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# Arithmetic and metric properties of $p$-adic Engel series expansions

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**Abstract.** We derive a characterization of rational numbers in terms of their unique $p$-adic Engel series expansions. Thereafter we investigate metric properties for the rational digits occurring in these $p$-adic Engel expansions. In particular, we obtain limiting distributions for the $p$-adic order of the digits and the $p$-adic order of approximation by the partial sums of the series expansions.

## 1. Introduction

Let $\mathbb{Q}$ be the field of rational numbers, $p$ a prime number and $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the $p$-adic absolute value $|\ |_p$ defined on $\mathbb{Q}$ by (cf. Koblitz [8] or Schikhof [12])

$$|0|_p = 0 \quad \text{and } |A|_p = p^{-a} \quad \text{if } A = p^a \frac{r}{s}, \quad \text{where } p \nmid rs. \tag{1.1}$$

The exponent $a$ in this definition is the $p$-adic valuation of $A$, which we denote by $v_p(A)$.

It is well known that every $A \in \mathbb{Q}_p$ has a unique series representation

$$A = \sum_{n=v_p(A)}^{\infty} c_n p^n, \quad c_n \in \{0, 1, 2, \dots, p-1\}.$$

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In the discussion below we call the finite sum $\langle A\rangle = \sum_{v_p(A)\le n\le 0} c_n p^n$ the *fractional* part of $A$. Then $\langle A\rangle \in S_p$, where we define $S_p = \{\langle A\rangle : A \in \mathbb{Q}_p\} \subset \mathbb{Q}$. This set $S_p$ is neither multiplicatively nor additively closed.

Recently the fractional part $\langle A\rangle$ was used by A. Knopfmacher and J. Knopfmacher [5, 6], to derive some new unique series expansions for any element $A \in \mathbb{Q}_p$, including in particular analogues of certain "Sylvester", "Engel" and "Lüroth" expansions of arbitrary *real* numbers into series with rational terms (cf. [10], Chap. IV). In the corresponding case of $p$-adic Lüroth type expansions ergodic and other metric properties have recently been investigated by A. and J. Knopfmacher [7]. For both the $p$-adic continued fractions and Lüroth expansions, ergodicity of the corresponding transformations were used to derive the results. However, in the case of Engel expansions the underlying transformation is not ergodic. The growth conditions satisfied by the digits suggest that an approach via Markov chains could be used. A similar approach was used to study metric properties of Engel expansions over the field of formal Laurent series over a finite field in [4].

Given $A \in \mathbb{Q}_p$, now note that $\langle A\rangle = a_0 \in S_p$ iff $v_p(A_1) \ge 1$ where $A_1 = A - a_0$. As in [5], if $A_n \ne 0$ $(n > 0)$ is already defined, we then let

$$a_n = \left\langle \frac{1}{A_n} \right\rangle \quad \text{and put} \quad A_{n+1} = a_n A_n - 1. \tag{1.2}$$

If some $A_m = 0$, this recursive process stops. It was shown in [5] that this algorithm leads to a finite or convergent (relative to $v_p$) Engel-type series expansion

$$A = a_0 + \frac{1}{a_1} + \sum_{r\ge 2} \frac{1}{a_1 \cdots a_r}, \tag{1.3}$$

where $a_r \in S_p$, $a_0 = \langle A\rangle$, and $v_p(a_{r+1}) \le v_p(a_r) - 1$ for $r \ge 1$. Furthermore this expansion is *unique* for $A$ subject to the preceding conditions on the "digits" $a_r$. For notational convenience we set

$$\frac{p_n}{q_n} = a_0 + \sum_{r=1}^{n} \frac{1}{a_1 \cdots a_r}, \quad \text{where} \quad q_n = a_1 \dots a_n.$$

In [6] it was claimed (see Proposition 4.2) that rational numbers are characterized by finite $p$-adic Engel expansions. However, the following example of an infinite $p$-adic Engel expansion for a rational number shows that this result is incorrect. For convenience in the sequel we denote the $p$-adic Engel expansion

$$\frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots \quad \text{by } (a_1, a_2, a_3, \ldots).$$

