*Publ. Math. Debrecen*
**63/3** (2003), 261–277

# Reverse order laws for the Drazin inverse of a triple matrix product

By YONGGE TIAN (Kingston)

**Abstract.** Necessary and sufficient conditions are given for the reverse order laws $(ABC)^D = C^D B^D A^D$, $(ABC)^D = C^D B^\dagger A^D$, $(ABC)^D = C^\dagger B^D A^\dagger$ and $(ABC)^D = C^\dagger B^\dagger A^\dagger$ to hold for the Drazin inverse of the triple matrix product $ABC$. Various consequences and related topics are also discussed.

## 1. Introduction

Suppose $A$ and $B$ are two invertible matrices of the same kind. Then the product $AB$ is also invertible, and the inverse of $AB$ can be simply written as the reverse order product $(AB)^{-1} = B^{-1}A^{-1}$ of $A^{-1}$ and $B^{-1}$. This law can be used to simplify various matrix expressions that involve inverses of matrix products. This identity is best known in linear algebra, and is called the reverse order law for the inverse of matrix product. This identity is, however, not valid in general for generalized inverses of products of matrices. For an $m \times n$ matrix $A$, the well-known Moore–Penrose inverse $A^\dagger$ is defined to be the unique solution of the following four Penrose equations

$$\text{(i)} \quad AXA = A, \qquad \text{(ii)} \quad XAX = X,$$

$$\text{(iii)} \quad (AX)^* = AX, \qquad \text{(iv)} \quad (XA)^* = XA,$$

*Mathematics Subject Classification:* Primary 15A09; Secondary 15A03.
*Key words and phrases:* Moore–Penrose inverse, Drazin inverse, reverse order law, rank equality.

where $(\cdot)^*$ denotes the conjugate transpose of a matrix. A matrix $X$ is called a g-inverse of $A$ if it satisfies (i) and is denoted by $A^-$, while the collection of all possible $A^-$ is denoted by $\{A^-\}$. In general, if a matrix $X$ satisfies the equations $i, \dots, j$ in (i)–(iv), then $X$ is called an $\{i, \dots, j\}$-inverse of $A$ and is denoted by $A^{(i,\dots,j)}$. For different types of generalized inverses of matrices, there are also different types of reverse order laws. For Moore–Penrose inverses, the standard reverse order law for the matrix product $AB$ is $(AB)^\dagger = B^\dagger A^\dagger$. For g-inverse of matrix, the standard reverse order law $(AB)^- = B^- A^-$ has some variations $B^-A^- \in \{(AB)^-\}$, $\{B^-A^-\} \subseteq \{(AB)^-\}$, $\{B^-A^-\} = \{(AB)^-\}$, etc. In addition, it is reasonable to consider various mixed-type reverse order laws, such as, $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$, $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$, $(AB)^\dagger = B^\dagger A^\dagger - B^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger$, $(ABC)^\dagger = (BC)^\dagger B(AB)^\dagger$, $(ABC)^\dagger = (B^\dagger BC)^\dagger B^\dagger (ABB^\dagger)^\dagger$, etc. Thus it is large work to investigate various reverse order laws for generalized inverses of products of matrices, which has been the object of many studies since 1960s. Various results related to reverse order laws for generalized inverses, reflexive generalized inverses, Moore–Penrose inverses, weighted Moore–Penrose inverses and Drazin inverses, etc. of matrix products can be found in the literature, see, e.g., [1]–[12], [14]–[17], [21]–[27].

A straightforward and effective method to investigate reverse order laws for generalized inverses of matrix products is the rank of matrix. It is obvious that any two matrices $A$ and $B$ of the same size are equal if and only if $\operatorname{rank}(A - B) = 0$. This statement seems quite trivial. If, however, one can find some nontrivial formulas for the rank of $A - B$, then necessary and sufficient conditions for $A = B$ can be derived from these rank formulas. This method can be applied to investigate any reverse order laws for generalized inverses of matrix products, or more generally, to investigate the relationship between any two matrix expressions that involve generalized inverses of matrices. Several interesting rank equalities found by the author are presented below

$$r(AA^\dagger - A^\dagger A) = 2r[A, A^*] - 2r(A),$$

$$r(A^kA^\dagger - A^\dagger A^k) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k, A^*] - 2r(A),$$

$$r(A^*A^\dagger - A^\dagger A^*) = r(AA^*A^2 - A^2A^*A),$$

$$r(AB - ABB^\dagger A^\dagger AB) = r[A^*, B] + r(AB) - r(A) - r(B),$$

$$r[(AB)^\dagger - B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger] = r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} + r[AB, AA^*AB] - 2r(AB),$$

$$r\left([A, B]^\dagger - \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix}\right) = r[AA^*B, BB^*A],$$

$$r\left([A, B]^\dagger[A, B] - \begin{bmatrix} A^\dagger A & 0 \\ 0 & B^\dagger B \end{bmatrix}\right) = r(A) + r(B) - r[A, B],$$

see [18], [19], [21], [22]. Let the right sides of the above rank equalities be zero. Then one can immediately obtain necessary and sufficient conditions such that the matrix expressions on the left sides are zero.

