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Harmonicity of maps between (indefinite) metric-f-manifolds and φ-pseudo harmonic morphisms

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Abstract. Conditions are investigated for maps to be harmonic between $M(P)$-f-manifolds with (semi-)Riemannian metrics. Also a geometrical condition, φ_r-pseudo horizontally weak conformality [which is weaker than horizontally weak conformality when they are comparable], is imposed on maps of a (semi-)Riemannian manifold into a metric $M(P)$-f-manifold with (semi-) Riemannian metric and harmonicity of such maps are discussed. Some results obtained by Loubeau and Lichnerowicz in the almost Hermitian case are given here in both almost Hermitian and almost para-Hermitian cases.

Introduction

Harmonic maps of semi-Riemannian manifolds (M^m, g) of signature $(\rho, m-\rho)$, ρ indicates the negative indices, sharply contrasts from the ones of Riemannian manifolds. The ones of semi-Riemannian manifolds are solutions of, so called, *(non-linear) ultrahyperbolic system* when $1 < \rho < m-1$ and *hyperbolic system* when $\rho = 1$ or $\rho = m-1$. On the other hand, harmonic maps of Riemannian manifolds are solutions of *(non-linear) elliptic system.* The later class of harmonic maps have been receiving much attention from geometers, but harmonic maps of semi-Riemannian manifolds have not got much, though one may cite e.g. [3], [8], [9], [11], [12], [19].

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However in other disciplines almost 150 articles so far have been produced on the topic (including the existence and uniqueness questions) some of which have applications in solving various type of problems. Here are few reasonings to call geometers' attentions to the topic:

1°) In general, harmonic maps of semi-Riemannian manifolds have significant roles to play in the areas of quantum field theory, solitons and scattering theory, general relativity and gauge field theory. For example harmonic maps ϕ of semi-Riemannian manifolds into $SO(3,2)/S(2,2)$ is connected to the $SL(2,\mathbb{R})$-gauge field theories. It is possible to understand these relations through some forms of Lagrangian density of ϕ [13].

Also harmonic maps of Lorentzian manifolds is closely connected to, so called *p-brane* (*membrane* when $m = 3$) [6].

2°) The pole density functions in crystallography are governed by the tension field of a function of the semi-Riemannian manifold $(\mathbb{R}^m, g)$. Exploiting this tension field may provide the means for:

i) checking the compatibility of experimental pole density functions.

ii) normalizing or completing incompletely measured pole density functions and

iii) calculating additional pole density functions directly without previous determination of a reasonable solution of the inverse problem.

Such means have been searched for long [17].

In this work we mainly investigated how to guarantee the harmonicity of maps and their further properties, namely being φ-pseudo harmonic morphism, between (semi)-Riemannian manifolds by putting extra structures on the manifolds and imposing some conditions on the maps. Our result recovers most of the known ones in the Riemannian cases and also improve significantly some of them. To be precise, Theorem 2.3 gives a result in the most general form so far: It recovers, as a special case, a well-known result of Lichnerowicz [14] that *every holomorphic map of a cosymplectic manifold into a quasi-Kaehlerian one is harmonic* and also it recovers various generalizations of this, e.g. see Bejan–Benyounes's [3], author's [9], Parmar's [19], Rawnsley's [20] results.

It should be noted here that in this work the structure φ_r can be taken to be

i) an almost complex structure J_r, in that case $r = 1$, so that (N, h, J_1) becomes an almost Hermitian or indefinite almost Hermitian manifold.

ii) an f-structure φ_1 so that (N, h, φ_1) becomes a metric f-manifold with a Riemannian or semi-Riemannian metric h.

iii) an almost paracomplex structure J_r, in that case $r = -1$, so that (N^n, h, J_{-1}) becomes an almost para-Hermitian manifold (with necessarily a semi-Riemannian metric h of signature $(n/2, n/2)$ i.e. a neutral metric).

iv) a para f-structure φ_{-1} so that (N, h, φ_{-1}) becomes a metric para f-manifold with necessarily a semi-Riemannian metric of any signature.

The case where (N, h, J_1) is an almost (indefinite) Hermitian, is treated in [15]. In the other cases, some more harmonic maps and morphisms can be produced.

We also impose a condition $[d\phi \circ (d\phi)^*, \varphi_r] = 0$ on maps $\phi : (M, g) \to (N, h, \varphi_r)$ from a semi-Riemannian manifold into a metric (para-)f-manifold (whose metric h is allowed to be semi-Riemannian as well as Riemannian) in an attempt to produce harmonic maps (see Theorem 3.2). This particular line of investigation was first taken by Loubeau, [15], for maps of Riemannian manifolds into Kaehlerian ones, even though the above condition was first considered in [2] in a study of stable harmonic maps. Following the terminology in [15], we call such maps satisfying the above condition *φ_r-pseudo horizontally weakly conformal* (φ_r-PHWC). Also, harmonic (φ_r-PHWC) map ϕ will be called *φ_r-pseudo harmonic morphism*, (φ_r-PHM). It turns out that a (φ_r-PHM) pulls certain harmonic maps, namely φ_r-pluriharmonic ones, back to harmonic maps, (see Section 4).

When the metrics g and h are Riemannian and φ_r is a complex structure then Theorem 3.2 gives Baird–Eells's result [1] (see Proposition 3.7 below). In passing that we also show how the Wood's method, used in generalizing a special case (namely, when $\dim N = 2$) of Baird–Eells's result for larger classes of maps, works when the metrics g and h are allowed to be semi-Riemannian.

1. Definitions, notations and some basic results

Let $\mathbb{K}$ denote either the complex number $\mathbb{C}$ or the paracomplex numbers $\mathbb{A} = \{x + \epsilon y : x, y \in \mathbb{R};\ \epsilon^2 = 1\}$. Let k denote $i = \sqrt{-1}$ when $\mathbb{K} = \mathbb{C}$ and ϵ when $\mathbb{K} = \mathbb{A}$, so that we may write $\mathbb{K} = \{x + ky : x, y \in \mathbb{R}\}$, (see [5] and the references therein for the paracomplex numbers and their introduction in Differential Geometry).

Let $(N^{2n+\ell}, h)$ be a (semi)-Riemannian manifold of dimension $2n + \ell$. For either $r = 1$ or $r = -1$, let φ_r denote a $(1,1)$-tensor field on N of rank $2n$ satisfying:

1°) $\varphi_r^3 + r\varphi_r = 0$

2°) $h(X, Y) = 0\ \forall X \in \mathcal{D}_r,\ \forall Y \in \mathcal{V}_r$

3°) $h(\varphi_r X, \varphi_r Y) = rh(X, Y);\ \forall X, Y \in \mathcal{D}_r$

4°) h is of constant signature (ξ, ζ) on $\mathcal{D}_r(p)$ for all $p \in N$.

(Here the first entry in the signature represents the negative indices.) Where $\mathcal{D}_r(p) = \varphi_r(T_pN)$ and $\mathcal{V}_r = \operatorname{Ker}\varphi$ which we call them *φ_r-horizontal* and *φ_r-vertical* distributions over N respectively.

For $r = 1$ [resp. $r = -1$] the (N, h, φ_r) will be called *(indefinite) metric-f-manifold* and abbreviated as M-f-manifold [resp. *metric para-f-manifold* and abbreviated as MP-f-manifold]. In the case of MP-f-manifold $(N^{2n+\ell}, h, \varphi_{-1})$, the metric h is necessarily of signature (n, n) on $\mathcal{D}$ in which case, h is said to induce a neutral metric on $\mathcal{D}$. We write *$M(P)$-f-manifold* to mean both M-f-manifold and MP-f-manifold. We shall drop the subindices r when there is no confusion arise.

The (para)-f-structure φ induces a decomposition:

$$TN \otimes \mathbb{K} = T^K N = T^\circ N \oplus T^+ N \oplus T^- N = \mathcal{V} \oplus D^+ \oplus D^-$$

into eingenbundles $T^\circ N = \mathcal{V}$, $T^+ N = \mathcal{D}^+$ and $T^- N = \mathcal{D}^-$ corresponding to the eigenvalues 0, k, $-k$ respectively. Note that

$$\mathcal{D}^+ = \{X - rk\varphi(X) : X \in \mathcal{D}\} \quad \text{and} \quad D^- = \overline{D^+}.$$

Let $\phi : (M, g) \to (N, h)$ be a map between (semi)-Riemannian manifolds throughout unless otherwise stated.

Definition 1.1. ϕ is said to be nondegenerate on a subset $\mathcal{U}$ of M if

i) $K_\phi(p) = \{X \in T_pM : d\phi_p(X) = 0\}$, the kernel of ϕ at p, is a nondegenerate subspace of T_pM for every $p \in \mathcal{U}$

ii) $d\phi(T_pM)$ is nondegenerate subspace of $T_{\phi(p)}N$ for all $p \in \mathcal{U}$.

