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Integrable solutions of a functional equation related to Wilson's equation

By BOGDAN CHOCZEWSKI (Kraków) and ZBIGNIEW POWĄZKA (Kraków)

Dedicated to Professor Zenon Moszner on his seventieth birthday and to Professor Lajos Tamássy on his eightieth birthday

Abstract. We look for solutions (f, g) of equation (1) on $\mathbb{R}$ in the case where f is locally integrable and g is continuous at the origin. In particular, among solutions exponential functions show up. The study is motivated by E. Wachnicki's paper [7] dealing with an integral mean value theorem.

1. Introduction

We consider the functional equation

$$af(x) + bf(y) = f(ax + by)g(y - x), \qquad x, y \in \mathbb{R}, \tag{1}$$

where a, b are some positive reals, $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, are unknown functions.

On putting $a = b = \frac{1}{2}$ in (1) we arrive at

$$f(x) + f(y) = 2f\left(\frac{x+y}{2}\right)g(y - x), \qquad x, y \in \mathbb{R}, \tag{2}$$

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which has appeared in E. Wachnicki's paper [7] in connection with an integral mean value theorem. Equation (2) reduces to Wilson's one which is dealt with in J. Aczél's monograph [1, pp. 165–171], cf. also [6]. Results on other generalization of (2) are found in [8].

In this paper we shall determine (in Sections 5–7) the solutions (f, g) of equation (1) belonging to the class $\mathcal{F} \times \mathcal{G}$ of functions, where

$$\mathcal{F} := \{f : \mathbb{R} \to \mathbb{R},\ f \neq 0 \text{ and } f \text{ is integrable on } \mathbb{R}\},$$

and

$$\mathcal{G} := \{g : \mathbb{R} \to \mathbb{R},\ g \text{ is continuous at the origin}\}.$$

2. Preliminaries

Let us first observe that, putting $x = y$ in (1), we get

$$(a+b)f(x) = g(0)f((a+b)x), \qquad x \in \mathbb{R}, \tag{3}$$

whence $f(0)(a + b - g(0)) = 0$. In the case where $g(0) \neq 0$ equation (3) becomes the simple Schröder's functional equation (for $x \in \mathbb{R}$)

$$f(px) = qf(x), \tag{4}$$

where $p = a + b$, $q = (a + b)/g(0)$.

The following facts on solutions of (4) are either found in [2] or they can be easily derived from the theory of the Schröder equation presented in [3], cf. also [5] (in particular, Theorem 6.1, p. 137 in [3]).

Lemma 1. *Let I be an interval containing zero and assume that*

$$0 < p < 1. \tag{P}$$

(A) *If $|q| > 1$ then the only solution $f : I \to \mathbb{R}$ of (4) which is continuous at zero is the zero function, $f(x) = 0$ for $x \in I$.*

(B) *If*

$$q = p, \tag{5}$$

then every C^1-solution of (4) in I is given by

$$f(x) = \alpha x, \qquad x \in I, \tag{6}$$

where $\alpha \in \mathbb{R}$ is a constant.

In the case where $g(x) = 1$ for $x \in \mathbb{R}$, equation (1) takes the form

$$af(x) + bf(y) = f(ax + by), \qquad x, y \in \mathbb{R}, \tag{7}$$

which is a special case of the equation considered in M. Kuczma's monograph [4]. From the Theorem 13.10.2 found there on p. 341 we obtain the following.

Proposition. *If a non-constant (Lebesgue) measurable function $f : \mathbb{R} \to \mathbb{R}$ fulfils equation (7) then there exist real numbers $\alpha \neq 0$ and β such that*

$$f(x) = \alpha x + \beta, \qquad x \in \mathbb{R},$$

and $\beta = 0$ when $a + b \neq 1$.

3. Case $a + b \neq 1$, constant solution f

We start with listing the cases in which f satisfying (1) is necessarily the constant function. If the constant is zero, then (1) is satisfied by any function g. We assume that

$$\text{(I)} \qquad p := a + b \neq 1 \quad (a > 0,\ b > 0).$$

Theorem 1. *Assume* (**I**). *The solutions (f, g) of equation* (1), *defined on $\mathbb{R}$, are the following:*

(i) *If $g(0) = 0$, then*

$$f(x) = 0,\ x \in \mathbb{R}, \quad g : \mathbb{R} \to \mathbb{R} \text{ is an arbitary function.} \tag{8}$$

(ii) *If $g(0) = p$ and f is continuous at zero, then either $c := f(0) \neq 0$ and*

$$f(x) = c, \qquad g(x) = p, \qquad x \in \mathbb{R},$$

or $c = 0$ and (8) *holds.*

(iii) *If $0 < |g(0)| < p < 1$ or $|g(0)| > p > 1$, and f is continuous at zero, then* (8) *holds.*

Proof. According to the introductory remark in Section 2 we may concentrate on determining f satisfying (3).