In this paper we study reverse order laws for Drazin inverses of matrix products. The Drazin inverse of a square matrix $A$ is defined to be the unique solution $X$ of the following three equations

$$\text{(i)}\ \ A^kXA = A^k, \qquad \text{(ii)}\ \ XAX = X, \qquad \text{(iii)}\ \ AX = XA,$$

and is often denoted by $X = A^D$, where $k$ is the index of $A$, i.e., the smallest nonnegative integer $k$ such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$. When $A$ is nonsingular, $A^D = A^{-1}$. It is quite easy to write out various reasonable reverse order laws for the Drazin inverse of matrix products, such as, $(AB)^D = B^DA^D$, $(AB)^D = B^D(A^DABB^D)^DA^D$, $(ABC)^D = C^DB^DA^D$, $(ABC)^D = (BC)^DB(AB)^D$, etc. In addition, it is worthwhile to consider reverse order laws combined both Moore–Penrose inverses and Drazin inverses, such as, $(ABC)^D = C^DB^\dagger A^D$, $(ABC)^D = C^\dagger B^DA^\dagger$ and $(ABC)^D = C^\dagger B^\dagger A^\dagger$, etc.

There is a close relationship between the Drazin inverse and the Moore-Penrose inverse of matrix. A well-known result asserts that the Drazin inverse of any square matrix $A$ with index $k$ can be expressed in the form

$$A^D = A^k(A^{2k+1})^\dagger A^k, \tag{1.1}$$

see, e.g., [3, p. 174]. Hence any question on Drazin inverses of matrices in fact is a question on Moore–Penrose inverses of matrices.

It is commonly seen in matrix analysis that a matrix is often written as a product of three matrices, such as, Smith form decompositions, singular value decompositions, eigenvalue decompositions, Schur decompositions, etc. Hence it is natural to investigate the Drazin inverse of a triple matrix product $ABC$. In this paper we show a set of rank equalities for the Drazin inverse of a triple matrix product, and then derive from them necessary and sufficient conditions for the reverse order laws

$$(ABC)^D = C^D B^D A^D, \qquad (ABC)^D = (BC)^D B (AB)^D,$$

$$(ABC)^D = C^D B^\dagger A^D, \qquad (ABC)^D = C^\dagger B^D A^\dagger, \qquad (ABC)^D = C^\dagger B^\dagger A^\dagger$$

to hold.

The matrices considered in this paper are all over the field $\mathbb{C}$ of complex numbers. For any $A \in \mathbb{C}^{m\times n}$, denote by $A^*$, $r(A)$ and $\mathscr{R}(A)$ the conjugate transpose, the rank and the range (column space) of $A$, respectively.

**Lemma 1.1** ([13]). *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{m\times k}, C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$ with $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$. Then*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A & 0 \\ 0 & D - CA^\dagger B \end{bmatrix} = r(A) + r(D - CA^\dagger B). \tag{1.2}$$

Let

$$C = [C_1, C_2], \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

and suppose that

$$\mathscr{R}(B_1) \subseteq \mathscr{R}(A_1),\ \mathscr{R}(B_2) \subseteq \mathscr{R}(A_2),\ \mathscr{R}(C_1^*) \subseteq \mathscr{R}(A_1^*),\ \mathscr{R}(C_2^*) \subseteq \mathscr{R}(A_2^*).$$

Then (1.2) becomes

$$r(D - C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2) = r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} - r(A_1) - r(A_2). \tag{1.3}$$

**Lemma 1.2** ([17]). *Suppose that $A_1$, $A_2$, $A_3$, $B_1$ and $B_2$ satisfy the range inclusions*

$$\mathscr{R}(B_i) \subseteq \mathscr{R}(A_{i+1}), \quad \text{and} \quad \mathscr{R}(B_i^*) \subseteq \mathscr{R}(A_i^*), \qquad i = 1, 2. \tag{1.4}$$

*Then*

$$\begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} A_3^{\dagger}B_2A_2^{\dagger}B_1A_1^{\dagger} & -A_3^{\dagger}B_2A_2^{\dagger} & A_3^{\dagger} \\ -A_2^{\dagger}B_1A_1^{\dagger} & A_2^{\dagger} & 0 \\ A_1^{\dagger} & 0 & 0 \end{bmatrix}, \tag{1.5}$$

*and*

$$A_3^{\dagger}B_2A_2^{\dagger}B_1A_1^{\dagger} = [I,\ 0,\ 0] \begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix}^{\dagger} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}. \tag{1.6}$$

Eq. (1.6) is from a general result in [17] for the product $A_{k+1}^{\dagger}B_kA_k^{\dagger}\cdots$ $\cdots B_1A_1^{\dagger}$ with $k = 2$.