For a nondegenerate ϕ on $\mathcal{U}$ define the adjoint map $d\phi^* : T_yN \to T_xM$ of $d\phi$ by:

$g(d\phi^*(v), w) = h(v, d\phi(w))$ for $v \in TN$, $w \in TM$ where $y = \phi(x)$. Set

$H_\phi = K_\phi^\perp$, the g-orthogonal complement of K_ϕ

$H_\phi^* = H_{\phi^*} = K_{\phi^*}^\perp$, the h-orthogonal complement of K_{ϕ^*}.

Note that the (semi-)Riemannian (also called *indefinite*) metrics g and h restrict to nondegenerate metrics on $H = H_\phi$ and $H^* = H_{\phi^*}$ respectively. Also $d\phi : H \to H^*$ and $d\phi^* : H^* \to H$ are one to one, onto.

Definition 1.2. The map $\phi : (M, g) \to (N, h, \varphi_r)$ into $M(P)$-f-manifold is said to be

1°) ([12]) horizontally weakly conformal (HWC) if

a°) for any $p \in M$ at which $K_\phi(p)$, with $K_\phi(p) \neq TM$, is nondegenerate we have that $d\phi_p$ is surjective and satisfies

$$h(d\phi_p(X), d\phi_p(Y)) = \lambda(p)g(X, Y);\ \ \forall X,\ Y \in H_p,\ \text{where } \lambda(p) \neq 0.$$

b°) for any $p \in M$ at which $K_\phi(p)$ is degenerate we have that $H_p \subseteq K_\phi(p)$.

2°) (c.f. [15]) φ_r-pseudo horizontally weakly conformal, abbreviated as φ_r-(PHWC), on an open set $\mathcal{U}$ of M if ϕ is nondegenerate on $\mathcal{U}$ and satisfies $[d\phi \circ d\phi^*, \varphi_r] = 0$.

It is not difficult to see that *every nondegenerate (HWC) map* $\phi : (M, g) \to (N, h, \varphi_r)$ *on* $\mathcal{U}$ *is also* φ_r*-(PHWC) on* $\mathcal{U}$*, for* $r = \pm 1$.

Remark 1.3. i) For a nondegenerate φ_r-(PHWC) map $\phi : (M, g) \to (N, h, \varphi_r)$ with $d\phi(TM) \subseteq \mathcal{D}^N$, we have that $d\phi(TM) = H^*$ is invariant under φ_r and therefore rank$(d\phi_p)$ is always even, since $H^* \subseteq \mathcal{D}^N$.

ii) The terminology φ_r-pseudo horizontal weak conformality is originated from LOUBEAU's work [15], in which he only deals with J-(PHWC) maps and calls them simply *pseudo horizontally weakly conformal maps*, where the metrics involved are Riemannian and (N, h, J) is Hermitian.

For a $M(P)$-f-manifold (N, h, φ), the fundamental 2-form of N is given by

$$\Omega(X, Y) = h(X, \varphi Y).$$

Definition 1.4. A $M(P)$-f-manifold $(N^{2n+\ell}, h, \varphi)$ is said to be

i) (1,2)-symplectic [resp. $(1,2)$-$\mathcal{D}$-symplectic] if the exterior derivative of Ω satisfies

$$d\Omega(X, \varphi X, Y) = 0, \ \ \forall X \in \mathcal{D} \text{ and } \forall Y \in TN \text{ [resp. } \forall X, Y \in \mathcal{D}]$$

ii) $(1,2)$-symplecticlike [resp. $(1,2)$-$\mathcal{D}$-symplecticlike] if

$$\sum_{t=1}^{n} h^{tt} d\Omega(e_t, \varphi e_t, Y) = 0$$

for an h-orthonormal frame field $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n\}$ for $\mathcal{D}$ and $\forall Y \in TN$, [resp. $\forall Y \in \mathcal{D}$], where $(h^{ts}) = (h(e_t, e_s))^{-1}$

iii) (c.f. [20]) satisfying the condition (A) if

$$\nabla_{\overline{U}} W \in \mathcal{D}^+; \quad \forall U, W \in \Gamma\mathcal{D}^+$$

where ∇ is the Levi–Civita connection on N.

Remark 1.5. Note that

i) Every (1,2)-symplectic $M(P)$-f-manifold is $(1,2)$-symplecticlike (and therefore $(1,2)$-$\mathcal{D}$-symplecticlike) and also it is $(1,2)$-$\mathcal{D}$-symplectic.

ii) Every $M(P)$-f-manifold (N, h, φ) satisfying the condition (A) is $(1,2)$-symplectic. The converse is not true in general. For example when $\operatorname{rank}(\varphi) < \dim N$, say $\operatorname{rank}(\varphi) = (\dim N) - 1$, consider a (para)contact (hyperbolic) metric manifold (N^{2n+1}, h, φ). This is $(1,2)$-symplectic since $d\Omega = 0$ and yet it can not satisfy the condition (A), (for detail see [9] and the references therein). Nevertheless the converse is true when $\operatorname{rank}(\varphi) = \dim N$, that is, *every $(1,2)$-symplectic almost para Hermitian or (indefinite) almost Hermitian manifold (N^{2n}, h, φ) satisfies the condition (A).* Also for an (indefinite) almost [para] Hermitian manifold (N^{2n}, h, J), the two notions $(1,2)$-symplecticlike and $(1,2)$-$\mathcal{D}$-symplecticlike [resp. $(1,2)$-symplectic and $(1,2)$-$\mathcal{D}$-symplectic] do coincide and in that case the manifold is also called *(indefinite) [para] cosymplectic* or *(indefinite) semi-[para-]Kaehler* [resp. *(indefinite) quasi-[para-]Kaehler*].

iii) Consider the pseudo-sphere

$$\mathbb{S}^6 = S^6_3 = \{X = (x_t) \in \mathbb{R}^7 : G(X,X) = 1\},$$

$$\text{where} \quad G(X,Y) = \sum_{t=1}^{4} x_t y_t - \sum_{t=5}^{7} x_t y_t.$$

The pseudo sphere $\mathbb{S}^6$, can be given an almost para-Hermitian structure $(G_1, \mathcal{P})$, by using Cayley split octaves. Now consider $\mathbb{S}^5$ as a totally geodesic submanifold of $\mathbb{S}^6$, given by $x_1 = 0$. The almost para-Hermitian structure on $\mathbb{S}^6$ induces an almost paracontact hyperbolic metric structure $(G', \mathcal{P}')$ on $\mathbb{S}^5$, with its second fundamental 2-form Ω, satisfying $d\Omega \neq 0$. This almost paracontact hyperbolic metric manifold $(\mathbb{S}^5, G', \mathcal{P}')$ (which is also metric para f-manifold) is in fact $(1,2)$-symplectic manifold and therefore it is $(1,2)$-symplecticlike (see [9], Lemma 4.4).

Set $S(X, \varphi X) = (\nabla_X \varphi)(\varphi X) - (\nabla_{(\varphi X)} \varphi) X$

Lemma 1.6. *For a $M(P)$-f-manifold $(N^{2n+\ell}, h, \varphi)$ and a vector field $X \in \Gamma(\mathcal{D})$ we have*

$$\nabla_X X + r \nabla_{\varphi X} (\varphi X) = -r\big\{S(X, \varphi X) + \varphi[X, \varphi X]\big\} \tag{1.1}$$

and

$$h(S(X, \varphi X), Z) = -d\Omega(X, \varphi X, Z), \quad \forall Z \in \Gamma(TN). \tag{1.2}$$

Consequently

i) *$S(X, \varphi X) = 0$, [resp. $S(X, \varphi X) \in \Gamma(\mathcal{V})$] if and only if N is $(1,2)$-symplectic [resp. $(1,2)$-$\mathcal{D}$-symplectic], $\forall X \in \Gamma(\mathcal{D})$.*

ii) *For a local h-orthonormal frame field $\{e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n\}$ for $\mathcal{D}$,*

$$\sum_{t=1}^{n} g^{tt} S(e_t, \varphi e_t) = 0, \quad \Big[\text{resp. } \sum_{t=1}^{n} g^{tt} S(e_t, \varphi e_t) \in \Gamma(\mathcal{V}^N)\Big]$$

if and only if N is $(1,2)$-symplecticlike [resp. $(1,2)$-$\mathcal{D}$-symplecticlike].