(i) From (3) (which now reads $pf(x) = 0$ for $x \in \mathbb{R}$) we get $f = 0$.

(ii) Equation (3) (now $f(x) = f(px)$) yields

$$f(x) = f(p^n x), \quad n \in \mathbb{N},\ x \in \mathbb{R}. \tag{9}$$

Thus, if $0 < p < 1$ and f is continuous at zero, on letting $n \to \infty$, we get $f(x) = f(0) =: c$ for $x \in \mathbb{R}$. Hence (1) and (**I**) yield $cp = cg(y - x)$, $x, y \in \mathbb{R}$. We see that $g(t) = p$ for $t \in \mathbb{R}$ if $c \neq 0$, whereas g is arbitrary if $c = 0$. In the case where $p > 1$ we rewrite (9) in the form

$$f(p^{-n}x) = f(x), \quad n \in \mathbb{N},\ x \in \mathbb{R},$$

and argue as above to get the same conclusion. The assertions of (ii) are proved.

(iii) In view of (3), f satisfies in $\mathbb{R}$ equation (4) with $q := p/g(0)$. If $0 < |g(0)| < p < 1$, then $|q| > 1$ and condition (P) holds, so that from Lemma 1(A) we get $f(x) = 0$ for $x \in \mathbb{R}$. When $|g(0)| > p > 1$ we write equation (4) in the form

$$f\left(\frac{1}{p}x\right) = \frac{1}{q}f(x), \qquad x \in \mathbb{R}.$$

Because of $0 < \frac{1}{p} < 1$ and $\left|\frac{1}{q}\right| > 1$ Lemma 1(A) again works, yielding $f = 0$. □

Remark 1. In the cases: $p < |g(0)| < 1$ or $p > |g(0)| > 1$ the solution f of (4) which is continuous in a neighborhood of the origin depends on an arbitrary function (cf. [1], and also [5, Theorem 3.1.3, p. 99]) and there is no way of finding solutions of (1) among them. In these cases necessarily $f(0) = 0$ (observe that since $f(p^n x) = q^n f(x)$, $n \in \mathbb{N},\ x \in \mathbb{R}$, the conditions $p < 1$ and $|q| < 1$ lead to a contradiction when $f(0) \neq 0$).

4. Regularity of solutions from the class $\mathcal{F} \times \mathcal{G}$

We shall prove the following.

Lemma 2. *If $a > 0$, $b > 0$ and $(f, g) \in \mathcal{F} \times \mathcal{G}$ is a solution to* (1)*, then*

(a) $f \in C^\infty(\mathbb{R})$,

(b) $g \in C^\infty(J)$, *where J is an open interval containing zero, and* $g'(0) = 0$.

Proof. We put again $p := a + b$ (> 0).

(a) For an arbitrarily fixed $x_0 > 0$ let $\delta > 0$ be such that f is integrable in the interval $V = V(x_0) := [x_0 - p\delta, x_0 + p\delta]$.

Let $t \in (0, \delta)$ and $x \in V$. Replacing x in (1) by $x - bt$ and y by $x + at$ we get

$$af(x - bt) + bf(x + at) = f(px)g(pt), \qquad x \in V,\ t \in (0, \delta).$$

Since $f \in \mathcal{F}$ is not the zero function, we have $g(0) \neq 0$ (cf. Theorem 1 (i)) and f satisfies (4) (with $q = p/g(0)$), whence

$$af(x - bt) + bf(x + at) = qf(x)g(pt), \quad x \in V,\ t \in (0, \delta). \tag{10}$$

In particular, the function $(0, \delta) \ni t \mapsto g(pt)$ is integrable. We integrate (10) with respect to t over the interval $[0, \delta]$ to get

$$a\int_0^\delta f(x - bt)dt + b\int_0^\delta f(x + at)dt = qf(x)\int_0^\delta g(pt)dt. \tag{11}$$