**Lemma 1.3.** *Let $A, X \in \mathbb{C}^{m\times m}$ with $\mathrm{Ind}(A) = k$. Then $X = A^D$ if and only if*

$$A^{k+1}X = A^k, \quad XA^{k+1} = A^k \qquad \textit{and} \quad r(X) = r(A^k). \tag{1.7}$$

This assertion can be easily proved through the Jordan canonical form of a matrix.

## 2. Main results

According to Lemma 1.2, the reverse order product $C^DB^DA^D$ can be expressed in terms of the Moore–Penrose inverse of a block matrix.

**Lemma 2.1.** *Let $A, B, C \in \mathbb{C}^{m\times m}$ with $\mathrm{Ind}(A) = k_1$, $\mathrm{Ind}(B) = k_2$ and $\mathrm{Ind}(C) = k_3$. Then the product $C^DB^DA^D$ can be expressed in the form*

$$\begin{aligned} C^DB^DA^D &= [C^{k_3},\ 0,\ 0] \begin{bmatrix} 0 & 0 & A^{2k_1+1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 \end{bmatrix}^{\dagger} \begin{bmatrix} A^{k_1} \\ 0 \\ 0 \end{bmatrix} \\ &:= PN^{\dagger}Q, \end{aligned} \tag{2.1}$$

*where* $P$, $N$ *and* $Q$ *satisfy the three properties*

$$\mathscr{R}(Q) \subseteq \mathscr{R}(N), \quad \mathscr{R}(P^*) \subseteq \mathscr{R}(N^*),$$
$$r(N) = r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}). \tag{2.2}$$

Proof. According to (1.1), the product $C^D B^D A^D$ can be written as

$$C^D B^D A^D = C^{k_3}(C^{2k_3+1})^{\dagger} C^{k_3} B^{k_2} (B^{2k_2+1})^{\dagger} B^{k_2} A^{k_1} (A^{2k_1+1})^{\dagger} A^{k_1}. \tag{2.3}$$

Note that

$$\mathscr{R}(A^{k_1}) = \mathscr{R}(A^{2k_1+1}), \qquad \mathscr{R}[(A^{k_1})^*] = \mathscr{R}[(A^{2k_1+1})^*], \tag{2.4}$$

$$\mathscr{R}(B^{k_2}) = \mathscr{R}(B^{2k_2+1}), \qquad \mathscr{R}[(B^{k_2})^*] = \mathscr{R}[(B^{2k_2+1})^*], \tag{2.5}$$

$$\mathscr{R}(C^{k_3}) = \mathscr{R}(C^{2k_3+1}), \qquad \mathscr{R}[(C^{k_3})^*] = \mathscr{R}[(C^{2k_3+1})^*]. \tag{2.6}$$

Hence by (1.6)

$$(C^{2k_3+1})^{\dagger} C^{k_3} B^{k_2} (B^{2k_2+1})^{\dagger} B^{k_2} A^{k_1} (A^{2k_1+1})^{\dagger}$$
$$= [I_m, \ 0, \ 0] \begin{bmatrix} 0 & 0 & A^{2k_1+1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 \end{bmatrix}^{\dagger} \begin{bmatrix} I_m \\ 0 \\ 0 \end{bmatrix}. \tag{2.7}$$

The combination of (2.3) and (2.7) yields (2.1). The three properties in (2.2) are straightforward from (2.4), (2.5) and (2.6). □

The main results of the paper are given in the following two theorems.

**Theorem 2.2.** *Let* $A, B, C \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k_1$, $\mathrm{Ind}(B) = k_2$ *and* $\mathrm{Ind}(C) = k_3$, *and denote* $M = ABC$ *with* $\mathrm{Ind}(M) = t$. *Then the reverse order law* $(ABC)^D = C^D B^D A^D$ *holds if and only if* $A, B$ *and* $C$ *satisfy the following three rank equalities*

$$r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 \\ M^{t+1}C^{k_3} & 0 & 0 & M^t \end{bmatrix}$$
$$= r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}), \tag{2.8}$$

$$r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1}M^{t+1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 \\ C^{k_3} & 0 & 0 & M^t \end{bmatrix}$$

$$= r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}), \tag{2.9}$$

$$r\begin{bmatrix} B^{2k_2+1} & B^{k_2}A^{k_1} \\ C^{k_3}B^{k_2} & 0 \end{bmatrix} = r(B^{k_2}) + r(M^t). \tag{2.10}$$

PROOF. Let $X = C^D B^D A^D$. Then we see by (1.7) that $X = M^D$ if and only if

$$M^{t+1}X = M^t, \quad XM^{t+1} = M^t, \quad r(X) = r(M^t),$$

which, in turn, are equivalent to

$$r(M^t - M^{t+1}X) = 0, \quad r(M^t - XM^{t+1}) = 0, \quad r(X) = r(M^t). \tag{2.11}$$

