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Projective flatness of complex Finsler metrics

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Dedicated to Professor Yoshihiro Ichijyō on the occasion of his 70th birthday

Abstract. In the previous papers [7], we have studied complex Finsler geometry from the view point of Kähler fibration, and have obtained the characterizations of flatness of complex Finsler metrics in terms of Finsler connection. In the present paper, we shall introduce the notion of projective flatness of Finsler connections, and characterize the projective flatness of complex Finsler metrics in terms of Finsler connections.

1. Introduction and preliminaries

Let $\pi : E \to M$ be a holomorphic vector bundle of $\operatorname{rank}(E) = r$ $(r \geq 2)$ over a connected complex manifold M of $\dim_{\mathbb{C}} M = n$. We denote by T_E and T_M the tangent bundle of the total space E and the base manifold M, and we also denote by $\Omega^1_\bullet$ the corresponding cotangent bundle. Moreover we denote by $T_{E/M} := \ker d\pi$ the relative tangent bundle of the morphism π. Then we have the fundamental sequence of vector bundles:

$$\mathbb{O} \to T_{E/M} \xrightarrow{i} T_E \xrightarrow{d\pi} \pi^{-1}T_M \to \mathbb{O}. \tag{1.1}$$

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A *connection* h_E in E is a smooth splitting in this sequence, that is, a smooth bundle morphism $h_E : \pi^{-1}T_M \to T_E$ such that $d\pi \circ h_E = \mathrm{Id}$. Then h_E induces an isomorphism $\mathcal{H} = h_E(\pi^{-1}T_M) \cong \pi^{-1}T_M$, and it defines a smooth decomposition

$$T_E = T_{E/M} \oplus \mathcal{H}. \tag{1.2}$$

The non-zero complex number field $\mathbb{C}^\times$ acts on E by multiplication. We denote by R_λ the action for $\lambda \in \mathbb{C}^\times$, that is, $R_\lambda v = (z, \lambda\xi)$ for $^\forall v = (z, \xi) \in E_z$ and $^\forall \lambda \in \mathbb{C}^\times$. We shall only consider homogeneous connections, that is, connections invariant under the action of R_λ.

The splitting (1.2) induces the dual splitting $\Omega^1_E = \Omega^1_{E/M} \oplus \mathcal{H}^*$, and so the differential operator $d_E : \mathcal{O}_E \to \Omega^1_E$ is decomposed as $d_E = d^v_E + d^h_E$ by the differential $d^h_E : \mathcal{O}_E \to \mathcal{H}^*$ along $\mathcal{H}$ and the differential $d^v_E : \mathcal{O}_E \to \Omega^1_{E/M}$ along vertical direction. We also decompose the operators ∂_E and $\bar\partial_E$ as $\partial_E = \partial^v_E + \partial^h_E$ and $\bar\partial_E = \bar\partial^v_E + \bar\partial^h_E$ respectively. We denote by $\mathcal{S}$ the sheaf of germs of linear functionals along the fibres of π. A connection h_E in the sequence (1.1) is determined by the action of ∂^h_E on $\mathcal{S}$ (cf. [16]). A connection h_E is said to be *compatible with the vector bundle structure* or simply *linear connection* if ∂^h_E sends $\mathcal{S}$ to $\mathcal{S}$, that is,

$$\partial^h_E \mathcal{S} \subset \mathcal{S} \otimes \mathcal{H}^*. \tag{1.3}$$

If a connection $h_E : \pi^{-1}T_M \to T_E$ is given in this sequence, we have to consider two cases. The one is the case where h_E is a *linear connection* and another one is the case where h_E is a *non-linear connection*.

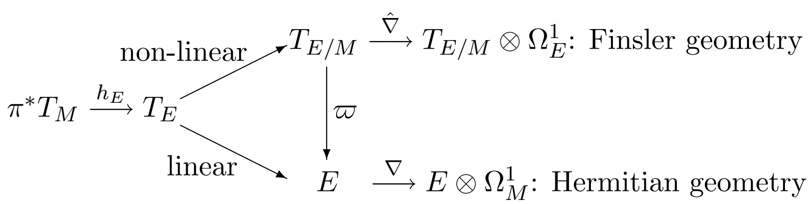
Throughout the present paper, we use the following local coordinate system on M and E. Let U be an open set in M with local coordinate $(z^1, \dots, z^n)$, and let $s_U = (s_1, \dots, s_r)$ be a local holomorphic frame field on U. The pair (U, s_U) induces a coordinate $(z^1, \dots, z^n, \xi^1, \dots, \xi^r)$ on $\pi^{-1}(U)$, where $(z^1, \dots, z^n)$ is lifted from M and $(\xi^1, \dots, \xi^r)$ is the fibre coordinate.

If a connection h_E is given in E, by definition, the condition $\partial^h_E \xi^i \in \mathcal{H}^*$ implies that there exists some local functions N^i_α on $\pi^{-1}(U)$ such that $\partial^h_E \xi^i = -\sum N^i_\alpha(z,\xi) dz^\alpha$. Since h_E is invariant by the action R_λ, these functions $\{N^i_\alpha\}$ satisfy the homogeneity

$$N^i_\alpha(z, \lambda\xi) = \lambda N^i_\alpha(z, \xi) \tag{1.4}$$

for all $\lambda \in \mathbb{C}$. These functions $\{N^i_\alpha\}$ are the *coefficients* of the connection h_E. If h_E is linear, then by definition (1.3), the coefficients N^i_α are linear in (ξ^i) along the fibre E_z, i.e., there exist some functions $\gamma^i_{j\alpha}(z)$ on U such that $N^i_\alpha(z,\xi) = \sum \gamma^i_{j\alpha}(z)\xi^j$. Then it is easily checked that the $(1,0)$-forms $\omega^i_j = \sum \gamma^i_{j\alpha}(z)dz^\alpha$ define a connection $\nabla : E \to E \otimes \Omega^1_M$. If E has a Hermitian metric, there exists a canonical connection ∇, and the Hermitian geometry on E is the differential geometry of the bundle E with the connection ∇.

On the other hand, if h_E is non-linear, then it induces a connection $\hat{\nabla} : T_{E/M} \to T_{E/M} \otimes \Omega^1_E$ in the relative tangent bundle $\varpi : T_{E/M} \to E$. Such a connection $\hat{\nabla}$ is naturally induced from a Bott connection D^E of the relative tangent bundle $T_{E/M}$. If E has a Finsler metric, then there exists a canonical connection $\hat{\nabla}$ in $T_{E/M}$, and the Finsler geometry on E is the differential geometry of the bundle $T_{E/M}$ with the connection $\hat{\nabla}$.



In the previous paper [7], we have studied Finsler geometry from the point of view of Kähler fibration, and characterized the flatness of complex Finsler metrics in terms of Finsler connection. The main purpose of this paper is to define the notion of *projective flatness* of Finsler metrics and to characterize it in terms of the *projective curvature* of $\hat{\nabla}$.