Proof. A simple calculation gives (1.1). For (1.2) we first show

Claim. For all $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(TN)$

$$2h\big((\triangledown_{(\varphi X)}\varphi)X, Z\big) = d\Omega(X, \varphi X, Z) - h\big(\mathcal{N}_\varphi(X, Z), X\big),$$

where the Nijenhius tensor $\mathcal{N}_\varphi$ of φ is given by (see [4], page 47),

$$\mathcal{N}_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

Indeed observe that

$$\ldots\quad d\Omega(X, \varphi X, Z) = X \cdot \Omega(\varphi X, Z) + (\varphi X) \cdot \Omega(Z, X) + Z \cdot \Omega(X, \varphi X) - \Omega([X, \varphi X], Z) - \Omega([Z, X], \varphi X) - \Omega([\varphi X, Z], X) \tag{1.3}$$

$$\ldots\quad 2h(\triangledown_Y X, Z) = Y \cdot h(X, Z) + X \cdot h(Y, Z) - Z \cdot h(X, Y) - h([X, Y], Z) + h([Z, Y], X) - h([X, Z], Y) \tag{1.4}$$

$$\ldots\quad 2h\big((\triangledown_{(\varphi X)}\varphi)X, Z\big) = 2h\big(\triangledown_{(\varphi X)}(\varphi X), Z\big) + 2h\big((\triangledown_{(\varphi X)}X), \varphi Z\big). \tag{1.5}$$

By using the identities (1.3) and (1.4) in (1.5) we get the claim. But then, putting $Y = \varphi X$, the claim gives

$$\begin{aligned} 2h((\triangledown_X\varphi)(\varphi X), Z) &= -2rh((\triangledown_{\varphi Y}\varphi)Y, Z) \\ &= -r\{d\Omega(Y, \varphi Y, Z) - h(\mathcal{N}_\varphi(Y, Z), Y)\} \\ &= -r\{rd\Omega(X, \varphi X, Z) - h(\mathcal{N}_\varphi(X, Z), X)\} \\ &= -d\Omega(X, \varphi X, Z) + rh(\mathcal{N}_\varphi(\varphi X, Z), \varphi X) \\ &= -d\Omega(X, \varphi X, Z) - rh(\varphi\mathcal{N}(X, Z), \varphi X) \\ &= -d\Omega(X, \varphi X, Z) - h(\mathcal{N}_\varphi(X, Z), X) \end{aligned}$$

by the fact that $\varphi\mathcal{N}_\varphi(X, Z) = v - \mathcal{N}_\varphi(\varphi X, Z)$; $\forall X \in \Gamma(\mathcal{D})$, $\forall Z \in \Gamma(TN)$ and for some $v \in \Gamma(\mathcal{V})$. Hence this, together with the claim, gives (1.2). The statements (i) and (ii) follow from (1.2) easily.

2. Harmonicity

Definition 2.1. The φ-vertical distribution $\mathcal{V}$ over $(N^{2n+\ell}, h, \varphi)$ is said to be minimal if

$$\left(\sum_{t=1}^{\ell} h^{tt} \bigtriangledown_{v_t} v_t\right) \in \mathcal{V}$$

where $\ell = \operatorname{rank} \mathcal{V}$ and $\{v_1, \ldots, v_\ell\}$ is a local h-orthonormal frame field for $\mathcal{V}$.

Let $\phi : (M^{m'}, g, \psi) \to (N^{n'}, h, \varphi)$ be a C^2 map between $M(P)$-f-manifolds of dimension m' and n' respectively.

Definition 2.2. The map ϕ is said to be i) (ψ, φ)-holomorphic [resp. (ψ, φ)-antiholomorphic] if

$$d\phi \circ \psi = \varphi \circ d\phi, \quad [\text{resp. } d\phi \circ \psi = -\varphi \circ d\phi].$$

We shall write $\pm(\psi, \varphi)$-holomorphic to mean (ψ, φ)-holomorphic or (ψ, φ)-antiholomorphic

ii) harmonic if the tension field

$$\tau(\phi) = \sum_{t=1}^{m'} g^{tt} \alpha_\phi(u_t, u_t) = \sum_{t=1}^{m'} g^{tt} \left\{ \widetilde{\bigtriangledown}_{u_t}(d\phi(u_t)) - d\phi(\bigtriangledown^M_{u_t} u_t) \right\}$$

which is the trace of the second fundamental form $\alpha_\phi = \bigtriangledown d\phi$ of ϕ, vanishes; where $\{u_1, \ldots, u_{m'}\}$ is a local g-orthonormal frame field for TM and $\widetilde{\bigtriangledown}$ is the pull-back of $\bigtriangledown^N$ (the Levi–Civita connection on N) under ϕ.

Theorem 2.3. *Let $\phi : (M^{2m+s}, g_r, \psi_r) \to (N^{2n+\ell}, h_r, \varphi_r)$ be a $\pm(\psi_r, \varphi_r)$-holomorphic map into $(1,2)$-symplectic $M(P)$-f-manifold N with $\mathcal{V}^M \subseteq \mathcal{K}_\phi$.*

i) *If the $\psi-$vertical distribution $\mathcal{V}^M$ is minimal (in particular ψ is parallel i.e. $\bigtriangledown\psi = 0$) and M is $(1,2)$-$\mathcal{D}$-symplecticlike then ϕ is harmonic.*

ii) *If ϕ is harmonic with $\mathcal{V}^M = K_\phi$ and $\mathcal{V}^M$ is minimal then M is $(1,2)$-$\mathcal{D}$-symplecticlike.*

iii) *If ϕ is harmonic with $\mathcal{V}^M = K_\phi$ and M is $(1,2)$-$\mathcal{D}$-symplecticlike then $\mathcal{V}^M$ is minimal.*

Proof. For a local g-orthonormal frame field $\{v_1,\dots,v_s\}$ for $\mathcal{V}^M$ and $\{e_1,\dots e_m,\psi e_1,\dots,\psi e_m\}$ for $\mathcal{D}^M$ we have $\tau(\phi)=\tau_{\mathcal{V}}(\phi)+\tau_{\mathcal{D}}(\phi)$, where

$$\tau_{\mathcal{V}}(\phi)=\sum_{t=1}^{s} g^{tt}\alpha_\phi(v_t,v_t) \quad \text{and} \quad \tau_{\mathcal{D}}(\phi)=\sum_{t=1}^{m} g^{tt}\big\{\alpha_\phi(e_t,e_t)+r\alpha_\phi(\psi e_t,\psi e_t)\big\}.$$

But

$$\tau_{\mathcal{D}}(\phi)=\sum_{t=1}^{m} g^{tt}\big\{\widetilde{\nabla}_{e_t}E_t+r\widetilde{\nabla}_{(\psi e_t)}(\varphi E_t)-d\phi\big(\nabla^M_{e_t}e_t+r\nabla^M_{(\psi e_t)}(\psi e_t)\big)\big\}.$$

By (1.1) we get

$$\begin{aligned}\tau_{\mathcal{D}}(\phi)&=-r\sum_{t=1}^{m} g^{tt}\big\{S(E_t,\varphi E_t)+\varphi[E_t,\varphi E_t]-d\phi\big(S(e_t,\psi e_t)+\psi[e_t,\psi e_t]\big)\big\}\\&=-r\sum_{t=1}^{m} g^{tt}\big\{S(E_t,\varphi E_t)-d\phi(S(e_t,\psi e_t))\big\}\\&\quad-r\sum_{t=1}^{m} g^{tt}\{\varphi[E_t,\varphi E_t]-d\phi(\psi[e_t,\psi e_t])\},\end{aligned}$$

where $E_t=d\phi(e_t)$. Since the later sum is zero by the (ψ,φ)-holomorphicity of ϕ we get

$$\tau_{\mathcal{D}}(\phi)=-r\bigg(\sum_{t=1}^{m} g^{tt}\{S(E_t,\varphi E_t)\}\bigg)+rd\phi\bigg(\sum_{t=1}^{m} g^{tt}\{S(e_t,\psi e_t)\}\bigg). \tag{2.1}$$

Since $\mathcal{V}^M\subseteq K_\phi$ we get

$$\dots \qquad \tau_{\mathcal{V}}(\phi)=-d\phi\bigg(\sum_{t=1}^{s} g^{tt}(\nabla^M_{v_t}v_t)\bigg). \tag{2.2}$$

Now for (i), observe that the first and the second sums in (2.1) are both zero by Lemma 1.1/(i and ii) since M is $(1,2)$-$\mathcal{D}$-symplecticlike and N is $(1,2)$-symplectic respectively. Thus $\tau_{\mathcal{D}}(\phi)=0$. Also (2.2) gives that $\tau_{\mathcal{V}}(\phi)=0$ since $\mathcal{V}^M$ is minimal and $\mathcal{V}^M\subseteq K_\phi$. Hence ϕ is harmonic. In particular observe that if ψ is parallel then M is (1,2)-symplectic (and therefore $(1,2)$-$\mathcal{D}$-symplectic) and $\mathcal{V}^M$ is minimal. So harmonicity of ϕ follows.