Replacing $X$ in (2.11) with $X = PN^{\dagger}Q$ in (2.1) and applying (1.2) to them, we find that

$$r(M^t - M^{t+1}X) = r(M^t - M^{t+1}PN^{\dagger}Q) = r\begin{bmatrix} N & Q \\ M^{t+1}P & M^t \end{bmatrix} - r(N),$$

$$r(M^t - XM^{t+1}) = r(M^t - PN^{\dagger}QM^{t+1}) = r\begin{bmatrix} N & QM^{t+1} \\ P & M^t \end{bmatrix} - r(N),$$

$$r(X) = r(PN^{\dagger}Q) = r\begin{bmatrix} N & Q \\ P & 0 \end{bmatrix} - r(N).$$

Hence (2.11) is equivalent to the following three rank equalities

$$r\begin{bmatrix} N & Q \\ M^{t+1}P & M^t \end{bmatrix} = r(N), \quad r\begin{bmatrix} N & QM^{t+1} \\ P & M^t \end{bmatrix} = r(N),$$

$$r\begin{bmatrix} N & Q \\ P & 0 \end{bmatrix} = r(N) + r(M^t).$$

Substituting $P$, $N$ and $Q$ in (2.1) into them and simplifying yield (2.8), (2.9) and (2.10). □

**Theorem 2.3.** *Let* $A, B, C \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k_1$, $\mathrm{Ind}(B) = k_2$ *and* $\mathrm{Ind}(C) = k_3$, *and denote* $M = ABC$ *with* $\mathrm{Ind}(M) = t$. *Then* $(ABC)^D = C^D B^D A^D$ *holds if and only if* $A$, $B$ *and* $C$ *satisfy the following rank equality*

$$r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1} & 0 \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 & 0 \\ C^{k_3} & 0 & 0 & 0 & M^t \\ 0 & 0 & 0 & M^t & -M^{2t+1} \end{bmatrix} = r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}) + r(M^t). \tag{2.12}$$

Proof. Applying (1.3) to $M^D - C^D B^D A^D = M^t(M^{2t+1})^{\dagger}M^t - PN^{\dagger}Q$ gives

$$r(M^D - C^D B^D A^D) = r[PN^{\dagger}Q - M^t(M^{2t+1})^{\dagger}M^t]$$

$$= r\begin{bmatrix} N & 0 & Q \\ 0 & -M^{2t+1} & M^t \\ P & M^t & 0 \end{bmatrix} - r(N) - r(M^t)$$

$$= r\begin{bmatrix} N & Q & 0 \\ P & 0 & M^t \\ 0 & M^t & -M^{2t+1} \end{bmatrix} - r(N) - r(M^t). \tag{2.13}$$

Substituting $P$, $N$ and $Q$ in (2.1) into the right side of (2.13) and letting it be zero gives (2.12). □

Two groups of necessary and sufficient conditions for $(ABC)^D = C^D B^D A^D$ to hold are given in Theorems 2.2 and 2.3. Although there are three rank equalities in (2.8)–(2.10), each of them is easier to simplify than (2.12) when $A$, $B$ and $C$ satisfy some conditions. Some consequences of the above two theorems are given below.

**Corollary 2.4.** *Let* $A, B, C \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}(B) = k$, $\mathrm{Ind}(ABC) = t$, *where* $A$ *and* $C$ *are nonsingular. Then* $(ABC)^D = C^{-1}B^D A^{-1}$ *if and only if*

$$\mathscr{R}[C(ABC)^t] = \mathscr{R}(B^k) \quad \text{and} \quad \mathscr{R}\{[(ABC)^t A]^*\} = \mathscr{R}[(B^k)^*]. \tag{2.14}$$

PROOF. Under the given conditions, (2.8), (2.9) and (2.10) become

$$r\begin{bmatrix} 0 & 0 & A & I_m \\ 0 & B^{2k+1} & B^k & 0 \\ C & B^k & 0 & 0 \\ M^{t+1} & 0 & 0 & M^t \end{bmatrix} = 2m + r(B^k), \tag{2.15}$$

$$r\begin{bmatrix} 0 & 0 & A & M^{t+1} \\ 0 & B^{2k+1} & B^k & 0 \\ C & B^k & 0 & 0 \\ I_m & 0 & 0 & M^t \end{bmatrix} = 2m + r(B^k), \tag{2.16}$$

$$r\begin{bmatrix} B^{2k+1} & B^k \\ B^k & 0 \end{bmatrix} = r(B^k) + r(M^t). \tag{2.17}$$