1.1. Projectively flat Hermitian metrics. We recall the notion of projective flatness of vector bundles and Hermitian metrics (for details, see [14]). We denote by $E^\times$ the open submanifold of a holomorphic vector bundle E consisting from non-zero elements. The multiplicative group $\mathbb{C}^\times = \mathbb{C} - \{0\}$ acts on $E^\times$ by scalar multiplication. The projective bundle $\mathbb{P}(E)$ associated with E is defined by $\mathbb{P}(E) = E^\times/\mathbb{C}^\times$ with the structure group $PGL(r,\mathbb{C}) := GL(r,\mathbb{C})/\mathbb{C}^\times I$. Then E is said to be *projectively flat* if $\mathbb{P}(E)$ admits a flat structure, i.e., E admits an open cover $\{U, s_U\}$ whose transition functions A_{UV} are of the forms

$$A_{UV} = c_{UV} \otimes C_{UV} \tag{1.5}$$

on $U \cap V$, where $\{c_{UV} : U \cap V \to \mathcal{O}^*_{U\cap V}\}$ are 1-cocycles and $C_{UV} : U \cap V \to GL(r, \mathbb{C})$ is locally constant. As a characterization of projectively flat bundles, the following is well-known (cf. Proposition 2.8 in p. 7 of [14]).

Proposition 1.1. *A complex vector bundle is projectively flat if and only if E admits a connection $\nabla : E \to E \otimes \Omega^1_M$ whose curvature Ω is of the form*

$$\Omega = \frac{1}{r}\mathrm{tr}(\Omega) \otimes I. \tag{1.6}$$

If the curvature form Ω of a connection ∇ is given by the form (1.6), then its connection form ω is given by

$$\omega = a \otimes I \tag{1.7}$$

for a local 1-form a with respect to certain open cover $\{U, s_U\}$ of E. In fact, if we take another local frame field $\tilde{s}_U = s_U A_U$ for some $A_U : U \to GL(r, \mathbb{C})$, the connection form $\tilde{\omega}$ relative to $\tilde{s}_U$ is given by $\tilde{\omega} = A_U^{-1} dA_U + A_U^{-1} \omega A_U$. Hence the condition $\tilde{\omega} = a \otimes I$ is equivalent to $aA_U = dA_U + \omega A_U$. The integrability condition $d(dA_U) \equiv 0$ for the existence of such A_U is given by the condition (1.6).

Definition 1.1. A connection ∇ in a complex vector bundle E is said to be *projectively flat* if its curvature Ω is of the form (1.6).

We shall explain this situation from classical view-point (cf. [19]). On each open set U with $s_U = (s_1, \ldots, s_r)$, we say that the direction of a section $\xi(t) = \sum \xi^i(t) s_i(t)$ along a smooth curve $c(t)$ is *projectively parallel* with respect to a connection ∇ if it satisfies $\nabla_{\dot{c}(t)}\xi = \lambda(\dot{c}(t))\xi$. If we put $\lambda(\dot{c}(t)) = \lambda(t)$, then this condition is written as

$$\frac{d\xi^i}{dt} + \sum \omega^i_j(\dot{c}(t))\xi^j = \lambda(t)\xi^i.$$

Suppose that the direction of ξ is also parallel with respect to another connection $\tilde{\nabla}$. This means that any section ξ satisfying

$$\xi^i \left(\frac{d\xi^h}{dt} + \sum \omega^h_j(\dot{c}(t))\xi^j \right) - \xi^h \left(\frac{d\xi^i}{dt} + \sum \omega^i_j(\dot{c}(t))\xi^j \right) = 0$$

also satisfies the following

$$\xi^i\left(\frac{d\xi^h}{dt}+\sum\tilde{\omega}^h_j(\dot{c}(t))\xi^j\right)-\xi^h\left(\frac{d\xi^i}{dt}+\sum\tilde{\omega}^i_j(\dot{c}(t))\xi^j\right)=0$$

for an arbitrary regular curve $c(t)$. Then we have $\xi^i\sum(\omega^h_j-\tilde{\omega}^h_j)\xi^j-\xi^h\sum(\omega^i_j-\tilde{\omega}^i_j)\xi^j=0$, and from this we get $\omega^i_j=\tilde{\omega}^i_j+a\delta^i_j$ with $a=\mathrm{tr}(\omega-\tilde{\omega})/r$. Hence there exists a 1-form a satisfying

$$\omega=\tilde{\omega}+a\otimes I$$

for the connections forms ω and $\tilde{\omega}$ of ∇ and $\tilde{\nabla}$ respectively. In this case, we say that ∇ is *projectively related* to $\tilde{\nabla}$. If ∇ is projectively related to a connection $\tilde{\nabla}$, the curvature Ω of ∇ is related to the one $\tilde{\Omega}$ of $\tilde{\nabla}$ by $\Omega=\tilde{\Omega}+A\otimes I$ for $A=da$. Since $A=\{\mathrm{tr}(\Omega)-\mathrm{tr}(\tilde{\Omega})\}/r$, the 2-form

$$\Theta=\Omega-\frac{1}{r}\mathrm{tr}(\Omega)\otimes I \tag{1.8}$$

is invariant by the projective change $\nabla\to\tilde{\nabla}$. This form Θ is called the *projective curvature* of ∇. From (1.7), a connection ∇ is projectively flat if and only if ∇ is projectively related to a flat connection $\tilde{\nabla}$. Moreover, from (1.8) we have

Proposition 1.2. *A connection ∇ is projectively flat if and only if its projective curvature Θ vanishes identically.*

A Hermitian metric g on E is said to be *projectively flat* if its Hermitian connection ∇ is projectively flat. If we denote by $g_{i\bar{j}}=g(s_i,s_j)$ the components of g with respect to the open cover $\{U,s_U\}$, the Hermitian connection ∇ is given by the $(1,0)$-form $\theta^i_j=\sum g^{i\bar{m}}g_{j\bar{m}}$, and its curvature $\Omega=(\Omega^i_j)$ is given by $\Omega^i_j=\bar{\partial}\theta^i_j$. Since the Ricci form $\mathrm{tr}(\Omega)$ of (E,g) is given by $\mathrm{tr}(\Omega)=\bar{\partial}\partial\log\det(g_{i\bar{j}})$, the condition (1.6) is written as

$$\Omega=\frac{1}{r}\bar{\partial}\partial\log\det(g_{i\bar{j}})\otimes I.$$

On each open set U, we put $\sigma_U=r^{-1}\log\det(g_{i\bar{j}})$. The metric $g_U:=e^{\sigma_U(z)}g$ is a flat metric on $E|_U$. Hence g is projectively flat if and only if g is (locally) conformally flat (cf. [15]).

Let $\mathcal{X}$ and M be connected complex manifolds of $\dim_{\mathbb{C}} \mathcal{X} = n + r$ and $\dim_{\mathbb{C}} M = n$, and let $p : \mathcal{X} \to M$ be a holomorphic map of maximal rank n everywhere. We suppose that each fibre $p^{-1}(z) = \mathcal{X}_z$ is connected. The family $\mathcal{X} = \{\mathcal{X}_z\}$ is considered as a family of complex manifold of $\dim_{\mathbb{C}} \mathcal{X}_z = r$ parameterized by $z \in M$. We say that $p : \mathcal{X} \to M$ a *Kähler fibration* if each fibre $\mathcal{X}_z$ is a Kähler manifold with a Kähler metric $\mathit{\Pi}_z$, where $\mathit{\Pi}_z$ is assumed to be parameterized smoothly by $z \in M$.