Then the rational number $-p$ has the infinite expansion

$$-p = \left( \frac{p^2-1}{p}, \frac{p^3-1}{p^2}, \frac{p^4-1}{p^3}, \ldots \right). \tag{1.4}$$

More generally, if $\beta, r \in \mathbb{N}$, then, if $p^{r+1} > \beta$, the rational number $-\frac{p^r}{\beta}$ has the expansion

$$-\frac{p^r}{\beta} = \left( \frac{p^{r+1}-\beta}{p^r}, \frac{p^{r+2}-\beta}{p^{r+1}}, \frac{p^{r+3}-\beta}{p^{r+2}}, \ldots \right). \tag{1.5}$$

These expansions follow by induction from the Engel algorithm above with $A = A_1 = -\frac{p^r}{\beta}$, $A_n = -\frac{p^{n+r-1}}{\beta}$, and $a_n = \frac{p^{r+n}-\beta}{p^{r+n-1}}$.

It turns out that rational numbers with infinite Engel expansions all have digits that ultimately follow the pattern in (1.5).

**Theorem 1.** *Let $A \in p\mathbb{Z}_p \setminus \{0\}$. Then $A$ is rational, $A = \frac{\alpha}{\beta}$, if and only if either the Engel expansion of $A$ is finite, or there exists an $n$ and an $r \geq n$, such that*

$$a_{n+j} = \frac{p^{r+j+1}-\gamma}{p^{r+j}} \quad \textit{for } j = 0, 1, 2, \ldots \tag{1.6}$$

*where $\gamma \mid \beta$.*

Let $I$ denote the valuation ideal $p\mathbb{Z}_p$ in the ring of $p$-adic integers $\mathbb{Z}_p$ and let $\mathbb{P}$ denote probability with respect to the Haar measure on $(\mathbb{Q}_p, +)$ normalized by $\mathbb{P}(I) = 1$. The Haar measure on $I$ is the product measure on $\{0, \ldots, p-1\}^{\mathbb{N}}$ defined by $\mathbb{P}\big(\{x\}\big) = p^{-1}$ for each factor and any element $x \in \{0, \ldots, p-1\}$.

We now state our main results.

**Theorem 2.** *The following assertions hold:*

(i) *The valuations of the Engel-digits $a_n$ obey a law of large numbers; more precisely, for almost all $x \in I$*

$$\lim_{n\to\infty} \frac{v_p(a_n)}{n} = -\frac{p}{p-1}.$$

(ii) *The valuations of the Engel-digits $a_n$ obey a central limit theorem:*

$$\lim_{n\to\infty} \mathbb{P}\left[x \in I : \frac{v_p(a_n) + \frac{p}{p-1}n}{\sqrt{np}/(p-1)} < t\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

(iii) *For almost all $x \in I$,*

$$\limsup_{n\to\infty} \frac{-v_p(a_{n+1}(x)) + v_p(a_n(x))}{\log_p n} = 1,$$

*and*

$$\liminf_{n\to\infty} \left(-v_p(a_{n+1}(x)) + v_p(a_n(x))\right) = 1.$$

(iv) $v_p(x - \frac{p_n}{q_n})$ *obeys a law of large number; more precisely, for almost all $x \in I$,*

$$\frac{1}{n^2} v_p\left(x - \frac{p_n}{q_n}\right) \to \frac{p}{2(p-1)}, \quad \text{as} \quad n \to \infty.$$

(v) $v_p(x - \frac{p_n}{q_n})$ *obeys central limit theorem*

$$\lim_{n\to\infty} \mathbb{P}\left[x \in I : \frac{v_p\left(x - \frac{p_n}{q_n}\right) - \frac{p}{p-1}\frac{(n+1)(n+2)}{2}}{\sqrt{\mathbb{V}(v_p(q_{n+1}))}} < t\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

*where* $\mathbb{V}(v_p(q_{n+1})) = \frac{(n+1)(n+2)(2n+3)}{6} \frac{p}{(p-1)^2}$.