By block Gaussian elimination, the left sides of (2.15), (2.16) and (2.17) can further be simplified to

$$\begin{aligned} r\begin{bmatrix} 0 & 0 & A & I_m \\ 0 & B^{2k+1} & B^k & 0 \\ C & B^k & 0 & 0 \\ M^{t+1} & 0 & 0 & M^t \end{bmatrix} &= r\begin{bmatrix} B^{2k+1} & B^k \\ M^{t+1}C^{-1}B^k & M^tA \end{bmatrix} + 2m \\ &= r\begin{bmatrix} B^k \\ M^tA \end{bmatrix} + 2m, \end{aligned}$$

$$\begin{aligned} r\begin{bmatrix} 0 & 0 & A & M^{t+1} \\ 0 & B^{2k+1} & B^k & 0 \\ C & B^k & 0 & 0 \\ I_m & 0 & 0 & M^t \end{bmatrix} &= r\begin{bmatrix} B^{2k+1} & B^kA^{-1}M^{t+1} \\ B^k & CM^t \end{bmatrix} + 2m \\ &= r[B^k,\ CM^t] + 2m, \end{aligned}$$

$$r\begin{bmatrix} B^{2k+1} & B^k \\ B^k & 0 \end{bmatrix} = r\begin{bmatrix} 0 & B^k \\ B^k & 0 \end{bmatrix} = 2r(B^k).$$

Hence (2.15), (2.16) and (2.17) are simplified to

$$r\begin{bmatrix} B^k \\ M^tA \end{bmatrix} = r[B^k, \quad CM^t] = r(M^t) = r(B^k),$$

which is, in turn, equivalent to (2.14). □

A matrix $X$ is called an outer inverse of $A$ if $XAX = X$. Notice that both $(ABC)^D$ and $C^{-1}B^DA^{-1}$ are outer inverses of $ABC$. Hence Corollary 2.4 can also be proved by the following rank formula

$$r(X_1 - X_2) = r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + r[X_1,\ X_2] - r(X_1) - r(X_2)$$

for any two outer inverses $X_1$ and $X_2$ of a matrix. For more details, see the author's recent paper [19].

**Corollary 2.5.** *Let* $A, B, C \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k_1$, $\mathrm{Ind}(B) = k_2$ *and* $\mathrm{Ind}(C) = k_3$, *and denote* $M = ABC$ *with* $\mathrm{Ind}(M) = t$. *Moreover suppose that*

$$AB = BA, \qquad AC = CA, \qquad BC = CB. \tag{2.18}$$

*Then* $(ABC)^D = C^DB^DA^D$ *if and only if* $A$, $B$ *and* $C$ *satisfy* (2.10).

Proof. Under (2.18), it follows that $A^{k_1}B^{k_2} = B^{k_2}A^{k_1}$, $A^{k_1}C^{k_3} = C^{k_3}A^{k_1}$, $B^{k_2}C^{k_3} = C^{k_3}B^{k_2}$ and $M^t = A^tB^tC^t$. By block Gaussian elimination, the left side of (2.12) is reduced to

$$r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 \\ M^{t+1}C^{k_3} & 0 & 0 & M^t \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 \\ M^{m+1}C^{k_3} & 0 & 0 & M^m \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 \\ A^{m+1}B^{m+1}C^{m+k_3+1} & 0 & 0 & A^mB^mC^m \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & 0 & 0 & A^{k_1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 \\ C^{2k_3+1} & 0 & 0 & 0 \\ 0 & -A^{m+1}B^{m+k_2+1}C^m & -A^{m+k_1+1}B^mC^m & 0 \end{bmatrix}$$

$$= r \begin{bmatrix} 0 & 0 & 0 & A^{k_1} \\ 0 & B^{2k_2+1} & 0 & 0 \\ C^{2k_3+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}).$$

Thus (2.8) is an identity under (2.18). It can be shown by a similar approach that (2.9) is an identity under (2.18). Thus we see from Theorem 2.2 that $(ABC)^D = C^D B^D A^D$ if and only if $A$, $B$ and $C$ satisfy (2.10). □

**Corollary 2.6.** *Let $A, B \in \mathbb{C}^{m\times m}$ with $\mathrm{Ind}(A) = k$, $\mathrm{Ind}(B) = l$ and $\mathrm{Ind}(AB) = t$. Then the following three statements are equivalent:*

(a) $(AB)^D = B^D A^D$.

(b) $r \begin{bmatrix} 0 & A^{2k+1} & A^k & 0 \\ B^{2l+1} & B^l A^k & 0 & 0 \\ B^l & 0 & 0 & (AB)^t \\ 0 & 0 & (AB)^t & (AB)^{2t+1} \end{bmatrix} = r(A^k) + r(B^l) + r[(AB)^t].$

(c) *$A$ and $B$ satisfy the following three rank equalities*

$$r[(AB)^t] = r(B^l A^k),$$

$$r \begin{bmatrix} 0 & A^{2k+1} & A^k \\ B^{2l+1} & B^l A^k & 0 \\ (AB)^{t+1} B^l & 0 & -(AB)^t \end{bmatrix} = r(A^k) + r(B^l),$$

$$r \begin{bmatrix} 0 & A^{2k+1} & A^k (AB)^{t+1} \\ B^{2l+1} & B^l A^k & 0 \\ B^l & 0 & -(AB)^t \end{bmatrix} = r(A^k) + r(B^l).$$

Proof. Taking $C = I_m$ in (2.12) and simplifying gives (b). Taking $B = I_m$ and replacing $C$ with $B$ in (2.8), (2.9) and (2.10) gives (c). □