For (ii), note that $\tau_{\mathcal{V}}(\phi) = 0$ since $\mathcal{V}^M$ is minimal and $\mathcal{V}^M = K_\phi$. So $\tau(\phi) = \tau_{\mathcal{D}}(\phi)$. On the other hand $S(E_t, \varphi E_t) = 0$; $\forall t = 1, \ldots, m$ since N is (1,2)-symplectic. So (2.1) and the harmonicity of ϕ give that

$$\tau(\phi) = rd\phi\bigg(\sum_{t=1}^{m} g^{tt}\{S(e_t, \psi e_t)\}\bigg) = 0.$$

So

$$\sum_{t=1}^{m} g^{tt}\{S(e_t, \psi e_t)\} \in \Gamma(K_\phi) = \Gamma(\mathcal{V}^M),$$

thus M is $(1,2)$-$\mathcal{D}$-symplecticlike.

iii) By the some reasoning as in (i), $\tau_{\mathcal{D}}(\phi) = 0$. So (2.2) and the harmonicity of ϕ give that

$$d\phi\bigg(\sum_{t=1}^{m} g^{tt}(\nabla^M_{\upsilon_t}\upsilon_t)\bigg) = 0.$$

Thus the minimality of $\mathcal{V}^M$ follows easily from $\mathcal{V}^M = K_\phi$. □

Remark 2.4.

i) Theorem 2.3 recovers

  a°) Rawnsley's result, ([20], Theorem 2.7) when g and h are both Riemannian.

  b°) Lichnerowcz's well known result, [14], that *every holomorphic map of a cosymplectic manifold into a quasi-Kaehlerian one is harmonic.*

ii) It is important here to note that "para" cases can produce extra harmonic maps. An other words, harmonicity of a map could be due to its $(\psi_{-1}, \varphi_{-1})$-holomorphicity rather than its (ψ_1, φ_1)-holomorphicity. For example, on $\mathbb{R}^4$, set the following: For $X = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$g(X, X) = (x_1^2 + x_2^2) - (x_3^2 + x_4^2)$$

and

$$\mathcal{J}(X) = (-x_2, x_1, -x_4, x_3), \quad \mathcal{P}(X) = (x_3, x_4, x_1, x_2).$$

Then $(\mathbb{R}^4, g, \mathcal{J})$ and $(\mathbb{R}^4, g, \mathcal{P})$ become indefinite Kaehler and para-Kaehler manifolds with the same neutral metric g respectively. Now define $\phi_J, \phi_P : (\mathbb{R}^4, g) \to (\mathbb{R}^4, g)$ by

$$\phi_J(X) = \big((x_1^2 - x_2^2), (2x_1x_2), (x_1x_3 - x_2x_4), (x_1x_4 + x_2x_3)\big)$$

and

$$\phi_P(X) = \big((x_1^2 + x_3^2), (x_1x_2 + x_3x_4), (2x_1x_3), (x_1x_4 + x_2x_3)\big)\,.$$

Then the map $\phi_J : (\mathbb{R}^4, g, \mathcal{J}) \to (\mathbb{R}^4, g, \mathcal{J})$ is holomorphic (i.e. $(\mathcal{J}, \mathcal{J})$-holomorphic) and therefore harmonic by Theorem 2.3/(ii). But it is not paraholomorphic (i.e. not $(\mathcal{P}, \mathcal{P})$-holomorphic). On the other hand, the map ϕ_P, is paraholomorphic and therefore harmonic and yet it is not holomorphic.

3. Revisiting φ-(PHWC) maps

Let $\phi : (M^{m'}, g) \to (N^{n'}, h, \varphi)$ be a nondegenerate, φ-(PHWC) map of a (semi)-Riemannian manifold M into $M(P)$-f-manifold N with the properties that

(3/i): $d\phi(TM) \subset \mathcal{D}^N$ and $d\phi(TM)$ is invariant under φ)

(3/ii): g and h are of constant signature (ξ, ζ) on $K_\phi(p)$ and (ξ', ζ') on $d\phi(T_pM)$ respectively for every $p \in M$.

Note that (ii) implies that $\dim(K_\phi(p))$ and $\mathrm{rank}(d\phi_p)$ are constant for every $p \in M$.

Unless otherwise stated ϕ will be as above throughout.

Define now ϕ-related $(1,1)$-tensor field $\mathcal{F}$ on M as follow:

$$\mathcal{F}(X) = \begin{cases} d\phi^* \circ \varphi \circ (d\phi^*)^{-1}(X), & \text{for } X \in H = K_\phi^\perp, \\ 0 & \text{for } X \in K_\phi. \end{cases}$$

Lemma 3.1. i) *$(M, \mathcal{F})$ is a (para-)f-manifold.*

ii) *$K_\phi = \mathcal{V}_\mathcal{F}^M$, the $\mathcal{F}$-vertical distribution, and therefore* $\mathrm{rank}\ \mathcal{F} = \mathrm{rank}(d\phi)$.

iii) *$(M^{m'}, g, \mathcal{F})$ is a metric (para-)f-manifold with* $\mathrm{rank}(\mathcal{F}) = \mathrm{rank}(\mathcal{D}^M) = 2m$, *where $m' = 2m + s$ with $s = \mathrm{rank}(\mathcal{V}^M)$.*

iv) *ϕ is $(\mathcal{F}, \varphi)$-holomorphic.*

Theorem 3.2. *Let $\phi : (M,g) \to (N,h,\varphi)$ be a φ-(PHWC) map of a (semi-)Riemannian manifold M into a $(1,2)$-symplectic $M(P)$-f-manifold N satisfying the conditions* 3/(i) *and* 3/(ii)*. Then*

i) *Any two of the following conditions imply the third:*

a°) *ϕ is harmonic,*

b°) *$\mathcal{V}^M$ is minimal or equivalently ϕ has minimal fibres,*

c°) *$(M,g,\mathcal{F})$ is $(1,2)$-$\mathcal{D}$-symplecticlike.*

ii) *If $\mathcal{F}$ is parallel then ϕ is harmonic.*

Proof. i) Note that, by Lemma 3.1, the map $\phi : (M,g,\mathcal{F}) \to (N,h,\varphi)$ satisfies all the hypothesis of Theorem 2.3. So (i) follows easily. For (ii), observe that $\mathcal{F}$ being parallel implies (i)/(b° and c°). So the harmonicity of ϕ follows from the part (i). □

For a $M(P)$-f-manifold (M,g,ψ) recall the notations: $\mathcal{V}_{\mathbb{K}} = \mathcal{V} \otimes \mathbb{K} = T^{\circ}M$ and $T^{\pm}M = \mathcal{D}^{\pm}_{\mathbb{K}} = \mathcal{D}^{\pm}$ which are the ψ-eigenbundles of $T^{\mathbb{K}}M = TM \otimes \mathbb{K}$ corresponding to the eigenvalues 0 and $\pm k$ of ψ. Set conditions

$$\nabla_X(T^{*+}M) \subset (T^{*+}M \oplus T^{*-}M) = \mathcal{D}^*_{\mathbb{K}}, \qquad \forall X \in \mathcal{V}_{\mathbb{K}}. \tag{$*$}$$

$$d\Omega^{1,2} = 0, \tag{$**$}$$

where $T^{*\pm}M$ is the dual bundle of $T^{\pm}M$, Ω is the fundamental 2-form of M and "$d\Omega^{1,2} = 0$" means that $d\Omega(u,v,w) = 0$ whenever u, v are of the same type and w is of a different one.

Remark 3.3. i) For $\mathbb{K} = \mathbb{C}$, the conditions $(*)$ and $(**)$ coincide with the ones in Loubeau's work ([15], conditions (7) and (8) in Theorem 3 respectively).

ii) Every $M(P)$-f-manifold (M,g,ψ) satisfying $(**)$ is $(1,2)$-$\mathcal{D}$-symplectic as we have that $d\Omega(w,\overline{w},u) = 0$, $\forall u,w \in \mathcal{D}^+$ if and only if $d\Omega(X,\psi X,Y) = 0\ \forall X,Y \in \mathcal{D}$. However the converse is not true in general as $d\Omega(u,v,w)$ need not to vanish, for example, for $u,v \in \mathcal{D}^+$ and $w \in \mathcal{V}_{\mathbb{K}}$ while M is $(1,2)$-$\mathcal{D}$-symplectic. Nevertheless, it is the standard fact that, when $\mathcal{V} = \{0\}$,

$$d\Omega(w,\overline{w},u) = 0 \text{ if and only if } d\Omega(w,\overline{v},u) = 0 \quad \text{for all } u,v,w \in \mathcal{D}^+.$$

In other words, *an almost (indefinite) Hermitian or almost para Hermitian manifold is* $(1,2)$*-*$\mathcal{D}$*-symplectic (i.e.* $(1,2)$*-symplectic) if and only if it satisfies the condition* $(**)$.