In particular we see from (i) that for almost all $x \in I$, $|a_n|_p^{1/n} \to p^{\frac{p}{p-1}}$, as $n \to \infty$. Regarding (i), (ii), (iv), and (v) above we note the similar but weaker results shown in [5] holding for all $x$ in $I$,

$$v_p(a_n) \le -n$$

and

$$\left|x - \frac{p_n}{q_n}\right|_p \le p^{-\frac{(n+1)(n+2)}{2}}, \quad n = 1, 2, 3, \ldots.$$

Furthermore, we consider the random variables $\left|\frac{a_{r+1}(x)}{a_r(x)}\right|_p \equiv p^{\Delta_r}$, $r = 1, 2, 3, \ldots$ . These are independent and identically distributed with infinite expectation. However, the following result holds.

**Theorem 3.** *For any fixed* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[x \in I : \left|\frac{1}{n\log_p n}\sum_{r=1}^{n}\left|\frac{a_{r+1}(x)}{a_r(x)}\right|_p - (p-1)\right| > \varepsilon\right] = 0$$

*i.e.* $\frac{1}{n\log_p n}\sum\limits_{r=1}^{n}\left|\frac{a_{r+1}(x)}{a_r(x)}\right|_p \to (p-1)$ *in probability over* $I$.

*Remark 1.* Since a theorem in Galambos [3] (p. 46), implies that either

$$\limsup_{n\to\infty} \frac{1}{n\log_p n}\sum_{r=1}^{n}\left|\frac{a_{r+1}(x)}{a_r(x)}\right|_p = \infty \quad \text{a.e.}$$

or

$$\liminf_{n\to\infty} \frac{1}{n\log_p n}\sum\left|\frac{a_{r+1}(x)}{a_r(x)}\right|_p = 0 \quad \text{a.e.},$$

the conclusion of Theorem 3 does not carry over to validity with probability one.

The paper is organized into sections, which split the proofs of the theorems. Section 2 treats rationality criteria and the proof of Theorem 1. Section 3 gives some elementary probabilities, which will be used in the subsequent proofs, Section 4 gives the proof of Theorem 2 and Section 5 gives the proof of Theorem 3.

## 2. Rationality criteria (Proof of Theorem 1)

Firstly, if $A \in p\mathbb{Z}_p \setminus \{0\}$ has an infinite Engel expansion, which satisfies (1.6) then

$$\begin{aligned} A &= \sum_{k=1}^{n-1}\frac{1}{a_1\cdots a_k} + \frac{1}{a_1\cdots a_{n-1}}\left(\frac{1}{a_n} + \frac{1}{a_n a_{n+1}} + \cdots\right) \\ &= \sum_{k=1}^{n-1}\frac{1}{a_1\cdots a_k} + \frac{1}{a_1\cdots a_{n-1}}\left(-\frac{p^r}{\beta}\right) \in \mathbb{Q}, \end{aligned}$$

using (1.5) and the uniqueness of the Engel expansion.

Now suppose $A = \frac{\alpha}{\beta} \in \mathbb{Q}$. By the algorithm for each $n \geq 1$, for which $A_n \neq 0$, $A_{n+1} = a_n A_n - 1$ and it follows that $A_n \in \mathbb{Q}$ for each $n \geq 1$. We will make use of the elementary inequality

$$0 < a_n < \sum_{k=0}^{\infty} (p-1)p^{-k} = p \quad \text{for all } a_n \in S_p. \tag{2.1}$$

By (2.1) we can write $a_n = b_n p^{v_p(a_n)}$, where $b_n \in \mathbb{N}$ and

$$0 < b_n < p^{-v_p(a_n)+1}. \tag{2.2}$$

Furthermore, each $A_n$ can be represented in the form $A_n = \frac{\alpha_n}{\beta_n} p^{-v_p(a_n)}$, where $\alpha_n \in \mathbb{Z}$, $\beta_n \in \mathbb{N}$, $(\alpha_n, \beta_n) = 1$, and $p \nmid \alpha_n \beta_n$. An analogous representation holds also for $A_{n+1}$, provided $A_{n+1} \neq 0$. Substituting these representations into (1.2) leads to

$$\frac{\alpha_{n+1} p^{-v_p(a_{n+1})}}{\beta_{n+1}} = \frac{b_n \alpha_n - \beta_n}{\beta_n}. \tag{2.3}$$