After the substitutions $s = x - bt$ and $s = x + at$ in the respective integrals equation (11) turns over

$$k(x) = cqf(x), \qquad x \in V, \tag{12}$$

where

$$k(x) := \frac{a}{b}\int_{x-b\delta}^{x} f(s)ds + \frac{b}{a}\int_x^{x+a\delta} f(s)ds, \qquad x \in V, \tag{13}$$

and

$$c := \int_0^\delta g(pt)dt.$$

Since f is integrable, (13) says that k is continuous in V. In turn, (12) implies that f is continuous in V, yielding, again by (13), the differentiability of k on V, etc. Therefore the function f is of class C^∞ on V, i.e., as x_0 was arbitrary in $\mathbb{R}$, $f \in C^\infty(\mathbb{R})$.

(b) By letting

$$u = ax + by, v = y - x, \qquad x, y \in \mathbb{R},$$

we see that (1) is equivalent to:

$$af\left(\frac{u - bv}{p}\right) + bf\left(\frac{u + av}{p}\right) = f(u)g(v), \qquad u, v \in \mathbb{R}. \tag{14}$$

Since f is not identically zero and it is of class C^∞ on $\mathbb{R}$, there is an open interval containing zero, say J, on which g is of class C^∞.

Taking derivatives in (14) with respect to v we obtain

$$\frac{-ab}{p} f'\left(\frac{u - bv}{p}\right) + \frac{ab}{p} f'\left(\frac{u + av}{p}\right) = f(u)g'(v), \qquad u \in \mathbb{R},\ v \in J.$$

Letting $v = 0$ here we get the equality $f(u)\ g'(0) = 0$, whence, as $f \neq 0$, we have $g'(0) = 0$. □

5. Case $a + b \neq 1$

In this case if $f \in \mathcal{F}$ has a non-zero derivative at zero then it is a linear function, and $g \in \mathcal{G}$ is a constant function.

Theorem 2. *Assume* (**I**). *If* $(f, g) \in \mathcal{F} \times \mathcal{G}$ *is a solution to* (1), f *is differentiable at zero, and*

$$f(0) = 0, \quad f'(0) \neq 0, \tag{15}$$

then

$$f(x) = \alpha x; \quad g(x) = 1, \qquad x \in \mathbb{R}, \tag{16}$$

where $\alpha \neq 0$ *is a real number.*

Proof. Assume (15). The function f satisfies equation (3), i.e.,

$$pf(x) = g(0)f(px), \quad x \in \mathbb{R}, \tag{17}$$

whence

$$p\frac{f(x)}{x} = g(0)p\frac{f(px)}{px}, \quad x \in \mathbb{R} \setminus \{0\}.$$

Since $f(0) = 0$ and f is differentiable at zero, we get $pf'(0)(g(0) - 1) = 0$ and by (**I**) and (15) we have

$$g(0) = 1. \tag{18}$$

Because of (18) the Schröder equation (17) for f becomes

$$f(px) = pf(x), \qquad x \in \mathbb{R}, \tag{19}$$

and, by Lemma 2 (a), f is of class $C^\infty(\mathbb{R})$. If $p < 1$ then conditions (P) and (5) are satisfied, Lemma 1 (B) works, and we get formula (6) (i.e., (16) for $f(x)$), whereas if $p > 1$ it is enough to write (19) in the form

$$f(p^{-1}x) = p^{-1}\, f(x), \qquad x \in \mathbb{R},$$

to have Lemma 1 (B) applicable again. Using (6) in (1) we get $g(x) = 1$ and formula (16) holds true. □

6. Case $a + b = 1$; a differential equation

We pass to the remaining case

(**II**) $$p := a + b = 1 \quad \text{and} \quad 0 < a < b.$$

(Solutions of equation (1) when $a = b = 1/2$ are described in the paper [6].)

We have the following

Lemma 3. *Assume* (**II**). *If* $(f, g) \in \mathcal{F} \times \mathcal{G}$ *is a solution to* (1), *then*

(a) *the function* g *fulfils condition* (18),

(b) *the function* f *satisfies the differential equation*

$$abf''(x) = f(x)g''(0), \qquad x \in \mathbb{R}. \tag{20}$$

Proof. (a) Because of (**II**) the function f satisfies (17) with $p = 1$, i.e., $f(x) = g(0)f(x)$ for $x \in \mathbb{R}$. Since f is not identically zero, we get $g(0) = 1$ that is condition (18).