If one of $A$, $B$ and $C$ is nilpotent, i.e., $A^k = 0$ or $B^k = 0$ or $C^k = 0$ for some positive integer $k$, then $A^D = 0$ or $B^D = 0$ or $C^D = 0$. In this case, the law $(ABC)^D = C^D B^D A^D = 0$ is trivial for consideration. If one of $A$, $B$ and $C$ is normal, then its Drazin inverse and Moore–Penrose inverse are the same. These cases motivate us to consider some mixed-type reverse order laws for Drazin inverses and Moore–Penrose inverses, such as, $(ABC)^D = C^D B^\dagger A^D$, $(ABC)^D = C^\dagger B^D A^\dagger$, $(ABC)^D = C^\dagger B^\dagger A^\dagger$ and so on.

**Theorem 2.7.** *Let* $A, B, C \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k_1$, $\mathrm{Ind}(B) = k_2$ *and* $\mathrm{Ind}(C) = k_3$, *and denote* $M = ABC$ *with* $\mathrm{Ind}(M) = t$. *Then*

(a) $(ABC)^D = C^D B^\dagger A^D$ *if and only if*

$$r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1} & 0 \\ 0 & B^*BB^* & B^*A^{k_1} & 0 & 0 \\ C^{2k_3+1} & C^{k_3}B^* & 0 & 0 & 0 \\ C^{k_3} & 0 & 0 & 0 & M^t \\ 0 & 0 & 0 & M^t & -M^{2t+1} \end{bmatrix} = r(A^{k_1}) + r(B) + r(C^{k_3}) + r(M^t). \tag{2.19}$$

(b) $(ABC)^D = C^\dagger B^D A^\dagger$ *if and only if*

$$r\begin{bmatrix} 0 & 0 & A^*AA^* & A^* & 0 \\ 0 & B^{2k+1} & B^kA^* & 0 & 0 \\ C^*CC^* & C^*B^k & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 & M^t \\ 0 & 0 & 0 & M^t & -M^{2t+1} \end{bmatrix} = r(A) + r(B^k) + r(C) + r(M^t). \tag{2.20}$$

(c) $(ABC)^D = C^\dagger B^\dagger A^\dagger$ *if and only if*

$$r\begin{bmatrix} 0 & 0 & A^*AA^* & A^* & 0 \\ 0 & B^*BB^* & B^*A^* & 0 & 0 \\ C^*CC^* & C^*B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 & M^t \\ 0 & 0 & 0 & M^t & -M^{2t+1} \end{bmatrix} = r(A) + r(B) + r(C) + r(M^t). \tag{2.21}$$

Proof. It is well known that $B^\dagger$ can be expressed in the form $B^\dagger = B^*(B^*BB^*)^\dagger B^*$ (see [28]). Hence the product $C^D B^\dagger A^D$ can be rewritten as

$$C^D B^\dagger A^D = C^{k_3}(C^{2k_3+1})^\dagger C^{k_3} B^*(B^*BB^*)^\dagger B^* A^{k_1}(A^{2k_1+1})^\dagger A^{k_1}.$$

Applying Lemma 1.2 to it gives

$$\begin{aligned} C^D B^\dagger A^D &= [C^{k_3},\ 0,\ 0] \begin{bmatrix} 0 & 0 & A^{2k_1+1} \\ 0 & B^*BB^* & B^*A^{k_1} \\ C^{2k_3+1} & C^{k_3}B^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} A^{k_1} \\ 0 \\ 0 \end{bmatrix} \\ &:= PN^\dagger Q, \end{aligned}$$

where $P$, $N$ and $Q$ satisfy the three properties

$$\mathscr{R}(Q) \subseteq \mathscr{R}(N), \quad \mathscr{R}(P^*) \subseteq \mathscr{R}(N^*), \quad r(N) = r(A^{k_1}) + r(B) + r(C^{k_3}).$$

Find the rank of $M^D - C^D B^\dagger A^D = M^t(M^{2t+1})^\dagger M^t - PN^\dagger Q$ by (1.3) and let it be zero to give (2.19). Parts (b) and (c) can be shown similarly. □

**Corollary 2.8.** *Let $A, B, C \in \mathbb{C}^{m\times m}$ with $\operatorname{Ind}(AB) = k$ and $\operatorname{Ind}(BC) = l$, and denote $M = ABC$ with $\operatorname{Ind}(M) = t$. Then $(ABC)^D = (BC)^D B(AB)^D$ if and only if*

$$\begin{aligned} &r\begin{bmatrix} 0 & (AB)^{2k+1} & (AB)^k & 0 \\ (BC)^{2l+1} & (BC)^l B(AB)^k & 0 & 0 \\ (BC)^l & 0 & 0 & M^t \\ 0 & 0 & M^t & M^{2t+1} \end{bmatrix} \\ &= r[(AB)^k] + r[(BC)^l] + r(M^t). \end{aligned}$$