Since $(\alpha_{n+1} p^{-v_p(a_{n+1})}, \beta_{n+1}) = 1$ it follows that $\beta_{n+1} \mid \beta_n$. Thus by (2.1) and using $v_p(a_n) \leq -n$ we have

$$\begin{aligned} |\alpha_{n+1}| &\leq p^{v_p(a_{n+1})}(b_n|\alpha_n| + \beta_n) \\ &< p^{v_p(a_{n+1}) - c_p(a_n)+1}|\alpha_n| + p^{v_p(a_{n+1})}\beta_n < |\alpha_n| + \beta_n p^{-n-1}. \end{aligned} \tag{2.4}$$

By choosing $N$ large enough, so that $|\beta_1| p^{-N-1} < 1$, we have for all $n \geq N$ that $|\alpha_{n+1}| \leq |\alpha_n|$. Suppose that $\alpha_n \neq 0$ for all $n$. Then for all $n$ sufficiently large we have $|\alpha_n| = \alpha$ and $\beta_n = \gamma$, where $\alpha, \gamma \in \mathbb{N}$ and $\gamma \mid \beta_1$. Substituting this into (2.3) yields

$$\pm \frac{\alpha p^{-v_p(a_{n+1})}}{\beta} = \frac{\pm b_n \alpha - \gamma}{\gamma}$$

which implies that

$$\pm b_n = \frac{\gamma}{\alpha} \pm p^{-v_p(a_{n+1})}.$$

Since $b_n \in \mathbb{N}$ and $(\alpha, \gamma) = 1$, we must have $\alpha = 1$.

In the case that $\alpha_n = +1$ we get that $b_n = \gamma \pm p^{-v_p(a_{n+1})}$. Since $-v_p(a_{n+1}) \geq n+1$ and $b_n > 0$ we conclude that

$$b_n = \gamma + p^{-v_p(a_{n+1})} \geq \gamma + p^{1-v_p(a_n)} > p^{1-v_p(a_n)},$$

which contradicts (2.2).

In the case $\alpha_n = -1$ it follows from (2.3) that $\alpha_{n+1} = -1$ as well and hence $b_n = p^{-v_p(a_{n+1})} - \gamma < p^{1-v_p(a_n)}$ only if $v_p(a_{n+1}) = v_p(a_n) - 1$ for $n \geq N$, since $\gamma p^{-v_p(a_N)-1} < 1$. Consequently, in order for an infinite Engel expansion to exist it is necessary that $\alpha_{N+j} = -1$ $(j = 0, 1, \ldots)$, $v_p(a_{N+j}) = v_p(a_N) - j$, $A_{N+j} = -\frac{p^{v_p(a_N)}}{\gamma}$, and $a_{N+j} = b_{N+j} p^{v_p(a_N+j)} = \frac{p^{-v_p(a_N)+j+1} - \gamma}{p^{-v_p(a_N)+j}}$, which proves (1.6) with $n = N$ and $r = -v_p(a_N) \geq n$.

*Remark 2.* If there exists $m \in \mathbb{N}$ such that $A_n < 0$ then by (2.3) $A_{n+j} < 0$ for every $j \geq 1$. In particular, this implies that every negative rational number $A \in p\mathbb{Z}_p$ has an infinite $p$-adic Engel expansion of type (1.6).

*Remark 3.* For positive rational numbers $A \in p\mathbb{Z}_p$, both terminating and infinite expansions can occur. However, in the special case, when $A \in \mathbb{N}$ we see by induction that $A_n \geq 0$ for all $n$ and it then follows from the proof of Theorem 1 that every positive integer $A$ has a finite Engel expansion.

## 3. Basic probabilities

We begin by deriving some basic probabilistic results concerning the digits in $p$-adic Engel expansions.