(b) Proceeding as in the proof of *Lemma 2 (b)* we obtain equation (14) with $p = 1$:

$$af(u - bv) + bf(u + av) = f(u)g(v), \qquad u, v \in \mathbb{R}. \tag{21}$$

According to Lemma 2 we may differentiate both sides of (21) with respect to v (in J) twice, to arrive at

$$ab^2 f''(u - bv) + a^2 b f''(u + av) = f(u)g''(v) \quad u \in \mathbb{R},\ v \in J.$$

With $v = 0$ ($\in J$) and $u = x$ here, thanks to (**II**) we get (20). □

7. Case $a + b = 1$; main result

In the theorem that follows exponential functions show up.

Theorem 3. *Assume* (**II**) *and let* $(f, g) \in \mathcal{F} \times \mathcal{G}$ *be a solution to* (1) *satisfying*

$$g''(0) > 0. \tag{$\star$}$$

If $f(0) \neq 0$ *then there is a non-zero* c *such that either*

$$f(x) = ce^{kx};\ g(x) = be^{akx} + ae^{-bkx}, \quad x \in \mathbb{R}, \tag{22}$$

or

$$f(x) = ce^{-kx};\ g(x) = ae^{bkx} + be^{-akx}, \quad x \in \mathbb{R}, \tag{23}$$

with

$$k := (g''(0)/ab)^{1/2}. \tag{24}$$

If $f(0) = 0$, *then* f *and* g *are given by* (8).

Proof. 1° If $(f, g) \in \mathcal{F} \times \mathcal{G}$ satisfy equation (1), then f fulfils equation (20) which, according to ($\star$) and (24), becomes $f''(x) = k^2 f(x)$ and

$$f(x) = Ae^{kx} + Be^{-kx}, \qquad x \in \mathbb{R}, \tag{25}$$

with some constants A and B.

2° Assume that $f(x) \neq 0$ in $\mathbb{R}$. Putting $x = 0$ in (1) we calculate (for $y \in \mathbb{R}$)

$$g(y) = \frac{af(0) + bf(y)}{f(by)} \tag{26}$$

and eliminate g from (1):

$$[af(x) + bf(y)]f[b(y - x)] = f(ax + by)[af(0) + bf(y - x)], \quad x, y \in \mathbb{R}.$$

We may substitute here $y = 0$. The resulting equation takes the form:

$$[af(x) + bd]f(-bx) = f(ax)[bf(-x) + ad], \quad x \in \mathbb{R} \quad (d := f(0)). \tag{27}$$

Using (25) in (27) we get the identity (with $t := kx$, for short):

$$\sum_{j=1}^{8} \alpha_j \exp(m_j t) \equiv 0, \tag{28}$$

where

$$m_1 = a, \quad m_2 = -a, \quad m_3 = b, \quad m_4 = -b,$$

$$m_5 = 1 + a, \quad m_6 = -(1 + a), \quad m_7 = 1 + b, \quad m_8 = -(1 + b)$$

and

$$\alpha_1 = aA(A - d), \qquad \alpha_2 = aB(B - d),$$

$$\alpha_3 = bB(d - B), \qquad \alpha_4 = bA(d - A),$$

$$\alpha_5 = \alpha_6 = bAB, \qquad \alpha_7 = \alpha_8 = aAB.$$

Since the exponents in (28) are mutually different, the corresponding exponential functions are linearly independent. The equalities $\alpha_5 = \cdots = \alpha_8 = 0$ yield $A = 0$ or $B = 0$. When $A = 0$, from the equalities $\alpha_2 = \alpha_3 = 0$ we get $B = d$, whereas when $B = 0$ there is $A = d$, because of $\alpha_1 = \alpha_4 = 0$. We have found formula (22), resp. (23), for f, with $C = d$.

We proceed with determining $g(x)$ from (26). At first we put $f(x) = de^{kx}$ there. We obtain by (**II**) $g(x) = be^{akx} + ae^{-bkx}$, in accordance with (22). Formula (23) for $g(x)$, corresponding to $f(x) = de^{-kx}$ is obtained in the same way.

A straightforward calculation shows that the functions f (which does not vanish on $\mathbb{R}$) and g given by (22) or (23) (with (24)) actually satisfy (1) and that they are from $\mathcal{F} \times \mathcal{G}$.