Lemma 3.4. *Let* (M^{2m+s}, g, ψ) *be a* $M(P)$*-*f*-manifold with its* ψ*-vertical and* ψ*-horizontal distributions* $\mathcal{V}$ *and* $\mathcal{D}$ *respectively. The distribution* $\mathcal{V}$ *is minimal if* M *satisfies the condition* $(*)$*.*

Proof. For any vector field $U \in \mathcal{V}$ consider $\bigtriangledown_U U = W_{\mathcal{D}} + W_{\mathcal{V}}$, where $W_{\mathcal{D}} \in \mathcal{D}_r$ and $W_{\mathcal{V}} \in \mathcal{V}_r$. Suppose $W_{\mathcal{D}} \neq 0$ and then choose a (local) vector field $X \in \mathcal{D}_r$ with $g(X, W_{\mathcal{D}}) \neq 0$. For $\mathcal{X} = X + kr\psi X \in \mathcal{D}_r^-$ define

$$\mathcal{X}'(\mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}), \quad \mathcal{Y} \in T^{\mathbb{K}}M.$$

Here g is $\mathbb{K}$-linearly extended to $T^{\mathbb{K}}M$. Note that $\mathcal{X}'$ is a nonzero 1-form in $\mathcal{D}_r^{*+} \subset (T^{\mathbb{K}}M)^*$ since for $\mathcal{W} = W_{\mathcal{D}} - kr\psi W_{\mathcal{D}} \in \mathcal{D}_r^+$,

$$\mathcal{X}'(\mathcal{W}) = g(\mathcal{X}, \mathcal{W}) = 2[g(X, W_{\mathcal{D}}) - kg(X, \psi W_D)] \neq 0.$$

Therefore $\mathcal{X}'(V) = 0$ for all $V \in \mathcal{V}_r$, in particular $\mathcal{X}'(U) = 0$. On the other hand, $(\bigtriangledown_U \mathcal{X}')(U) = 0$ too since $(\bigtriangledown_U \mathcal{X}') \in (\mathcal{D}_r^{\mathbb{K}})^*$ by the assumption that M satisfies the $(*)$. Thus we have

$$0 = \bigtriangledown_U(\mathcal{X}'(U)) = (\bigtriangledown_U \mathcal{X}')(U) + \mathcal{X}'(\bigtriangledown_U U) = \mathcal{X}'(\bigtriangledown_U U).$$

But then

$$0 = \mathcal{X}'(\bigtriangledown_U U) = \mathcal{X}'(W_D + W_{\mathcal{V}}) = \mathcal{X}'(W_D) = g(\mathcal{X}, W_D) \neq 0.$$

This contradiction is due to the assumption that $W_{\mathcal{D}} \neq 0$. Thus $W_{\mathcal{D}} = 0$, so $\bigtriangledown_U U \in \mathcal{V}_r$. Hence

$$\left(\sum_{t=1}^{s} g^{tt} \bigtriangledown_{u_t} u_t \right) \in \mathcal{V}_r$$

for any g-orthonormal local frame field $\{u_1, \ldots, u_s\}$ for $\mathcal{V}$. This completes the proof. □

Remark 3.3 and Lemma 3.4 would help us to appreciate the improvements and the generalizations made on the earlier results obtained, e.g.

Theorem 3.5 ([15], Theorem 2 and Theorem 3)**.** *Let $\phi : (M, g) \to (N, h, J)$ be a J-(PHWC) map of a Riemannian manifold into a Kaehlerian one (with* $\operatorname{rank} d\phi_p$ *constant for all $p \in M$).*

i) *Further suppose $(M, g, \mathcal{F})$ satisfies the conditions $(*)$ and $(**)$ and also $\mathcal{F}$ is integrable. Then ϕ is harmonic.*

ii) *If $\mathcal{F}$ is parallel then ϕ is harmonic.*

Remark 3.6. We may higlight the improvements and generalizations to Theorem 3.5 as follows:

i) The metrics g and h are allowed to be semi-Riemannian.

ii) The rank of the tensor field J need not to be equal to $\dim N$. It is allowed that $\operatorname{rank} J \leq \dim N$. Also J is allowed to be para-f-structure as well as f-structure (in particular, it is allowed to be almost paracomplex structure as well as almost complex one).

iii) The target manifold (N, h, J) with its fundamental 2-form Ω^N need not to be Kaehler. That is, neither J need to be integrable nor $d\Omega^N$ need to vanish.

iv) The integrability condition imposed on the J-related f-structure $\mathcal{F}$ is dropped altogether.

v) The condition $(**)$ imposed on $(M, g, \mathcal{F})$ is relaxed, so that $(M, g, \mathcal{F})$ is only required to be $(1, 2)$-$\mathcal{D}$-symplectic (see Remark 3.3).

vi) The condition $(*)$ imposed on $(M, g, \mathcal{F})$ is relaxed to the requirement that the $\mathcal{F}$-vertical distribution $\mathcal{V}^M$ is minimal (see Lemma 3.4), which is equivalent to saying that "ϕ has minimal fibres" since $\mathcal{V} = K_\phi$ under the circumstances.

For a φ-(PHWC) map $\phi : (M, g) \to (N, h, \varphi)$ of a (semi-)Riemannian manifold into $M(P)$-f-manifold N with $\dim N = 2$, satisfying 3/(ii) we necessarily have that

i) $\operatorname{rank} \varphi = 2$ and so that (N^2, h, φ) is either Kaehler or para-Kaehler. That is, N is either a Riemann surface or a Lorentz surface.

ii) ϕ is horizontally weakly conformal (HWC) and thus ϕ is a submersion. Also g restricts on $H = K_\phi^\perp$ to a Riemannian metric when h is Riemannian and to a neutral metric (of signature $(1, 1)$) when h is neutral.

Theorem 3.2 overlaps with the result of Baird–Eells when the metrics involved are Riemannian and the target manifold (N, h, J) is semi-Kaehler:

Proposition 3.7 ([1])**.** *Let $\phi : (M, g) \to (N, h)$ be a horizontally conformal submersion between Riemannian manifolds with dilation λ. Then for $n \neq 2$*

i) *Any two of the following conditions imply the third:*
   a°) *ϕ is harmonic,*
   b°) *ϕ has minimal fibres,*
   c°) $\operatorname{grad} \lambda^2$ *is vertical or equivalently ϕ is horizontally homothetic (i.e. λ is constant along horizontal curves).*

ii) *For $\dim N = 2$ we have that ϕ is harmonic if and only if ϕ has minimal fibres.*

Wood, [22], has shown that the conclusion of Proposition 3.7/(ii) is still valid for a larger class of mappings $\phi : (M^{m'}, g) \to (N^2, h)$ from a Riemannian manifold into a Riemann surface, namely, those which have horizontally holomorphic quadratic differentials. We will further extend his result to the cases where g and h are allowed to be semi-Riemannian as well as Riemannian.

Let $\phi : (M^{m'}, g) \to (N^2, h)$ be a map of a (semi-)Riemannian manifold into a surface with Riemannian or semi-Riemannian metric, (in the later case h is necessarily neutral). Let $\mathcal{U}$ be a dense subset of M on which ϕ is a submersion with g of constant signature on $H = K_\phi^\perp$. (Note here that when g is Riemannian and ϕ is real analytic then $\mathcal{U} = \{p \in M : \operatorname{rank}(d\phi_p) = 2\}$, is always dense.) At each $p \in \mathcal{U}$, orient the horizontal space H_p so that $d\phi_p : H_p \to T_{\phi(p)}N$ is an orientation preserving map with respect to a chosen (locally) orientation on N. The orientation and the metric on H defines an endomorphism J_r on H which is a complex structure if $G = g|_H$ is Riemannian, a paracomplex structure if G is semi-Riemannian. More precisely, for a G -orthonormal oriented frame field $\{e_1, e_2\}$ for H over an open subset $\mathcal{U}'$ in $\mathcal{U}$ set $J_r(e_1) = e_2$ and $J_r(e_2) = -re_1$, where $r = 1$ when G is Riemannian and $r = -1$ when G is semi-Riemannian. Define a section η of the symmetric square bundle $\odot^2(H^+)^*$ over $\mathcal{U}$:

$$\eta(Z, W) = h'\big(d^{\mathbb{K}}\phi(Z), d^{\mathbb{K}}\phi(W)\big) \in \mathbb{K}$$

for $Z, W \in H^+ = \{X - rkJ_rX : X \in H\}$, where $d^{\mathbb{K}}\phi : T^{\mathbb{K}}M = TM \otimes \mathbb{K} \to T^{\mathbb{K}}N$ and h' are the $\mathbb{K}$-linear extension of $d\phi$ and h respectively. (As before $\mathbb{K} = \mathbb{C}$, $k = i$ for $r = 1$ and $\mathbb{K} = \mathbb{A}$, $k = \epsilon$ for $r = -1$.)