**Lemma 1.** *The digits $a_n \in S_p$ form a Markov chain with initial probabilities*

$$\mathbb{P}\left[v_p(a_1) = -j\right] = (p-1)p^{-j}, \tag{3.1}$$

*and transition probabilities*

$$\mathbb{P}\left[v_p(a_{n+1}) = -k \mid v_p(a_n) = -j\right] = \begin{cases} (p-1)p^{j-k} & \text{for } k > j \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. Firstly by the Engel algorithm $A_1 = x \in I$. Then using the definition of Haar measure $\mathbb{P}\big[v_p(A_1) > j\big] = \mathbb{P}[v_p(a_1) < -j] = p^{-j}$. Thus $\mathbb{P}[v_p(a_1) = -j] = \mathbb{P}[v_p(a_1) < -(j-1)] - \mathbb{P}[v_p(a_1) < -j] = (p-1)p^{-j}$.

Next, $A_2$ is obtained from $A_1$ by a system of linear congruences to successively higher powers of $p$, arising from the relation $A_2 = a_1 A_1 - 1$. From this it follows that $A_2$ is uniformly distributed in $p^j I$ where $j = -v_p(a_1)$. Inductively, if $v_p(a_n) = -j$ then $A_{n+1}$ is uniformly distributed in $p^j I$ for all $n > 1$. Since the event $v_p(a_{n+1}) < -k$ under the condition that $v_p(a_n) = -j$ is just a cylinder set in the infinite product space $\{0, \ldots, p-1\}^{\mathbb{N}}$, which is described by fixing $k-j$ of the digits in the $p$-adic expansion of $A_2$ equal to 0, we conclude that

$$\mathbb{P}\left[v_p(a_{n+1}) < -k \mid v_p(a_n) = -j\right] = p^{j-k}$$

and (3.2) follows immediately. □

*Remark 4.* Since the probability in (3.2) depends only on the difference $k-j$ this implies that the random variables $v_p(a_n) - v_p(a_{n+1})$ are independent and identically distributed. Thus for

$$n_1 < n_2 < \cdots < n_j \quad \text{and} \quad k_i \geq 1,\ i = 1, 2, \ldots j,$$

$$\begin{aligned}\mathbb{P}\Big[&v_p(a_{n_j+1}) = v_p(a_{n_j}) - k_j, v_p(a_{n_{j-1}+1}) = v_p(a_{n_{j-1}}) - k_{j-1}, \ldots,\\ &v_p(a_{n_1+1}) = v_p(a_{n_1}) - k_1\Big] = (p-1)^j p^{-(k_1+\cdots+k_j)}.\end{aligned} \tag{3.3}$$

**Corollary 4.** *Let $\Delta_n = \Delta_n(x)$ denote the random variable $v_p(a_n) - v_p(a_{n+1})$, with $\Delta_0 = -v_p(a_1)$. Then*

$$\mathbb{P}\Big[\#\big\{1 \leq \ell \leq n \mid \Delta_\ell = 1\big\} = k\Big] = \binom{n}{k}\left(1 - \frac{1}{p}\right)^k p^{k-n}.$$

*Thus the number of times that degrees of consecutive digits increase by 1 has a binomial distribution with mean value $n\left(1 - \frac{1}{p}\right)$ and variance $n\,\frac{p-1}{p^2}$.*

In particular the lim inf result of part (ii) of Theorem 2 follows immediately.

**Corollary 5.** *The random variables $\Delta_n$ have mean value and variance*

$$\mathbb{E}(\Delta_n) = \frac{p}{p-1}$$

*and*

$$\mathbb{V}(\Delta_n) = \frac{p}{(p-1)^2}.$$

Proof. By Lemma 1

$$\mathbb{E}(\Delta_n) = \sum_{\ell=1}^{\infty} \ell\,\mathbb{P}[v_p(a_n) - v_p(a_{n+1}) = \ell] = (p-1)\sum_{\ell=1}^{\infty} \ell p^{-\ell} = \frac{p}{p-1}.$$

Similarly

$$\mathbb{E}(\Delta_n^2) = (p-1)\sum_{\ell=1}^{\infty} \ell^2 p^{-\ell} = \frac{p}{p-1} + 2\,\frac{p}{(p-1)^2}$$

from which the formula for $\mathbb{V}(\Delta_n)$ is immediate. □

**Lemma 2.** *The following equations hold:*

(i)

$$\mathbb{P}[v_p(a_n) = t] = (p-1)^n p^{-t}\binom{t-1}{n-1} \tag{3.4}$$

*and therefore*

$$\mathbb{P}\left[\,\exists n : v_p(a_n) = t\right] = 1 - \frac{1}{p}. \tag{3.5}$$