3° In turn, let $f(0) \neq 0$ and $f(r) = 0$ for an $r > 0$. We make use of (21) putting $u = r$, $v = t$ there:

$$af(r - bt) + bf(r + at) = 0, \quad -\frac{r}{a} \leq t \leq \frac{r}{b}.$$

On replacing t by $-t$ we obtain

$$af(r + bt) + bf(r - at) = 0, \quad -\frac{r}{b} \leq t \leq \frac{r}{a}.$$

The equalities when first added then subtracted side by side yield

$$a\varphi(t) + b\varphi\left(\frac{a}{b}t\right) = 0, \quad a\psi(t) - b\psi\left(\frac{a}{b}t\right) = 0, \quad t \in I_r := \left[-\frac{r}{b}, \frac{r}{b}\right], \quad (29)$$

where we have put

$$\begin{aligned} &\varphi(t) := f(r + bt) + f(r - bt), \\ &\psi(t) := f(r + bt) - f(r - bt), \quad t \in I_r. \end{aligned} \quad (30)$$

Equations (29) are the Schröder equations (4). We rewrite the first equation in (29) as

$$\varphi\left(\frac{a}{b}t\right) = -\frac{a}{b}\varphi(t), \qquad t \in I_r,$$

and replace t by $ab^{-1}t$. Hence we get

$$\varphi\left(\frac{a^2}{b^2}t\right) = -\frac{a}{b}\varphi\left(\frac{a}{b}t\right) = \frac{a^2}{b^2}\varphi(t), \qquad t \in I_r,$$

i.e., equation (4) with $0 < p = q = a^2b^{-2} < 1$ (cf. (**II**)). In turn, cf. (29),

$$\psi\left(\frac{a}{b}t\right) = \frac{a}{b}\psi(t), \qquad t \in I_r,$$

i.e., equation (4) with $0 < p = q = ab^{-1} < 1$ (cf. (**II**)). By Lemma 2 the function f is of class $C^1(\mathbb{R})$, whence so are φ and ψ given by (30), in the interval I. Lemma 1(B) then implies that there are real constants α_1 and α_2 such that $\varphi(t) = \alpha_1 t$, $\psi(t) = \alpha_2 t$, $t \in I_r$. From relations (30) we see that $2f(r+bt)=\varphi(t)+\psi(t)=(\alpha_1+\alpha_2)t$ for $t\in I_r$. This means that, whenever $x\in[0,2r]$, we have

$$f(x)=\gamma(x-r), \tag{31}$$

where $\gamma:=\frac{1}{2}(\alpha_1+\alpha_2)$. Coming back to equation (1) we take $x,y\in[0,2r]$ there. Since the convex combination $ax+by$ of x and y (cf. (**II**)) is also in $[0,2r]$, we have by (31):

$$a\beta\cdot(x-r)+b\beta\cdot(y-r)=[\beta\cdot(ax+by)-r]g(y-x).$$

As $a+b=1$, this yields $g(y-x)=1$ for $x,y\in[0,2r]$, that is $g(t)=1$ for $|t|\le 2r$. Therefore $g''(0)=0$, which contradicts assumption $(\star)$.

If $f(r)=0$ for an $r<0$, the proof runs the same way.

4° Finally, in the case where $f(0)=0$ we get from (25) the formula

$$f(x)=C\sinh kx, \qquad x\in\mathbb{R}, \tag{32}$$

where $C:=2A=-2B$. Assume that $C\neq 0$. Thus $f(x)\neq 0$ for $x\neq 0$ and formula (26) works for $y\neq 0$, yielding

$$g(y)=\frac{bf(y)}{f(by)}=\frac{b\ \sinh(ky)}{\sinh(kby)}, \qquad y\neq 0. \tag{33}$$

On letting $y=0$ in (1) and taking into account that $f(0)=0$ we obtain

$$af(x)=f(ax)g(-x).$$

Substituting here (32) and (33) we have $a\sinh(kbx)=b\sinh(kax)$ for $x\neq 0$ which is not an identity when (**II**) is assumed. Therefore $C=0$ in (32), whence f is the zero function and g is arbitrary, i.e., formula (8) describes all the solutions of (1). □

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BOGDAN CHOCZEWSKI
FACULTY OF APPLIED MATHEMATICS
UNIVERSITY OF MINING AND METALLURGY
AL. MICKIEWICZA 30, 30-059 KRAKÓW
POLAND

E-mail: smchocze@cyf-kr.edu.pl

ZBIGNIEW POWĄZKA
INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
PODCHORĄŻYCH 2, 30-084 KRAKÓW
POLAND

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