A typical example of Kähler fibration is the projective bundle $\mathbb{P}(E) \to M$ associated to an Hermitian bundle (E, g) over M. In a Hermitian vector bundle (E, g), if we put $F(z, \xi) = \sum g_{i\bar{j}}(z)\xi^i\bar{\xi}^j$, we have a Kähler fibration $\pi : \mathbb{P}(E) \to M$ with Kähler metrics

$$\mathit{\Pi}_z = \sqrt{-1}\,\frac{\partial^2 \log F}{\partial\xi^i\partial\bar{\xi}^j} d\xi^i \wedge d\bar{\xi}^j. \tag{1.9}$$

Since, if we fix a point $z_0 \in M$, we can take an orthonormal frame s_{z_0} at z_0, the Kähler metric $\mathit{\Pi}_{z_0}$ can be written as $\mathit{\Pi}_{z_0} = \sqrt{-1}\partial\bar{\partial}\log\left(\sum \delta_{ij}\xi^i\bar{\xi}^j\right) = \sqrt{-1}\partial\bar{\partial}\log\left(\sum|\xi^i|^2\right)$, the Fubini-Study metric $\mathit{\Pi}_{FS}$ on $\mathbb{P}_{z_0} = \mathbb{P}^{r-1}$. We can not, however, take a frame field s_U on U so that $\mathit{\Pi}_z = \mathit{\Pi}_{FS}$ at every point $z \in U$.

We suppose that (E, g) is projectively flat. Since the projective-flatness of g is equivalent to the local conformal-flatness, there exists an open cover $\{U, s_U\}$ of E and local functions σ_U on each U such that $\tilde{g}_U = e^{\sigma_U(z)}g$ defines a flat metric on E_U. Then, if we take a suitable frame field s_U on U, we may assume that $\tilde{g}_{i\bar{j}} = \delta_{ij}$ and $\tilde{F}_U = e^{\sigma_U(z)}F$ is given by $\tilde{F}_U = \sum \tilde{g}_{i\bar{j}}\xi^i\bar{\xi}^j = \sum|\xi^i|^2$ at each point on U. Since $\log\left(\sum|\xi^i|^2\right) = \sigma_U(z) + \log F(z, \xi)$ and the Kähler metrics $\mathit{\Pi}_z$ are given by (1.9), the Kähler metrics on $\mathbb{P}_z$ induced from $\log F$ and $\log \tilde{F}_U$ coincide each other, i.e., $\mathit{\Pi}_z = \mathit{\Pi}_{FS}$, and thus $\mathit{\Pi}_z$ is independent of base point $z \in U$.

Definition 1.2. We say that a Kähler fibration $p : \mathcal{X} \to M$ is *flat* if, at each point $z \in M$ there exists an open neighborhood U of z so that we can choose Kähler potentials for $\mathit{\Pi}_z$ which is independent of $z \in U$. Such a pseudo-Kähler metric $\mathit{\Pi}_{\mathcal{X}} = \{\mathit{\Pi}_z\}$ is said to be *flat*.

Then, from the discussion above we have

Proposition 1.3. *If a holomorphic vector bundle E admits a projectively flat Hermitian metric, then its projective bundle $\mathbb{P}(E)$ is a flat Kähler fibration.*

By Proposition 2.2 in the below, the metric on E corresponding to a pseudo-Kähler metric $\mathit{\Pi}_{\mathbb{P}(E)}$ on $\mathbb{P}(E)$ is a Finsler metric, not a Hermitian metric in general. The converse of Proposition 1.3 will be proved in the last section.

1.2. Bott connections. A connection h_E in the sequence (1.1) does not necessarily define a connection ∇ in the bundle E so long as h_E is not linear. However, any connection h_E defines a connection $\nabla : T_{E/M} \to T_{E/M} \otimes \Omega^1_E$ in the relative tangent bundle $\varpi : T_{E/M} \to E$. To show this, we recall the notion of *partial connection*. A morphism $D^E : T_{E/M} \to T_{E/M} \otimes \mathcal{H}^*$ is called a *partial connection* if the Leibnitz condition $D^E(fs) = d^h_E f \otimes s + f D^E s$ is satisfied for $^\forall s \in T_{E/M}$ and $^\forall f \in C^\infty(E)$. A connection h_E in the sequence (1.1) defines a partial connection D^E on the relative tangent bundle $T_{E/M}$.

Definition 1.3. Let h_E be a connection in the sequence (1.1). The *Bott connection* of h_E is a partial connection $D^E : T_{E/M} \to T_{E/M} \otimes \mathcal{H}^*$ of $(1,0)$-type defined by

$$D^E_X Y = \langle [X, Y] \rangle \tag{1.10}$$

for all $X \in \mathcal{H}$ and $Y \in T_{E/M}$, where $\langle \cdot \rangle : T_E \to T_{E/M}$ is the natural projection.

By direct calculations, for $Y = \sum Y^i (\partial/\partial \xi^i) \in T_{E/M}$, we have

$$D^E Y = \sum \left(d^h_E Y^i + \sum \hat{\omega}^i_j Y^j \right) \otimes \frac{\partial}{\partial \xi^i} \tag{1.11}$$

for the $(1,0)$-form $\hat{\omega}^i_j$ defined by the horizontal $(1,0)$-form $\hat{\omega}^i_j = \sum \mathit{\Gamma}^i_{j\alpha} dz^\alpha$, where we put

$$\mathit{\Gamma}^i_{j\alpha} = \frac{\partial N^i_\alpha}{\partial \xi^j} \tag{1.12}$$

for the coefficients $\{N^i_\alpha\}$ of h_E. By the homogeneity (1.4), the connection form $\hat{\omega} = (\hat{\omega}^i_j)$ satisfies the homogeneity $\hat{\omega}(z, \lambda\xi) = \hat{\omega}(z, \xi)$. In terms of $\hat{\omega}$, the connection h_E is expressed as

$$\partial^h \xi^i = -\sum \hat{\omega}^i_j \xi^j. \tag{1.13}$$

The Bott connection D^E defined by a connection h_E in the sequence (1.1) is extended to an ordinary connection $\hat{\nabla}$ in $T_{E/M}$. In fact, since $T_{E/M} \cong \pi^{-1}E$, the relative tangent bundle $T_{E/M}$ admits a relatively flat connection $D^0 : T_{E/M} \to T_{E/M} \otimes \Omega^1_{E/M}$ defined by $D^0(\pi^{-1}s) = 0$ for every $s \in E$. The connection $\hat{\nabla} : T_{E/M} \to T_{E/M} \otimes \Omega^1_E$ is given by

$$\hat{\nabla} = D^E \oplus D^0. \tag{1.14}$$

For any section $Y = \sum Y^i(\partial/\partial\xi^i) \in T_{E/M}$, the covariant differential $\hat{\nabla}Y$ is given by

$$\hat{\nabla}Y = \sum \left(d_E Y^i + \sum \hat{\omega}^i_j Y^j \right) \otimes \frac{\partial}{\partial \xi^i}.$$

Since the curvature Ω^D of D^E is defined by $\Omega^D = d^h_E\hat{\omega} + \hat{\omega} \wedge \hat{\omega}$, the curvature $\Omega^{\hat{\nabla}}$ of $\hat{\nabla}$ is given by

$$\Omega^{\hat{\nabla}} = \Omega^D + d^v_E\hat{\omega}. \tag{1.15}$$

2. Finsler geometry

2.1. Finsler metrics. A *Finsler metric* or *Minkowski metric* $f(\xi) = f(\xi^1, \ldots, \xi^r)$ on $\mathbb{C}^r$ is a function satisfying the following conditions:

(1) $f(\xi) \geq 0$ for all $\xi \in \mathbb{C}^r$, and the equality holds if and only if $\xi = 0$,

(2) f is smooth on $\mathbb{C}^r - \{0\}$,

(3) $f(\lambda\xi) = |\lambda|^2 f(\xi)$ for all $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{C}^r$,

(4) f is pluri-subharmonic, that is, $\sqrt{-1}\partial\bar{\partial}f > 0$.