Proof. Writing $ABC = (AB)B^\dagger(BC)$ and applying Theorem 2.7(a) to it, we see that $(ABC)^D = (BC)^D(B^\dagger)^\dagger(AB)^D = (BC)^D B(AB)^D$ if and only if

$$\begin{aligned} &r\begin{bmatrix} 0 & 0 & (AB)^{2k+1} & (AB)^k & 0 \\ 0 & (B^\dagger)^*B^\dagger(B^\dagger)^* & (B^\dagger)^*(AB)^k & 0 & 0 \\ (BC)^{2l+1} & (BC)^l(B^\dagger)^* & 0 & 0 & 0 \\ (BC)^l & 0 & 0 & 0 & M^t \\ 0 & 0 & 0 & M^t & -M^{2t+1} \end{bmatrix} \\ &= r[(AB)^k] + r(B) + r[(BC)^l] + r(M^t). \end{aligned}$$

Note that $(B^{\dagger})^*[(B^{\dagger})^*B^{\dagger}(B^{\dagger})^*]^{\dagger}(B^{\dagger})^* = B$. Hence the left side of the above equality can be simplified by (1.2) to

$$r\begin{bmatrix} 0 & 0 & (AB)^{2k+1} & (AB)^k & 0 \\ 0 & (B^{\dagger})^*B^{\dagger}(B^{\dagger})^* & (B^{\dagger})^*(AB)^k & 0 & 0 \\ (BC)^{2l+1} & (BC)^l(B^{\dagger})^* & 0 & 0 & 0 \\ (BC)^l & 0 & 0 & 0 & M^t \\ 0 & 0 & 0 & M^t & -M^{2t+1} \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & 0 & (AB)^{2k+1} & (AB)^k & 0 \\ 0 & (B^{\dagger})^*B^{\dagger}(B^{\dagger})^* & 0 & 0 & 0 \\ (BC)^{2l+1} & 0 & -(BC)^lB(AB)^k & 0 & 0 \\ (BC)^l & 0 & 0 & 0 & M^t \\ 0 & 0 & 0 & M^t & -M^{2t+1} \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & (AB)^{2k+1} & (AB)^k & 0 \\ (BC)^{2l+1} & (BC)^lB(AB)^k & 0 & 0 \\ (BC)^l & 0 & 0 & M^t \\ 0 & 0 & M^t & M^{2t+1} \end{bmatrix} + r(B).$$

Thus the result in the corollary follows. □

*Remarks.* Many rank equalities for Drazin inverses of matrices can be established by (1.1), (1.2) and (1.3). From them, one can derive necessary and sufficient conditions for various equalities that involve Drazin inverses of matrices to hold, such as, $(AA^D)^* = AA^D$, $A^*A^D = A^DA^*$, $A^{\dagger}A^D = A^DA^{\dagger}$, $A^*A^DA = AA^DA^*$, $A^DA^*A = AA^*A^D$, $AA^D = BB^D$, etc. The results obtained illustrate many new properties for Drazin inverses of matrices. For more details, see the author's recent paper [20]. We now can summarize the work in the paper and [20] as a general topic: Given two matrix expressions $p(A_1^D, \dots, A_k^D)$ and $q(B_1^D, \dots, B_l^D)$ of the same size involving matrices and their Drazin inverses. Then determine necessary and sufficient conditions such that

$$p(A_1^D, \dots, A_k^D) = q(B_1^D, \dots, B_l^D).$$

Obviously, the equality is equivalent to

$$r[p(A_1^D, \ldots, A_k^D) - q(B_1^D, \ldots, B_l^D)] = 0.$$

If some rank formulas can be established for the matrix expression on the left side of the above equality, then necessary and sufficient conditions can be derived from the rank formulas for $p(A_1^D, \ldots, A_k^D) = q(B_1^D, \ldots, B_l^D)$ to hold. This method has been proved to be quite effective for characterizing various equalities for generalized inverses of matrices. Using this method, the author gave in a previous paper [17] necessary and sufficient conditions for $A_k^\dagger \cdots A_1^\dagger \in \{A^-\}$, $A^\dagger = A_k^\dagger \cdots A_1^\dagger$, $AA^\dagger = AA_k^\dagger \cdots A_1^\dagger$, etc. to hold, where $A = A_1 \cdots A_k$. In a recent paper [23], the author gives a necessary and sufficient condition for $\{A_k^- \cdots A_1^-\} \subseteq \{A^-\}$ to hold through determining the maximal rank of $A - AA_k^- \cdots A_1^- A$ with respect to $A_1^-, \ldots, A_k^-$. For square matrices $A_1, \ldots, A_k$ of the same order and their product $A = A_1 \cdots A_k$, one can also use rank method to characterize various equalities for their Drazin inverses, such as, $A^t A_k^D \cdots A_1^D A = A^t$, $A_k^D \cdots A_1^D A A_k^D \cdots A_1^D = A_k^D \cdots A_1^D$, $AA_k^D \cdots A_1^D = A_k^D \cdots A_1^D A$, $A^D = A_k^D \cdots A_1^D$, $A^D = A_k^\dagger \cdots A_1^\dagger$, etc. where the reverse order law $A^D = A_k^D \cdots$ $\cdots A_1^D$ is investigated in Wang [24] by rank method.