(ii) $$\mathbb{P}[v_p(a_{n+m}) = t \mid v_p(a_n) = s] = (p-1)^m p^{s-t}\binom{t-s-1}{m-1}.$$

Proof. First we prove statement (i). Since the sequence of degrees of the digits $a_1, a_2, \ldots$ is strictly increasing we have by Lemma 1,

$$\begin{aligned}
\mathbb{P}\left[v_p(a_n) = -t\right] &= \sum_{1\le j_1<j_2<\cdots<j_{n-1}<t} \mathbb{P}\left[v_p(a_n) = -t \mid v_p(a_{n-1}) = -j_{n-1}\right] \\
&\times \mathbb{P}\left[v_p(a_{n-1}) = -j_{n-1} \mid v_p(a_{n-2}) = -j_{n-1}\right] \\
&\cdots \mathbb{P}\left[v_p(a_2) = -j_2 \mid v_p(a_1) = -j_1\right]\mathbb{P}\left[v_p(a_1) = -j_1\right] \\
&= (p-1)^n \sum_{1\le j_1<j_2<\cdots<j_{n-1}<t} p^{j_{n-1}-t} p^{j_{n-2}-j_{n-1}} \ldots p^{j_1-j_2} p^{-j_1}, \\
&= (p-1)^n p^{-t} \sum_{1\le j_1<j_2<\cdots<j_{n-1}<t} 1 = (p-1)^n p^{-t}\binom{t-1}{n-1}.
\end{aligned}$$

Thus we have

$$\begin{aligned}\mathbb{P}\,[\,\exists n: v_p(a_n)=t\,] &= \sum_{n=1}^{\infty}(p-1)^n p^{-t}\binom{t-1}{n-1}\\ &= (p-1)p^{-t}\sum_{\ell=0}^{t-1}(p-1)^{\ell}\binom{t-1}{\ell} = 1-\frac{1}{p}.\end{aligned}$$

For the proof of (ii) we find

$$\begin{aligned}&\mathbb{P}\,[v_p(a_{n+m})=-t \mid v_p(a_n)=-s]\\ &\quad= \sum_{s<j_1<j_2<\cdots<j_{m-1}<t} \mathbb{P}\,[v_p(a_{n+m})=-\,t \mid v_p(a_{n+m-1})=-\,j_{m-1}]\\ &\qquad\cdots\mathbb{P}\,[v_p(a_{n+2})=-j_2 \mid v_p(a_{n+1})=-j_1]\\ &\qquad\times\mathbb{P}\,[v_p(a_{n+1})=-j_1 \mid v_p(a_n)=-s]\\ &\quad=(p-1)^m p^{s-t}\sum_{s<j_1<j_2<\cdots<j_{m-1}<t} 1\\ &\quad=(p-1)^m p^{s-t}\binom{t-s-1}{m-1}.\end{aligned}$$

□

*Remark 5.* From the proof of (i) we can also deduce the joint probability distribution

$$\mathbb{P}\,[v_p(a_1)=-j_1,\ldots,v_p(a_n)=-j_n]=(p-1)^n p^{-j_n},$$

provided that the growth condition $v_p(a_i)\leq -i$ holds for each $i=1,2,\ldots,n$. Otherwise the joint probability distribution has value 0.

**Lemma 3.** *Let $X_n$ be a sequence of independent, identically distributed random variables with $\mathbb{E}X_n=\mu$ and $\mathbb{V}X_n=\sigma^2$. Then*

$$\lim_{n\to\infty}\frac{2}{n(n+1)}\sum_{k=1}^{n}(n+1-k)X_k=\mu \quad \textit{almost surely.} \tag{3.6}$$

Proof. Under these hypotheses the law of large numbers

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}X_k=\mu \quad \text{almost surely}$$

holds. Since (3.6) is just the second order Césaro mean of the random variables $X_k$. Since the first order Césaro mean exists almost surely and equals $\mu$, so does the second order mean. □

## 4. Proof of Theorem 2

Since we can write $v_p(a_n)$ as the sum of independent random variables

$$v_p(a_n) = \sum_{i=1}^{n-1}(v_p(a_{i+1}) - v_p(a_i)) + v_p(a_1) = -\sum_{i=0}^{n-1} \Delta_i,$$

it follows from Corollary 2 that $v_p(a_n)$ has mean and variance

$$\mathbb{E}(v_p(a_n)) = -\frac{p}{p-1}\, n$$

and

$$\mathbb{V}(v_p(a_n)) = n\, \frac{p}{(p-1)^2},$$

respectively.