We show that any Finsler metric on $\mathbb{C}^r$ $(r \geq 2)$ induces a Kähler metric on the complex projective space $\mathbb{P}^{r-1}$. We denote by $\rho : \mathbb{C}^r - \{0\} \to \mathbb{P}^{r-1}$ the natural projection. The tangent bundle $T_{\mathbb{P}^{r-1}}$ is locally spanned by the vector fields $\{\rho_*(\partial/\partial\xi^i)\}$ with the relation

$$\rho_* \left(\sum xi^i \frac{\partial}{\partial \xi^i} \right) = 0. \tag{2.1}$$

Let $H_{\mathbb{P}^{r-1}} = \mathcal{O}_{\mathbb{P}^{r-1}}(1)$ be the hyperplane bundle over $\mathbb{P}^{r-1}$. We identify the fibre $H_{[\xi]} = \mathcal{O}_{[\xi]}(1)$ over $[\xi] \in \mathbb{P}^n$ with the set of homogeneous functions of order 1 on $\rho^{-1}([\xi])$. For the tautological line bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ over $\mathbb{P}^{r-1}$, the Euler sequence $0 \longrightarrow \mathcal{O}_{\mathbb{P}^{r-1}}(-1) \longrightarrow \mathcal{O}^{\oplus r} \longrightarrow \mathcal{O}_{\mathbb{P}^{r-1}}(-1) \otimes T_{\mathbb{P}^{r-1}} \longrightarrow 0$ implies

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{r-1}} \xrightarrow{\ i\ } H^{\oplus r}_{\mathbb{P}^{r-1}} \xrightarrow{\ \sigma\ } T_{\mathbb{P}^{r-1}} \longrightarrow 0, \tag{2.2}$$

where the surjective morphism $\sigma : H^{\oplus r}_{\mathbb{P}^{r-1}} \to T_{\mathbb{P}^{r-1}}$ is defined by

$$\sigma(X^1, \ldots, X^r) = \rho_* \left(\sum X^i(\xi) \frac{\partial}{\partial \xi^i} \right).$$

By the relation (2.1), the bundle $\mathcal{O}_{\mathbb{P}^{r-1}}$ is the trivial line bundle locally spanned by $\mathcal{E} = (\xi^1, \ldots, \xi^r)$. Since f satisfies $\sqrt{-1}\partial\bar{\partial} f > 0$, we define a Hermitian metric $\langle \,\cdot\, , \,\cdot\, \rangle$ on $H^{\oplus r}_{\mathbb{P}^{r-1}}$ by

$$\langle X, Y \rangle = \frac{1}{f(\xi)} \sum \frac{\partial^2 f}{\partial \xi^i \partial \bar{\xi}^j} X^i \overline{Y^j}$$

for sections $X = (X^1, \ldots, X^r)$ and $Y = (Y^1, \ldots, Y^r)$ of $H^{\oplus r}_{\mathbb{P}^{r-1}}$. With respect to this Hermitian metric, we get an orthogonal decomposition $H^{\oplus r}_{\mathbb{P}^{r-1}} = T_{\mathbb{P}^{r-1}} \oplus \mathcal{O}_{\mathbb{P}^{r-1}}$. Since $\langle \mathcal{E}, \mathcal{E} \rangle = 1$, we decompose $\sigma(X) = \tilde{X}$ orthogonally as $\tilde{X} = X - \langle X, \mathcal{E} \rangle \mathcal{E}$. Then it induces a Hermitian metric $\langle \,\cdot\, , \,\cdot\, \rangle_{\mathbb{P}^{r-1}}$ by $\langle \tilde{X}, \tilde{Y} \rangle_{\mathbb{P}^r} = \langle X, Y \rangle - \langle X, \mathcal{E} \rangle \langle \mathcal{E}, Y \rangle$ which is written as

$$\langle \tilde{X}, \tilde{Y} \rangle_{\mathbb{P}^{r-1}} = \big(\partial\bar{\partial} \log f\big)\big(\tilde{X}, \overline{\tilde{Y}}\big).$$

Hence any Finsler metric f on $\mathbb{C}^r$ determines a Kähler metric on the projective space $\mathbb{P}^{r-1}$. Let $(\zeta^1, \ldots, \zeta^{r-1})$ be the inhomogeneous coordinate on $U_j = \big\{[\xi] \in \mathbb{P}^{r-1} \mid \xi^i \neq 0\big\}$. We put $g_j(\zeta^1, \ldots, \zeta^{r-1}) := \log f(\xi) - \log |\xi^j|^2$ on U_j. Since $\sqrt{-1}\partial\bar{\partial} g_i = \sqrt{-1}\partial\bar{\partial} g_j$ on $U_i \cap U_j$, the real $(1,1)$-form $\sqrt{-1}\partial\bar{\partial} g_i$ defines the Kähler metric $\langle \,\cdot\, , \,\cdot\, \rangle_{\mathbb{P}^{r-1}}$. The functions $\{g_j\}$ are called the *Kähler potentials* of $\langle \,\cdot\, , \,\cdot\, \rangle_{\mathbb{P}^{r-1}}$. We note that the functions $G_j = f(\xi)|\xi^j|^{-2}$ satisfy $|\xi^i|^2 G_i(\xi) = |\xi^j|^2 G_j$ on $U_i \cap U_j$, and thus the family $\{G_j\}$ defines a Hermitian metric on H with positive curvature.