We say that η *is horizontally holomorphic* [resp. *paraholomorphic*] *if* $\nabla_{\overline{Z}}\eta = 0$ *for* $Z \in H_p^+$ *and for all* $p \in \mathcal{U}$ *and* $r = 1$ [resp. $r = -1$]. The section η will be called *horizontal quadratic differential of* ϕ.

Theorem 3.8 (c.f. [22]). *Let* $(M, g), (N^2, h)$ *and* $\phi : M \to N$ *with* $\mathcal{U} \subset M$ *be as above such that* $H_p \subset K_\phi(p)$ *for all* $p \in M \setminus \mathcal{U}$. *Then*

i) $\eta \equiv 0$ *on* $\mathcal{U}$ *if and only if* ϕ *is (HWC) on* M.

ii) *Any two of the following conditions imply the other one:*

a°) ϕ *is harmonic on* M,

b°) ϕ *has minimal fibres,*

c°) η *is horizontally (para) holomorphic on* $\mathcal{U}$.

Proof. It is the same as in [22] with some minor modifications where necessary. □

Remark 3.9. In the above theorem note that

i) If $\eta \equiv 0$ then the metric G and h are necessarily both Riemannian or both semi-Riemannian.

ii) Otherwise we can have the mixture of cases, namely:

a°) When G is Riemannian on H (while g may still be semi-Riemannian on TM), h can be Riemannian or semi-Riemannian.

b°) When G is semi-Riemannian, again h can be Riemannian or semi-Riemannian.

4. φ-pseudo harmonic morphism

Definition 4.1. i) (c.f. [15]) A map $\phi : (M, g) \to (N^{2n+\ell}, h, \varphi_r)$ of a (semi-)Riemannian manifold into a $M(P)$-f-manifold is said to be φ_r-pseudo harmonic morphism if ϕ is harmonic and φ_r-(PHWC).

ii) (c.f. [7]) A map $\phi : (M^{2m+s}, g, \psi_r) \to (N, h)$ of a $M(P)$-f-manifold M into a (semi-)Riemannian manifold N is said to be ψ_r-pluriharmonic [resp. $\mathcal{D}$-pluriharmonic] if for every $X, Y \in TM$ [resp. for every $X, Y \in \mathcal{D}_r$] we have

$$\beta_\phi(X, Y) = \alpha_\phi(X, Y) + r\alpha_\phi(\psi_r X, \psi_r Y) = 0,$$

where $\alpha_\phi = \nabla d\phi$, the second fundamental form of ϕ.

Remark 4.2. i) When (M, g) is a Riemannian manifold and (N, h, φ_r) is a Hermitian one (i.e. h is Riemannian, $r = 1, \operatorname{rank}\varphi_1 = \dim N$ and φ_1 is integrable) then the notion of being φ_1-pseudo harmonic morphism coincides with the one introduced by Loubeau in [15], in which he simply calls it *pseudo harmonic morphism.*

ii) When the metrics g and h involved are Riemannian and (M^{2m+s}, g, ψ_r) is a M-f-manifold (i.e $r = 1, \operatorname{rank}(\psi_1) = 2m < \dim M$) then the notions ψ-pluriharmonicity and $\mathcal{D}$-pluriharmonicity coincide with the ones introduced in [7].

iii) For $s = 0$, so that (M^{2m}, g, ψ_r) is an (indefinite) almost (para) Hermitian manifold, we have that ϕ is ψ_r-pluriharmonic if and only if

$$\alpha_\phi(\overline{Z}, W) = 0 \quad \text{for every } Z, W \in T^+M.$$

In this case ϕ is also called ψ_r-$(1,1)$-*geodesic.* In the literature, for $r = 1$, ψ_1-$(1,1)$-geodesic map is simply called $(1,1)$-*geodesic.* However the word *pluriharmonic* is mostly reserved for maps ϕ satisfying: For every $Z, W \in T^+M$

$$\alpha'_\phi(\overline{Z}, W) = \widetilde{\nabla}_{\overline{Z}} d\phi(W) - d\phi(\overline{\partial}_{\overline{Z}} W) = 0 \tag{4.1}$$

when ψ_1 is integrable and g is Riemannian so that (M, g, ψ_1) is a Hermitian manifold. Here $\overline{\partial}$ is the usual $\overline{\partial}$-operator of a complex manifold. Note that the condition (4.1) is still meaningful for maps of indefinite Hermitian or para-Hermitian manifolds. Thus the definition of pluriharmonicity may be extended for maps of those manifolds, (for the $\overline{\partial}$-operator in "para" cases see [10]). We shall be calling pluriharmonic maps in the 'para' cases, *para-pluriharmonic.* Here is an example of para-pluriharmonic map:

For a Riemannian manifold (M, g) let $\pi : TM \to M$ be its tangent bundle. For a local coordinate frame field $\{\partial_{x_i} = \frac{\partial}{\partial x_i}\}$, let $\{\partial_{x_i}, \partial_{y_i}\}$ denote the induced local coordinate frame field on TM, where $\{x_i\}$ is a local coordinate system on M and $\{x_i, y_i\}$, is the induced one on TM. Then we have the following definitions ([21]):

1°) For a smooth map $f : M \to \mathbb{R}^n$, $f = (f^1, \ldots, f^n)$, *the vertical lift* $f^v : TM \to \mathbb{R}^n$ of f is given by

$$f^v = f \circ \pi = (f^1 \circ \pi, \ldots, f^n \circ \pi).$$

2°) For a vector field $X = \sum a_j \partial_{x_j} \in \Gamma(TM)$ on M, $p \in M$ and $W_p \in T_pM$

a°) *the vertical lift* $X^v \in \Gamma(TTM)$ of X, is given by

$$X^v(p, W_p) = \sum a_j^v(p, W_p)\partial_{y_j}(p, W_p),$$

b°) *the complete lift* $X^c \in \Gamma(TTM)$ of X, is given by

$$X^c(p, W_p) = \sum a_j^v(p, W_p)\partial_{x_j}(p, W_p) + da_j(W_p)\partial_{y_j}(p, W_p),$$

3°) *the complete lift* g^c of a Riemannian metric g on M is characterized by $\forall\, X, Y \in \Gamma(TM)$

$$g^c(X^c, Y^c) = (g(X, Y))^c, \quad g^c(X^v, Y^v) = 0$$

$$g^c(X^c, Y^v) = g^c(X^v, Y^c) = (g(X, Y))^v.$$

The complete lift g^c is a neutral metric, that is, it is a semi-Riemannian metric of signature (m, m) on TM.

4°) the almost paracomplex structure $\mathcal{P}$ on TM is characterized by

$$\mathcal{P}X^v = X^v \quad \text{and} \quad \mathcal{P}X^c = 2X^v - X^c.$$

Then $(TM, \mathcal{P}, g^c)$ provides an example of an almost para-Hermitian manifold. Moreover, it is para-Kaehler if and only if M is flat, ([3], Proposition 4.1).

Now let $f : (M, g) \to (N, h)$ be a totally geodesic map between Riemannian manifolds and consider the differential map

$$df = F : (TM, \mathcal{P}_M, g^c) \to (TN, \mathcal{P}_N, h^c).$$

Since F is also totally geodesic ([3], Theorem 4.3) it trivially becomes $\mathcal{P}_M$-$(1,1)$-geodesic. In the cases where M is flat, the manifold $(TM, \mathcal{P}, g^c)$ is para-Kaehler. Thus, by the next Lemma 4.3, F becomes para-pluriharmonic.