Acknowledgement. The author would like to thank the referee for many valuable suggestions.

## References

[1] E. Arghiriade, Remarques sur l'inverse généralisée d'un produit de matrices, *Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. Ser.* VIII **42** (1967), 621–625.

[2] D. T. Barwick and J. D. Gilbert, Generalization of the reverse order law with related results, *Linear Algebra Appl.* **8** (1974), 345–349.

[3] A. Ben–Israel and T. N. E. Greville, Generalized Inverses: Theory and Applications, *R. E. Krieger Publishing Company, New York*, 1980.

[4] R. Bouldin, The pseudo-inverse of a matrix product, *SIAM. J. Appl. Math.* **24** (1973), 489–495.

[5] R. E. Cline, Note on the generalized inverse of the product of matrices, *SIAM Rev.* **6** (1964), 57–58.

[6] R. E. Cline and T. N. E. Greville, An extension of the generalized inverse of a matrix, *SIAM J. Appl. Math.* **19** (1970), 682–688.

[7] S. L. Campbell and C. D. Meyer, Generalized Inverses of Linear Transformations, *Pitman*, *London*, 1979.

[8] I. Erdelyi, On the "reverse order law" related to the generalized inverse of matrix products, *J. Assoc. Comput. Mach.* **13** (1966), 439–433.

[9] A. M. Galperin and Z. Waksman, On pseudo-inverses of operator products, *Linear Algebra Appl.* **33** (1980), 123–131.

[10] M. C. Gouvreia and R. Puystiens, About the group inverse and Moore–Penrose inverse of a product, *Linear Algebra Appl.* **150** (1991), 361–369.

[11] T. N. E. Greville, Note on the generalized inverse of a matrix product, *SIAM Rev.* **8** (1966), 518–521.

[12] R. E. Hartwig, The reverse order law revisited, *Linear Algebra Appl.* **76** (1986), 241–246.

[13] G. Marsaglia and G. P. H. Styan, Rank conditions for generalized inverses of partitioned matrices, *Sankhyā Ser. A* **36** (1974), 437–442.

[14] W. Sun and Y. Wei, Inverse order rule for weighted generalized inverse, *SIAM J. Matrix Anal. Appl.* **19** (1998), 772–775.

[15] H. Tian, On the reverse order law $(AB)^D = B^D A^D$, *J. Math. Res. Expo.* **19** (1999), 355–359.

[16] Y. Tian, The Moore–Penrose inverse of a matrix product, *Math. Practice and Theory* **22**(1) (1992), 64–70 (in *Chinese*).

[17] Y. Tian, Reverse order laws for the generalized inverses of multiple matrix products, *Linear Algebra Appl.* **211** (1994), 185–200.

[18] Y. Tian, How to characterize equalities for the Moore–Penrose inverse of a matrix, *Kyungpook Math. J.* **41** (2001), 1–15.

[19] Y. Tian, Rank equalities related to outer inverses of matrices and applications, *Linear and Multilinear Algebra* **49** (2002), 269–288.

[20] Y. Tian, How to characterize commutativity equalities for Drazin inverses of matrices, *Archivum Mathematicum*, (in press).

[21] Y. Tian, Using rank formulas to characterize equalities for Moore–Penrose inverses of matrix products, *Applied Mathematics and Computation*, (in press).

[22] Y. Tian, Rank equalities for block matrices and their Moore–Penrose inverses, *Houston J. Math.*, (in press).

[23] Y. Tian, Generalized inverses of multiple matrix products, (submitted).

[24] G. Wang, The reverse order law for the Drazin inverses of multiple matrix products, *Linear Algebra Appl.* **348** (2002), 265–272.

[25] M. Wei, Reverse order laws for generalized inverses of multiple matrix products, *Linear Algebra Appl.* **293** (1999), 273–288.

[26] H. J. Werner, When is $B^- A^-$ a generalized inverse of $AB$, *Linear Algebra Appl.* **210** (1994), 255–263.

[27] E. A. Wibker, R. B. Howe and J. D. Gilbert, Explicit solution to the reverse order law $(AB)^{+} = B^{-}_{mr}A^{-}_{lr}$, *Linear Algebra Appl.* **25** (1979), 107–114.

[28] S. Zlobec, An explicit form of the Moore–Penrose inverse of an arbitrary complex matrix, *SIAM Rev.* **12** (1970), 132–134.

YONGGE TIAN
DEPARTMENT OF MATHEMATICS AND STATISTICS
QUEEN'S UNIVERSITY
KINGSTON, ONTARIO, K7L 3N6
CANADA

*E-mail:* ytian@mast.queensu.ca

*(Received November 15, 2000; revised March 3, 2003)*