Conversely, from any Kähler metric $\sqrt{-1}\partial\bar{\partial} g_j$ on $\mathbb{P}^{r-1}$, we get a Finsler metric f on $\mathbb{C}^r$. In fact, since $H^1(\mathbb{P}^{r-1}, \mathcal{O}^{r-1}_{\mathbb{P}}) = 0$, we can take g_j satisfying $|\xi^i|^2 \exp g_i = |\xi^j|^2 \exp g_j$ on $U_i \cap U_j$. Then the function $f(\xi) = |\xi^j|^2 \exp g_j$

defines a convex Finsler metric on $\mathbb{C}^r$. We suppose that we get another Finsler metric $\tilde{f}$ from another Kähler potential $\{\tilde{g}_j\}$. Then, since $\sqrt{-1}\partial\bar{\partial}\tilde{g}_j = \sqrt{-1}\partial\bar{\partial}g_j$, the function $\log\tilde{f} - \log f$ is pluri-harmonic function on $\mathbb{P}^{r-1}$. If we denote by $\mathcal{F}$ the sheaf of germs of pluri-harmonic functions on $\mathbb{P}^{r-1}$, the exact sequence $0 \longrightarrow \mathbb{R} \overset{\times\sqrt{-1}}{\longrightarrow} \mathcal{O} \overset{Re}{\longrightarrow} \mathcal{F} \longrightarrow 0$ of sheaves on $\mathbb{P}^n$ implies the long exact sequence of cohomology groups

$$0 \longrightarrow H^0(\mathbb{P}^{r-1},\mathbb{R}) \longrightarrow H^0(\mathbb{P}^{r-1},\mathcal{O}_{\mathbb{P}^{r-1}})$$
$$\longrightarrow H^0(\mathbb{P}^{r-1},\mathcal{F}) \longrightarrow H^1(\mathbb{P}^{r-1},\mathbb{R}) \longrightarrow \dots .$$

The identifications $H^0(\mathbb{P}^{r-1},\mathbb{R}) \cong \mathbb{R}$, $H^0(\mathbb{P}^{r-1},\mathcal{O}_{\mathbb{P}^{r-1}}) \cong \mathbb{C}$ and $H^1(\mathbb{P}^{r-1},\mathbb{R}) \cong \mathbb{R}$ imply the identification $H^0(\mathbb{P}^{r-1},\mathcal{F}) = \mathbb{R}$. Hence any pluri-harmonic function on $\mathbb{P}^{r-1}$ is a constant c. Consequently we have $\tilde{f} = e^c f$. Hence we have

Proposition 2.1 ([7]). *Any Kähler metric on the complex projective space $\mathbb{P}^{r-1}$ determines a Finsler metric on $\mathbb{C}^r$ uniquely up to the multiple by a positive constant.*

Example 2.1. If $f(\xi) = \sum|\xi^i|^2$, then it induces a flat metric $ds^2 = \sqrt{-1}\sum d\xi^i \wedge d\bar{\xi}^i$ on $\mathbb{C}^r$. The induced Kähler metric $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{P}^{r-1}}$ is called the *Fubini-Study metric* and given by the form

$$\Pi_{FS} = \sqrt{-1}\partial\bar{\partial}\log\left(1+\sum_{i=1}^{r-1}|\zeta^i|^2\right).$$

Conversely, the Fubini-Study metric on $\mathbb{P}^{r-1}$ induces a flat Hermitian metric on $\mathbb{C}^r$ uniquely up to a positive constant.

A complex Finsler metric on a vector bundle is defined as follows.

Definition 2.1. A *Finsler metric* F on a homomorphic vector bundle $\pi : E \to M$ is a smooth assignment of Finsler metrics f_z to each fibre $E_z \cong \mathbb{C}^r$. The pair (E, F) is called a *Finsler bundle.*

It is easily shown that if a Finsler metric F is given on E, then $\mathbb{P}(E)$ admits a pseudo-Kähler form $\Pi_{\mathbb{P}(E)} = \sqrt{-1}\partial\bar{\partial}\log F$. We shall show that the converse is also true. For this purpose, we take an open covering $\{(U, s_U)\}$

on E which induces complex coordinate systems $(z^1, \dots, z^n, \xi^1, \dots, \xi^r)$ on $\pi^{-1}(U)$ and $(z^1, \dots, z^n, \zeta_j^1, \dots, \zeta_j^{r-1})$ on $U_j = \{(z, [\xi]) \in p^{-1}(U) \mid \xi^j \neq 0\}$.

Let $L = \mathcal{O}_{\mathbb{P}(E)}(-1)$ the tautological line bundle over $\mathbb{P}(E)$. The restriction of L to each fibre $\mathbb{P}_z$ is the tautological line bundle $\mathcal{O}_{\mathbb{P}_z}(-1)$ over $\mathbb{P}_z \cong \mathbb{P}^{r-1}$. The sequence (2.2) is true for each fibre $\mathbb{P}_z$, that is,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_z} \xrightarrow{\ i\ } H_{\mathbb{P}_z}^{\oplus r} \xrightarrow{\ \sigma\ } T_{\mathbb{P}_z} \longrightarrow 0.$$

Thus, on each $\mathbb{P}_z$, we can construct a Kähler form $\mathit{\Pi}_z$ on $\mathbb{P}_z$. If each $\mathit{\Pi}_z$ depends on $z \in M$ smoothly, then the family $\{\mathit{\Pi}_z\}$ defines a pseudo-Kähler form $\mathit{\Pi}_{\mathbb{P}(E)}$ on $\mathbb{P}(E)$. If we put $\mathit{\Pi}_{\mathbb{P}(E)} = \sqrt{-1}\partial\bar{\partial} g_j$ on $\mathbb{P}(E)$, then we can construct a Finsler metric F on E by

$$F(z, \xi) = |\xi^j|^2 \exp g_j(z, [\xi]).$$

We note that another Kähler potential $\{\tilde{g}_j\}$ for $\mathit{\Pi}$ which induces the Kähler metric $\mathit{\Pi}_z$ on each $\mathbb{P}_z$ is given by

$$\tilde{g}_j(z, [\xi]) = \sigma_U(z) + g_j(z, [\xi]) \tag{2.3}$$

for some functions $\sigma_U(z)$ defined on U. Hence the Finsler metric $\tilde{F}$ determined from the potential $\{\tilde{g}_j\}$ is connected to the function F by the relation $\tilde{F} = e^{\sigma_U(z)} F$ on each U. Consequently we have

Proposition 2.2. *Any pseudo-Kähler metric on $\mathbb{P}(E)$ determines a Finsler metric on E uniquely up to the multiple by a positive function on M.*

2.2. Finsler connections. Each fibre of a Finsler bundle (E, F) is a vector space $E_z \cong \mathbb{C}^r$ with a Finsler metric f_z. By definition f_z is parameterized smoothly by points of the base manifold M. Since $F(z, \xi) = f_z(\xi)$, the real $(1,1)$-form $\sqrt{-1}\partial\bar{\partial} F$ defines a pseudo-Kähler form on E which induces a Kähler metric on each fibre E_z. Hence the bundle $\pi_E : E \to M$ is a Kähler fibration with a pseudo-Kähler form $\sqrt{-1}\partial\bar{\partial} F$. We put

$$F_{i\bar{j}}(z, \xi) = \frac{\partial^2 F}{\partial \xi^i \partial \bar{\xi}^j}.$$

Then, since the locally $\partial\bar{\partial}$-exact real $(1,1)$-form $\sqrt{-1}\partial\bar{\partial} F$ is positive definite on each fibre E_z, the Hermitian matrix $(F_{i\bar{j}})$ defines a Hermitian metric G on the bundle $\varpi : T_{E/M} \to E$ by

$$G\left(\frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j}\right) = F_{i\bar{j}}.$$

In the sequel we consider the bundle $\varpi : T_{E/M} \to E$ with the Hermitian metric G.

$$\begin{array}{ccc} T_{E/M} \cong \pi^* E & \longrightarrow & E \\ \varpi \downarrow & & \downarrow \pi \\ E & \xrightarrow{\pi} & M \end{array}$$

