (a'). See also Theorems 31.3 and 31.4 of [6].

Remark 2. In Theorem 28.3 of [6] there is constructed a 2-generated torsion-free simple group G such that every proper subgroup of G is cyclic and every two maximal subgroups have trivial intersection. It is clear that this group satisfies the condition (a) and does not satisfy the condition (b).

Remark 3. If F is a (finitely generated) free nonabelian group then it is torsion-free, satisfies the condition (a') (but not (a)), and certainly does not satisfy the condition (b).

Remark 4. Let G be finitely generated and suppose that G satisfies the conditions (a) and (b). If we suppose G to contain torsion elements, then by (b), G must be a p-group having exactly one subgroup of order p. Because this subgroup is cyclic and normal then, by (a), G is finite and, as it is well known, G is either a cyclic p-group or a generalized quaternion group Q_{2^n} for some $n \geqslant 3$.

On the other hand, for each sufficiently large prime p Ol'shanskii constructed an infinite p-group $G(p)$ whose subgroups are cyclic of order p. This group satisfies (a) and does not satisfy (b). The group $G(p)$ can be extended to a p-group containing exactly one subgroup of order p and whose every subgroup is cyclic ([6], Theorem 31.8). Obviously this extension satisfies (b) and beside the unique subgroup of order p all subgroups satisfy (a).

References

[1] S. I. Adjan, Certain torsion-free groups, *Izv. Acad. Nauk SSSR Ser. Mat.* **35** (1971), 459–468.

[2] P. Dowbor, Stabilizer conjecture for representation-tame Galois coverings of algebras, *J. Algebra* **239** (2001), 112–149.

[3] L. Fuchs, Infinite Abelian Groups, Vol. I, II, *Acacemic Press*, *London*, 1970, 1973.

[4] J. M. Gorchakov, Groups with Finite Conjugacy Classes, *Moscow*, 1978.

[5] H. Kurzweil and B. Stellmacher, Theorie der endlichen Gruppen, *Springer-Verlag*, *Berlin*, 1998.

[6] A. Yu. Ol'shanskii, Geometry of Defining Relations in Groups, *Nauka*, *Moscow*, 1989.

[7] D. S. Passman, Infinite Group Rings, *Marcel Dekker*, *New York*, 1971.

[8] D. J. S. Robinson, A Course in the Theory of Groups, *Springer-Verlag*, *New York*, 1996.

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