Lemma 4.3. *Let $\phi : (M^{2m}, g, J_r) \to (N, h)$ be a map from an (indefinite) Kaehler [resp. para-Kaehler] manifold into a (semi-)Riemannian manifold. Then the two concepts; being J_1-$(1,1)$-geodesic [resp. $J_{(-1)}$-$(1,1)$-geodesic] and J_1-pluriharmonicity [resp. $J_{(-1)}$-pluriharmonicty] coincide, that is, $\alpha_\phi = \alpha'_\phi$.*

Proof. It is the standard fact that (see [16], [18]) "$\bigtriangledown^{0,1}_{\overline{Z}} = \overline{\partial}_{\overline{Z}}$ *if and only if M is (indefinite) Kaehler*" where

$$\bigtriangledown^{0,1} : \Gamma(T^{-}M) \times \Gamma(T^{C}M) \to \Gamma(T^{C}M)$$

is the $(0,1)$ part of the $\mathbb{C}$-linear extension of the Levi–Civita connection on (M,g). The above statement is also true for the 'para' cases. (Its proof goes formally exactly the same as that of (indefinite) Kaehlerian case, (see [10], [19]).) Thus, the result will immediately follow from this fact. □

Lemma 4.4. *Recalling the adjoints $\varphi^* : TN \to TN$, $d\phi^* : TN \to TM$ and $d\pi^* : TB \to TN$ we have*

i) $\varphi^* = -\varphi$

ii) $(d\pi \circ d\phi)^* = d\phi^* \circ d\pi^*$

iii) *The following are equivalent:*

a°) $d\pi \circ \varphi = \Xi \circ d\pi$, *i.e. π is (φ, Ξ)-holomorphic*

b°) $\varphi \circ (d\pi)^* = (d\pi)^* \circ \Xi$.

Proposition 4.5. *Let $\phi : (M, g, \psi) \to (N, h)$ be a map of a M(P)-f-manifold into a (semi-)Riemannian one.*

i) *ϕ is harmonic if it is ψ-pluriharmonic.*

ii) *ϕ is harmonic if it is $\mathcal{D}$-pluriharmonic and ψ-vertical distribution $\mathcal{V}^M$ is minimal with $\mathcal{V}^M \subset K_\phi$.*

By a *local map* we shall mean a map defined on an open set.

Proposition 4.6. *Let (M, g) be a semi-Riemannian manifold, (N, h, φ) a (para-)Hermitian manifold and $\phi : M \to N$ a nondegenerate map. Then ϕ is φ-(PHWC) if and only if it pulls back local $\pm(\varphi, \Xi)$-holomorphic maps $\pi : (N, h, \varphi) \to (B, \mu, \Xi)$ onto Ξ-(PHWC) maps, where (B, μ, Ξ) is a (para-)Hermitian manifold.*

Proof. Let ϕ be φ-(PHWC) map. Then setting $\mathcal{G} = \pi \circ \phi$ we have

$$\Xi \circ d\mathcal{G} \circ d\mathcal{G}^* = \Xi \circ d\pi \circ d\phi \circ d\phi^* \circ d\pi^* = \pm d\pi \circ \varphi \circ (d\phi \circ d\phi^*) \circ d\pi^*$$

since π is $\pm(\varphi, \Xi)$-holomorphic.

$$\begin{aligned} &= \pm d\pi \circ (d\phi \circ d\phi^*) \circ \varphi \circ d\pi^* = (d\pi \circ d\phi) \circ (d\pi \circ d\phi)^* \circ \Xi \\ &= d\mathcal{G} \circ d\mathcal{G}^* \circ \Xi, \end{aligned}$$

which says that $\mathcal{G}$ is a local Ξ-(PHWC) map.

Conversely assume that $\mathcal{G}$ is Ξ-(PHWC) with a local $\pm(\varphi,\Xi)$-holomorphic map π. Then, by Lemma 4.4

$$\begin{aligned}&\pm\, d\pi\circ\varphi\circ(d\phi\circ d\phi^*)\circ d\pi^* = \Xi\circ(d\pi\circ d\phi)\circ(d\pi\circ d\phi)^* = \Xi\circ d\mathcal{G}\circ d\mathcal{G}^*\\ &= d\mathcal{G}\circ d\mathcal{G}^*\circ\Xi = (d\pi\circ d\phi)\circ(d\pi\circ d\phi)^*\circ\Xi = d\pi\circ(d\phi\circ d\phi^*)\circ d\pi^*\circ\Xi\\ &= \pm d\pi\circ(d\phi\circ d\phi^*)\circ\varphi\circ d\pi^*.\end{aligned}$$

That is

$$d\pi\circ\varphi\circ(d\phi\circ d\phi^*)\circ d\pi^* = d\pi\circ(d\phi\circ d\phi^*)\circ\varphi\circ d\pi^*.$$

Noting that $d\pi : H_\pi \to H_{\pi^*} = d\pi(TN)$ and $d\pi^* : H_{\pi^*} \to H_\pi$ are invertible, we get

$$d\phi\circ d\phi^*\circ\varphi = \varphi\circ d\phi\circ d\phi^* \quad \text{on} \quad d\phi(TM)\cap H_\pi.$$

But for every $X \in d\phi(TM)$ one can construct a local $\pm(\varphi,\Xi)$-holomorphic map π , by means of (para) holomorphic charts of N and B with $X \in H_\pi$ since (N,φ) and (B,Ξ) are (para) complex manifolds. Thus φ-pseudo horizontal weak conformality of ϕ follows. □

Proposition 4.7. *Let $\pi : (N,h,\varphi) \to (B,\mu,\Xi)$ be a $\pm(\varphi,\Xi)$-holomorphic map of $(1,2)$-$\mathcal{D}$-symplectic $M(P)$-f-manifold N into $(1,2)$-$\mathcal{D}$-symplectic B. Then*

i) *π is $\mathcal{D}$-pluriharmonic*

ii) *π is φ-pluriharmonic (and therefore harmonic) if further $d\pi(TN) \subset \mathcal{D}^B$ and $\mathcal{V}_\varphi$ is a minimal distribution on N.*

Proof. For $X \in \mathcal{D}^N$.

$$\begin{aligned}\alpha_\pi(X,X) + r\alpha_\pi(\varphi X,\varphi X) &= \widetilde{\nabla}_X d\pi(X) + r\widetilde{\nabla}_{\varphi X} d\pi(\varphi X)\\ &\quad - d\pi\big(\nabla_X X + r\,\nabla_{(\varphi X)}(\varphi X)\big)\\ &= -r\big\{\Xi[d\pi(X),\Xi d\pi(X)] - d\pi\big(\varphi[X,\varphi X]\big)\big\}\end{aligned}$$

since N,B are $(1,2)$-$\mathcal{D}$-symplectic and π is $\pm(\varphi,\Xi)$-holomorhic. But then

$$\Xi[d\pi(X),\Xi d\pi(X)] = d\pi\big(\varphi[X,\varphi X]\big).$$

So we get

$$\beta_\pi(X,X) = \alpha_\pi(X,X) + r\alpha_\pi(\varphi X,\varphi X) = 0.$$

Hence, for any $X, Y \in \mathcal{D}^N$,

$$\beta_\pi(X,Y) = \frac{1}{4}\big\{\beta_\pi(X+Y, X+Y) - \beta_\pi(X-Y, X-Y)\big\} = 0,$$

which completes the proof. □

This proposition has the following two corollaries as its particular cases.

Corollary 4.8. *Let $\pi : (N, h, J) \to (B, \mu, \Xi)$ be a $\pm(J, \Xi)$-holomorphic map of a $(1,2)$-symplectic almost (para-)Hermitian (i.e. quasi-(para-)Kaehler) manifold into $(1,2)$-symplectic $M(P)$-f-manifold. Then π is J-pluriharmonic (and therefore harmonic).*

We say that ([9]) *an almost (para) contact (hyperbolic) metric manifold (N^{2n+1}, h, φ) satisfies the geodesic condition if $\nabla_\xi \xi = 0$*, where $\mathrm{span}_{\mathbb{R}}\{\xi\} = \mathcal{V}^N$ and in which case ξ is called *the characteristic vector field of N*. Thus *(N^{2n+1}, h, φ) satisfies the geodesic condition if and only if $\mathcal{V}^N$ is a minimal distribution over N.*

Corollary 4.9. *Let $\pi : (N^{2n+1}, h, \varphi) \to (B^{2b+1}, \mu, \Xi)$ be a $\pm(\varphi, \Xi)$-holomorphic map of an $(1,2)$-$\mathcal{D}$-symplectic almost (para) contact (hyperbolic) metric manifold N whose distribution $\mathcal{V}^N$ is minimal (or equivalently, N being satisfying the geodesic condition) into $(1,2)$-symplectic almost (para) contact (hyperbolic) metric manifold B with $d\pi(TN) \subset \mathcal{D}^B$. Then π is φ-pluriharmonic (and therefore harmonic).*

This result overlaps in great deal with an other result obtained by the author:

Proposition 4.10 ([9])**.** *Let $\pi : (N^{2n+1}, h, \varphi) \to (B^{2b+1}, \mu, \Xi)$ be a $\pm(\varphi, \Xi)$-holomorphic map between almost (para) contact (hyperbolic) metric manifolds with both N and B satisfying the geodesic condition. If further N is $(1,2)$-symplecticlike with nonintegrable distribution $\mathcal{D}^N$ and B is $(1,2)$-symplectic then π is harmonic.*

Proposition 4.7 also recovers, as a special case, a recent result of Duggal–Ianus–Pastore ([7], Proposition 6.1/(a)).