We shall determine a connection $h_E : \pi^* T_M \to T_E$ in the sequence (1.1) so that the induced connection $\hat{\nabla}$ in $T_{E/M}$ satisfies the metrical condition

$$d_E^h G(Y, Z) = G(\hat{\nabla} Y, Z) + G(Y, \hat{\nabla} Z) \tag{2.4}$$

for all $Y, Z \in T_{E/M}$. Since $\hat{\nabla}$ is of $(1,0)$-type, we have $d_E^h F_{i\bar{j}} = \sum F_{m\bar{j}} \hat{\omega}_i^m + F_{i\bar{m}} \overline{\hat{\omega}_j^m}$. Hence the connection form $\hat{\omega} = (\hat{\omega}_j^i)$ of $\hat{\nabla}$ is given by $\hat{\omega}_j^i = \sum F^{i\bar{m}} \partial_E^h F_{j\bar{m}}$. From (1.13), the coefficients of h_E is defined by $\sum N_\alpha^i dz^\alpha = \sum \hat{\omega}_j^i \xi^j$, and thus we have

$$N_\alpha^i = \sum F^{i\bar{m}} \frac{\partial F_{j\bar{m}}}{\partial z^\alpha} \xi^j. \tag{2.5}$$

The horizontal lifts X_α of $\partial/\partial z^\alpha$ are given by

$$X_\alpha = \frac{\partial}{\partial z^\alpha} - \sum N_\alpha^m \frac{\partial}{\partial \xi^m}.$$

We denote by $X_{\bar{\alpha}}$ its complex conjugate $\overline{X_\alpha}$. The connection $\hat{\nabla} : T_{E/M} \to T_{E/M} \otimes \Omega_E^1$ induced from the connection h_E of (2.5) is called the *Finsler connection* of (E, F). We also denote by $\mathcal{E}$ the section of $T_{E/M}$ which spans the line bundle $\ker\{\rho_* : T_{E/M} \to T_{\mathbb{P}(E)}\}$, i.e., $\mathcal{E} = \sum \xi^m (\partial/\partial \xi^m)$. Then, the equation (1.13) shows that $\hat{\nabla}\mathcal{E} \equiv 0$, and from $F(z, \xi) = G(\mathcal{E}, \mathcal{E})$ and (2.4) we have $d_E^h F = \sum X_\alpha F dz^\alpha + \sum X_{\bar{\alpha}} F d\bar{z}^\alpha \equiv 0$. Then we have

Lemma 2.1 ([3]). *Let (E, F) be a Finsler bundle, and $\hat{\omega}$ the connection form of the Finsler connection $\hat{\nabla}$. Then we have*

(1) $\partial^h \hat{\omega} + \hat{\omega} \wedge \hat{\omega} \equiv 0.$

(2) $\Omega^D = \bar{\partial}^h_E \hat{\omega}$.

(3) $\Omega^{\hat{\nabla}} = \Omega^D + d^v_E \hat{\omega}$.

A Finsler bundle (E, F) is said to be *modeled on a complex Minkowski space* $(\mathbb{C}^r, f)$ if the connection h_E defined by N^i_α in (2.5) is linear(cf. [11]). Then h_E induces a connection ∇ in E. We recall some results from [1] and [2].

We fix a point $z_0 \in M$ and identify the fibre (E_{z_0}, f_{z_0}) as a Minkowski space $(\mathbb{C}^r, f)$. If we set

$$G = \left\{ A \in GL(r, \mathbb{C}) \mid f(A\xi) = f(\xi),\ {}^\forall \xi \in \mathbb{C}^r \right\},$$

then G is a compact subgroup of unitary group $U(r)$. Let $z \in M$ be an arbitrary point and $c = c(t)$ be a smooth curve connecting $z_0 = c(0)$ and $z = c(1)$. We can assume without loss of generality that the points z and z_0 are contained in a neighborhood (U, s_U). Let $\xi(t)$ be a parallel field of E along the curve c. Since

$$\frac{d\xi^i}{dt} + \sum_{j,\alpha} \xi^j \Gamma^i_{j\alpha}(c(t)) \frac{dz^\alpha}{dt} = 0, \tag{2.6}$$

we have

$$\frac{d}{dt}\|\xi(t)\|^2 = \frac{d}{dt} F(c(t), \xi(t)) = \sum \left(X_\alpha F \frac{dz^\alpha}{dt} + X_{\bar{\alpha}} F \frac{d\bar{z}^\alpha}{dt} \right) = 0.$$

This shows that the parallel displacement P_c along the curve c is norm-preserving. Hence each fibre (E_z, f_z) is congruent to a fixed Minkowski space $(\mathbb{C}^r, f)$, and the holonomy group H is a subgroup of the compact Lie group G. Then there exists a $GL(r, \mathbb{C})$-valued function $A_U : U \to GL(r, \mathbb{C})$ satisfying

$$F(z, \xi) = f(A_U(z)\xi). \tag{2.7}$$

Since $f(A_U(z)\xi) = f(A_V(x)\xi)$ on $U \cap V$, we see that the local frame fields $\{\tilde{s}_U = s_U A_U^{-1}\}$ define a G-structure on E.

On the other hand, by using Szabó's idea (cf. [18]), we can construct a Hermitian metric g_F compatible with the connection $\hat{\nabla}$. In fact, for an arbitrary Hermitian inner product $(\cdot, \cdot)$ in E_{z_0}, we define a G-invariant Hermitian inner product $\langle \cdot\, , \cdot \rangle_0$ in E_{z_0} by

$$\langle \eta, \zeta \rangle_0 = \int_G (g\eta, g\zeta) dg$$

for all $\eta, \zeta \in E_0$ and for a bi-invariant Haar measure dg of G. Then, since the holonomy group H with reference point z_0 is contained in G, this Hermitian inner product $\langle \cdot, \cdot \rangle_0$ is extended to a Hermitian metric g_F on E defined on the whole of M by

$$g_F(\xi, \eta) = \langle P_c^{-1}\xi, P_c^{-1}\eta \rangle_0.$$

This Hermitian metric g_F is compatible with $\hat{\nabla}$. Hence we have

Theorem 2.1 ([1]). *We suppose that a Finsler bundle (E, F) is modeled on a complex Minkowski space $(\mathbb{C}^r, f)$. Then*

(1) *the metric F is of the form* (2.7),

(2) *the structure group of E is reducible to the Lie group G,*

(3) *there exists a Hermitian metric g_F on E which is compatible with the connection $\hat{\nabla}$.*

A Finsler metric F on E which is modeled on a complex Minkowski space $(\mathbb{C}^r, f)$ can be written in the form (2.7). We consider the case where the connection $\hat{\nabla}$ is flat. In this case, we can assume that the neighborhoods $\{U, s_U\}$ can be chosen so that the connection form $\omega^i_j = \sum \Gamma^i_{j\alpha}(z)dz^\alpha$ vanishes on each U. Hence the differential equation (2.6) is simplified as $d\xi^i/dt = 0$, and thus the components $\xi^i(t)$ of parallel field $\xi(t)$ along a curve $c(t)$ are constant on $c(t)$. Hence the function $A_U : U \to GL(r, \mathbb{C})$ in (2.7) is constant. Consequently, with respect to such a neighborhood $(U, \tilde{s}_U)$, the metric F is independent of the base point $z \in M$. The following definition is a generalization of real case in [17].