Theorem 4.11. *Let* $(M,g) \xrightarrow{\phi} (N^{2n+\ell},h,\varphi) \xrightarrow{\pi} (B^{2b+\rho},\mu,\Xi)$ *and* $\mathcal{G}$ *be as above with* $d\phi(TM)\subset \mathcal{D}^N$. *Then*

i) *ϕ pulls back any local $\mathcal{D}$-pluriharmonic map π to a local harmonic map $\mathcal{G}$ if ϕ is a φ-pseudo harmonic morphism.*

ii) *Suppose also that N and B are $(1,2)$-$\mathcal{D}$-symplectic [in particular, N and B are (indefinite) quasi-(para)-Kaehler] manifolds. Then ϕ pulls back every local $\pm(\varphi,\Xi)$-holomorphic map π to a local Ξ-pseudo harmonic morphism $\mathcal{G}$ if ϕ is a φ-pseudo harmonic morphism.*

iii) *Suppose further that N and B are (indefinite) (para-)Kaehler manifolds. Then ϕ is a φ-pseudo harmonic morphism if and only if ϕ pulls back each local $\pm(\varphi,\Xi)$-holomorphic map π to a local harmonic map $\mathcal{G}$.*

Proof. i) Let ϕ be a φ-pseudo harmonic morphism. Then $\phi:(M^{2m+s},g,\mathcal{F})\to(N,h,\varphi)$ is $(\mathcal{F},\varphi)$-holomorphic, where $\mathcal{F}$ is the ϕ-related f-structure. For a g-orthonormal frame field $\{e_1,\dots,e_m,\mathcal{F}e_1,\dots$ $\dots,\mathcal{F}e_m,v_1,\dots,v_s\}$ for TM with $\{v_1,\dots,v_s\}\subset\mathcal{V}_{\mathcal{F}}$ and the rest in $\mathcal{D}_{\mathcal{F}}$, we have $d\phi(v_t)=0$, $t=1,\dots,s$ since ϕ is $(\mathcal{F},\varphi)$-holomorphic. Thus

$$\tau(\mathcal{G}) = d\pi(\tau(\phi)) + \sum_{t=1}^{m} g^{tt}\big(\alpha_\pi\big(d\phi(e_t),d\phi(e_t)\big) + r\alpha_\pi\big(d\phi(\mathcal{F}e_t),d\phi(\mathcal{F}e_t)\big)\big).$$

By the harmonicity and holomorphicity of ϕ, this becomes

$$\begin{aligned}\tau(\mathcal{G}) &= \sum_{t=1}^{m} g^{tt}\big(\alpha_\pi\big(d\phi(e_t),d\phi(e_t)\big) + r\alpha_\pi\big(\varphi d\phi(e_t),\varphi d\phi(e_t)\big)\big)\\ &= \sum_{t=1}^{m} g^{tt}\beta_\pi\big(d\phi(e_t),d\phi(e_t)\big).\end{aligned}$$

But $\beta_\pi(X,X)=0$, $\forall X\in\mathcal{D}^N$ since π is $\mathcal{D}$-pluriharmonic map. So, we get $\tau(\mathcal{G})=0$, the harmonicity of $\mathcal{G}$.

ii) Since ϕ is φ-(PHWC) by the assumption, $\mathcal{G}$ is Ξ-(PHWC) by Proposition 4.6. So, it is left to show that $\mathcal{G}$ is a locally harmonic map. Indeed, by Proposition 4.7, π is $\mathcal{D}$-pluriharmonic since π is $\pm(\varphi,\Xi)$-holomorphic. Then, the harmonicity follows from Part (i).

iii) "only if" part of (iii) is a special case of Part (ii). For the "if" part of the statement, we quote the proof provided for ([15], Proposition 3) as it extends "indefinite" and "para" cases with no difficulty. □

Corollary 4.12. *Let* $(M,g) \xrightarrow{\phi} (N^{2n},h,\varphi) \xrightarrow{\pi} (B^{2b},\mu,\Xi)$ *be as above with (indefinite) (para-)Kaehler manifolds* N *and* B*. Then* ϕ *is a* φ*-pseudo harmonic morphism if and only if* ϕ *pulls back each local* φ*-pluriharmonic map* π *to a local harmonic map* $\mathcal{G} = \pi \circ \phi$*.*

Proof. Observing that, under the circumstances, every $\pm(\varphi,\Xi)$-holomorphic map is φ-pluriharmonic by Proposition 4.7, corollary follows easily from Theorem 4.11. □

Remark 4.13. The above corollary recovers the results obtained in [15] (Propositions 3 and 4 and Corollary 1).

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References

[1] P Baird and J. Eells, A conservation law for harmonic maps, Lecture Notes in Math., Vol. 894, 1981, 1–25.

[2] P. de Bartolomeis, D. Burns F. E. Burstall and J Rawnsley, Stability of harmonic maps of Kaehler manifolds, *J. Diff. Geom.* **30** (1989), 579–594.

[3] C. L. Bejan and M. Benyounes, Harmonic maps between para-Hermitian manifolds, New developments in differential geometry, Budapest (1996), 67–76, *Kluwer Acad. Publ.*, *Dordrecth*, 1999.

[4] D. Blair, Contact manifolds in Riemannian Geometry, Lecture Notes in Math. 509, *Springer-Verlag*, 1976.

[5] V. Cruceanu P. Fortuny and P. M. Gadea, A survey on paracomplex geometry, *Rocky Mountain J. Math.* **26** (1996), 83–115.

[6] M. J. Duff and J. X. Lu, Black and super p-branes in diverse dimensions, *Nuclear Phys.* **B416** (1994), 301.

[7] K. L. Duggal, S. Ianus and A. M. Pastore, Maps interchanging f-structures and their harmonicity, *Acta. Appl. Math.* **67**, no. 1 (2001), 91–115.

[8] K. L. Duggal, S. Ianus and A. M. Pastore, Harmonic maps on f-manifolds with semi-Riemannian metrics, Proc. of the 23rd Conf. on Geom. and Topology, 47–55.

[9] S. Erdem, On almost (para) contact hyperbolic metric manifolds and harmonicity of (φ,φ')-holomorphic maps between them, *Houston J. Math.* **28**, no. 1 (2002), 21–45.

[10] S. Erdem, Paraholomorphic structures and connections of vector bundles over paracomplex manifolds, *New Zealand J. Math.* **30** (2001), 41–50.

[11] S. Erdem, Paracomplex projective models and harmonic maps into them, *Beitrage Algebra Geom.* **40**, no. 2 (1999), 385–398.

[12] B. Fuglede, Harmonic morphism between semi-Riemannian manifolds, *Annales Academiae Scientiarum Fennicae Mathematica* **21** (1996), 31–50.

[13] D. Lambert and J. Rembielinski, From Godel quarternions to non-linear sigma models, *J. Phys. A: Math Gen.* **21** (1988), 2677–2691.

[14] A. Lichnerowicz, Applications harmoniques at varieties Kahleriennes, Sympos. Math., Roma 3, probl. Evolut. Sist. solare, Nov. 1968 e Geometria, Febb. 1969, 1970, 341–402.

[15] E. Loubeau, Pseudo harmonic morphisms, *International Journal of Mathematics* **8**, no. 7 (1997), 943–957.

[16] E. Loubeau, Pluriharmonic morphisms, *Math. Scand.* **84**, no. 2 (1999), 165–178.

[17] D. I. Nikolayev and H. Schaeben, Characteristic of the ultrahyperbolic differential equation governing pole density functions, *Inverse Problems* **15**, no. 6 (1999), 1603–1619.

[18] Y. Ohnita and S. Udagawa, Complex-Analyticity of Pluriharmonic maps and their constructions, Lecture Notes In Maths, Vol. 1468, 1991.

[19] V. K. Parmar, Harmonic morphisims between semi-Riemannian manifolds, Ph.D. Thesis, *School of Mathematics, The University of Leeds*, *U.K.*, 1991.

[20] J. H. Rawnsley, F-Structures, F-twistor spaces and harmonic maps, Lecture Notes in Maths, Vol. 1164, 1985.

[21] K. Yano and S. Ishihara, Tangent and Contangent bundles, *Marcel Dekker, Inc.*, 1973.

[22] J. C. Wood, Harmonic morphisms, Foliations and Gauss map, *Contemp. Math.* **49**, *Amer. Math. Soc., Providence, R.I.* 1986.

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