Definition 2.2. A Finsler bundle (E, F) is said to be *flat* or *locally Minkowski* if it is locally isometric to a Minkowski space $(\mathbb{C}^r, f)$, i.e., E admits an open cover $\{(U, s_U)\}$ with respect to which the metric F depends only on the fibre point ξ not on the base point z.

If (E, F) is flat, then from (2.5) it is easily shown that its Finsler connection $\hat{\nabla}$ is flat. In the previous papers [3] and [7], we have shown the following:

Proposition 2.3. *A Finsler bundle (E, F) is flat if and only if its Finsler connection $\hat{\nabla}$ is flat, i.e., (E, F) is modeled on a complex Minkowski space and its associated Hermitian metric g_F is flat.*

3. Projectively flat Finsler metrics

3.1. Projectively flat Finsler metrics. Similarly to (1.8), the projective curvature $\hat{\Theta}$ of Finsler connection $\hat{\nabla}$ is defined by

$$\hat{\Theta} = \Omega^{\hat{\nabla}} - \frac{1}{r}\mathrm{tr}(\Omega^{\hat{\nabla}}) \otimes I.$$

Definition 3.1. A Finsler metric F is said to be *projectively flat* if its projective curvature $\hat{\Theta}$ vanishes identically.

We suppose that F is projectively flat, i.e., $\hat{\Theta} \equiv 0$. If we put

$$\begin{aligned}\frac{1}{r}\mathrm{tr}(\Omega^{\hat{\nabla}}) &= \sum A_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \sum A_{\alpha k} dz^\alpha \wedge \theta^k + \sum A_{\alpha\bar{k}} dz^\alpha \wedge \bar{\theta}^k \\ &:= A,\end{aligned}$$

the curvature $\Omega^{\hat{\nabla}}$ is given in the form $\Omega^{\hat{\nabla}} = A \otimes I$. To investigate the projective-flatness of F in local coordinates, we compute the curvature $\Omega^{\hat{\nabla}} = \bar{\partial}^h \hat{\omega} + d^v \hat{\omega}$. The components $\hat{\Omega}^i_j = \bar{\partial}^h \hat{\omega}^i_j + \partial^v \hat{\omega}^i_j + \bar{\partial}^v \hat{\omega}^i_j$ of $\Omega^{\hat{\nabla}}$ are given by

$$\bar{\partial}^h \hat{\omega}^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \quad \partial^v \hat{\omega}^i_j = \sum R^i_{j\alpha k} dz^\alpha \wedge \theta^k,$$
$$\bar{\partial}^v \hat{\omega}^i_j = \sum R^i_{j\alpha\bar{k}} dz^\alpha \wedge \bar{\theta}^k,$$

where the coefficients of $\hat{\Omega}^i_j$ are given by

$$R^i_{j\alpha\bar{\beta}} = X_{\bar{\beta}} \Gamma^i_{j\alpha}, \quad R^i_{j\alpha k} = \frac{\partial \Gamma^i_{j\alpha}}{\partial \xi^k}, \quad R^i_{j\alpha\bar{k}} = \frac{\partial \Gamma^i_{j\alpha}}{\partial \bar{\xi}^k}.$$

For local computations, we shall state some formulas. From the homogeneity (1.4), differentiating (1.4) with respect to λ and $\bar{\lambda}$ respectively, we have

$$\sum \frac{\partial N^i_\alpha}{\partial \xi^j} \xi^j = N^i_\alpha, \quad \sum \frac{\partial N^i_\alpha}{\partial \bar{\xi}^j} \bar{\xi}^j = 0. \tag{3.1}$$

Moreover, from the homogeneity $\Gamma^i_{j\alpha}(z, \lambda\xi) = \Gamma^i_{j\alpha}(z, \xi)$, we have

$$\sum \frac{\partial \Gamma^i_{j\alpha}}{\partial \xi^k} \xi^k = 0, \quad \sum \frac{\partial \Gamma^i_{j\alpha}}{\partial \bar{\xi}^k} \bar{\xi}^k = 0. \tag{3.2}$$

Lemma 3.1. *If the Finsler connection $\hat{\nabla}$ in (E,F) is projectively flat, then (E,F) is modeled on a complex Minkowski space.*

Proof. From $R^i_{j\alpha k} = A_{\alpha k}\delta^i_j$ and the first identity of (3.2) we have

$$A_{\alpha k}\xi^i = \sum \xi^j \frac{\partial \Gamma^i_{j\alpha}}{\partial \xi^k} = \sum \xi^j \frac{\partial \Gamma^i_{k\alpha}}{\partial \xi^j} = 0.$$

Here we used the homogeneity $\Gamma^i_{j\alpha}(z,\lambda\xi) = \Gamma^i_{j\alpha}(z,\xi)$ for $\lambda \in \mathbb{C}$. Hence we get $A_{\alpha k} = 0$, and thus $R^i_{j\alpha k} = 0$. Moreover, from $R^i_{j\alpha\bar{k}} = A_{\alpha\bar{k}}\delta^i_j$ we have

$$A_{\alpha\bar{k}}\xi^i = \sum \xi^j R^i_{j\alpha\bar{k}} = \sum \xi^j \frac{\partial \Gamma^i_{j\alpha}}{\partial \bar{\xi}^k} = \frac{\partial N^i_\alpha}{\partial \bar{\xi}^k}. \tag{3.3}$$

Hence, from the second identity of (3.1) we have $\left(\sum A_{\alpha\bar{k}}\bar{\xi}^k\right)\xi^i = 0$, and thus we get $\sum A_{\alpha\bar{k}}\bar{\xi}^k = 0$. We also consider the tensor field N_α on each fibre E_z defined by

$$N_\alpha = \sum \frac{\partial N^i_\alpha}{\partial \bar{\xi}^j} d\bar{\xi}^j \otimes \frac{\partial}{\partial \xi^i}.$$

Since each fibre E_z has a Hermitian metric $(F_{i\bar{j}})$, the norm $\|N_\alpha\|_z$ of N_α is naturally defined. Then, by the condition (3.3), we have

$$\left\|N_\alpha\right\|^2_z = \sum \left(A_{\alpha\bar{j}}\xi^i\right)\overline{A_{\alpha\bar{i}}\xi^j} = \sum (A_{\alpha\bar{j}}\bar{\xi}^j)\cdot \overline{(A_{\alpha\bar{i}}\bar{\xi}^i)} = 0,$$

from which we have $N_\alpha = 0$, and so $A_{\alpha\bar{i}} = 0$ from (3.3). Consequently we get $R^i_{j\alpha k} = R^i_{j\alpha\bar{k}} = 0$. Thus $d^v\omega = 0$, that is, (E,F) is modeled on a complex Minkowski space. □

On the other hand, the condition $\hat{\Theta} \equiv 0$ is equivalent to that there exists a suitable open covering $\{(U, s_U)\}$ of E such that the connection form $\hat{\omega}$ of the Finsler connection $\hat{\nabla}$ is of the form $\hat{\omega} = a_U \otimes I$ for a $(1,0)$-form a_U on each $\pi^{-1}(U)$. Then we have

Lemma 3.2. *If the Finsler connection $\hat{\nabla}$ in (E,F) is projectively flat, then there exists a local function $\sigma_U : U \to \mathbb{R}$ such that $\Omega^{\hat{\nabla}} = \partial\bar{\partial}\sigma_U \otimes I$ on U.*

Proof. By Lemma 3.1, if $\hat{\nabla}$ is projectively flat, then (E,F) is modeled on a complex Minkowski space. Hence, by Theorem 2.1, there exists a Hermitian metric $g_F = (g_{i\bar{j}}(z))$ on E such that the connection form $\hat{\omega}^i_j$ of $\hat{\nabla}$ is given by $\hat{\omega}^i_j = \sum g^{i\bar{m}} \partial g_{j\bar{m}}$ and the form a_U is given by

$$a_U = \frac{1}{r}\,\mathrm{tr}(\hat{\omega}) = \frac{1}{r}\partial \log\big(\det(g_{i\bar{j}})\big).$$

If we put $\sigma_U = r^{-1}\log\det(g_{i\bar{j}})$ on each U, the connection form is given by $\hat{\omega}^i_j = \partial\sigma_U \otimes \delta^i_j$, and its curvature $\Omega^{\hat{\nabla}}$ is given by $\Omega^{\hat{\nabla}} = \partial\bar{\partial}\sigma_U \otimes I$. □

3.2. Main theorems. We suppose that (E, F) is projectively flat. Then, by the proof of Lemma 3.2, there exists a local function $\sigma_U(z)$ on U such that the curvature $\hat{\Omega}$ is written as $\hat{\Omega} = \partial\bar{\partial}\sigma_U(z) \otimes I$. Then we can show hat the local metric $\tilde{F}_U = e^{\sigma_U(z)}F(z,\xi)$ is a flat Finsler metric on U, i.e., $F(z,\xi) = e^{-\sigma_U(z)}\tilde{F}(\xi)$. Then, from (1.9), the Kähler metrics Π_z are given by $\Pi_z = \sqrt{-1}\partial\bar{\partial}\log\tilde{F}(\xi)$ which shows that $\{\mathbb{P}_z, \Pi_z\}$ is a flat Kähler fibration.

Conversely we suppose that F is induced from a flat pseudo-Kähler metric on $\mathbb{P}(E)$. Then, from (2.3), there exists a local function $\sigma_U(z)$ on each U such that

$$\tilde{g}_j = \log\left(\frac{1}{|\xi^j|^2}F(z,\xi)\right) - \sigma_U(z) \tag{3.4}$$

is independent of the base point $z \in M$. If we put $\tilde{F}_U(\xi) = |\xi^j|^2 \exp\tilde{g}_j([\xi])$, we have

$$F(z,\xi) = e^{\sigma_U(z)}\tilde{F}_U(\xi) \tag{3.5}$$

on each $\pi^{-1}(U)$. In this case, the connection $h : \pi^*T_M \to T_E$ is given by

$$N^i_\alpha(z,\xi) = \frac{\partial\sigma_U}{\partial z^\alpha}\xi^i,$$

and the Finsler connection $\hat{\nabla}$ in (E, F) is given by $\hat{\omega}^i_j = \partial\sigma_U \otimes \delta^i_j$ on each $\pi^{-1}(U)$. Hence its curvature $\Omega^{\hat{\nabla}}$ is given by the form $\hat{\Omega}^i_j = \partial\bar{\partial}\sigma_U \otimes \delta^i_j$. This shows that the Finsler connection $\hat{\nabla}$ is projectively flat.

Theorem 3.1. *A complex Finsler metric F is projectively flat if and only F is induced from a flat pseudo-Kähler metric on $\mathbb{P}(E)$.*

By Lemma 3.1 and 3.2, we have also proved the following.

Corollary 3.1. *A Finsler bundle (E, F) is projectively flat if and only if it is modeled on a complex Minkowski space and its associated Hermitian metric g_F is projectively flat.*

Corollary 3.1, Proposition 2.3 and Proposition 1.3 imply the following.

Theorem 3.2. *Let E be a holomorphic vector bundle over a complex manifold M. The projective bundle $p : \mathbb{P}(E) \to M$ is a flat Kähler fibration if and only if E admits a projectively flat Hermitian metric.*

From (3.5), the projective flatness of Finsler metrics is equivalent to the conformal-flatness in the sense of [3]. Then we get an example of projectively flat Finsler metrics.

Example 3.1 (cf. [4]). Let $M = \mathbb{C}^{\times n}/\lambda_{\mathbb{Z}}$ be the Hopf manifold. The tangent bundle T_M admits a projectively flat Finsler metric. In fact, for an arbitrary Finsler metric $f : \mathbb{C}^n \to \mathbb{R}$, the function $F : T_M \to \mathbb{R}$ given by

$$F_0(z, \xi) = e^{-\log \|z\|^2} f(\xi)$$

defines a projectively flat Finsler metric on T_M, and an associated Hermitian metric is given by the Boothby metric $ds^2 = e^{-\log \|z\|^2} \sum dz^j \otimes d\bar{z}^j$. The projective bundle $\mathbb{P}(T_M) \to M$ is a flat Kähler fibration.

4. Some remarks

In this last section, we shall consider the case where M is a compact Riemann surface and $f : \mathcal{X} \to M$ a geometrically ruled surface. Every geometrically ruled surface over M is isomorphic to $\mathbb{P}(E)$ for some holomorphic vector bundle $E \to M$ of rank two.

For the degree $\int_M c_1(E) := \deg(E)$ of a holomorphic vector bundle E over M, its *degree/rank ratio* of E is defined by $\mu(E) := \deg(E)/\mathrm{rank}(E)$. A holomorphic vector bundle E is said to be *stable* (in the sense of Mumford) if it satisfies $\mu(E') < \mu(E)$ for an arbitrary proper sub-bundle E' satisfying $0 < \mathrm{rank}(E') < \mathrm{rank}(E)$.

We fix a Kähler metric $g = g_{1\bar{1}} dz \otimes d\bar{z}$ of M. A Hermitian metric h on E is said to be *weak Einstein–Hermitian* if its curvature form $\Omega^i_j = R^i_{j1\bar{1}} dz \wedge d\bar{z}$ satisfies $g^{\bar{1}1} R^i_{j1\bar{1}} = \varphi \delta^i_j$ for a function φ. Hence the curvature form is written as the form (1.6) for the 2-form $A = \varphi g_{1\bar{1}} dz \wedge d\bar{z}$, which shows that (E, h) is projectively flat. The converse is also true. On the other hand, by [8], a holomorphic vector bundle E is stable if and only if it admits a projectively flat Hermitian metric h. Hence the following three conditions are equivalent (see (2.7) Theorem on p. 140 of [14]):

1. E is stable in the sense of Mumford,
2. E admits a weak Einstein–Hermitian metric h,
3. E admits a projectively flat Hermitian metric h.

On the other hand, by Theorem 3.2, a holomorphic vector bundle E admits a projectively flat Hermitian metric if and only if $\mathbb{P}(E)$ admits a flat pseudo-Kähler metric $\Pi_{\mathbb{P}(E)}$. Hence the statement above, we have

Proposition 4.1. *A geometrically ruled surface $f : \mathcal{X} = \mathbb{P}(E) \to M$ is a flat Kähler fibration if and only if the bundle E is stable in the sense of Mumford.*

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