Hence by the law of large numbers and the central limit theorem (see e.g. Feller [2, pp. 244, 253]) parts (i) and (ii) of Theorem 2 follow.

For the proof of (iii) we note that the events $v_p(a_n) - v_p(a_{n+1}) > k(n)$ are independent with probabilities $\mathbb{P}\big[\Delta_n > k(n)\big] = p^{-k(n)}$. The Borel–Cantelli lemmas then yield

$$\mathbb{P}\big[\Delta_n > k(n) \text{ for infinitely many } n\big] = \begin{cases} 0, & \text{if } \sum\limits_{n=1}^{\infty} p^{-k(n)} \text{ converges} \\ 1 & \text{if } \sum\limits_{n=1}^{\infty} p^{-k(n)} \text{ diverges.} \end{cases}$$

By choosing $k(n) = c \log_p n$ we see that with probability 1 the events $\frac{(v_p(a_n) - v_p(a_{n+1}))}{\log_p n} > c$ occur infinitely often if $c \le 1$ and only finitely often if $c > 1$. The lim sup result then follows. The corresponding lim inf result was already shown in Section 2.

(iv), (v) We first compute the mean and variance of $v_p(x - \frac{p_n}{q_n})$. In [5] it is shown that

$$\left| x - \frac{p_n}{q_n} \right|_p = p^{-v_p(q_{n+1})}.$$

Now

$$\mathbb{E}(v_p(q_{n+1})) = \sum_{r=1}^{n+1} \mathbb{E}(v_p(a_n)) = \frac{p}{p-1}\, \frac{(n+1)(n+2)}{2}.$$

To compute the variance we make use of the fact that

$$v_p(q_{n+1}) = \sum_{r=1}^{n+1} v_p(a_r) = -\sum_{r=1}^{n+1}\sum_{l=0}^{r-1} \Delta_l \tag{4.1}$$

$$= -\sum_{l=0}^{n} \Delta_l (n+1-l). \tag{4.2}$$

We now remark that the last sum has the same distribution as the sum

$$-\sum_{l=0}^{n} (l+1)\Delta_l.$$

Thus we have for the variance

$$\mathbb{V}\left(v_p(q_{n+1})\right) = \sum_{l=0}^{n} (l+1)^2 \mathbb{V}\Delta_l = \frac{(n+1)(n+2)(2n+3)}{6} \frac{p}{(p-1)^2}.$$

Assertion (iv) now follows form Lemma 3.

For the proof of (v) we check that the random variables $(l+1)\Delta_l$ satisfy Lindeberg's condition (cf. [2, p. 256]): since $s_n^2 = \mathbb{V}(v_p(q_{n+1}))$ is of order of magnitude $n^3$, we have to compute the integrals

$$\int\limits_{|y|\geq tn^{3/2}} y^2\, dF_k(y) = (k+1)^2 \int_{|x|\geq \frac{tn^{3/2}}{k+1}} x^2\, dF(x)$$

$$\leq (k+1)^2 \int_{|x|\geq \frac{t}{2}\sqrt{n}} x^2\, dF(x),$$

where $F_k$ is the distribution function of $(k+1)(\Delta_k - \frac{p}{p-1})$ and $F = F_0$.

Thus the last integral is equal to the sum

$$\sum_{k\geq \frac{p}{p-1}+\frac{t}{2}\sqrt{n}} \left(k - \frac{p}{p-1}\right)^2 p^{-k} = O\left(np^{-\frac{t}{2}\sqrt{n}}\right)$$

for $n$ sufficiently large, and we have

$$\frac{1}{s_n^2} \sum_{k=0}^{n} \int\limits_{|y|\geq ts_n} y^2\, dF_k(y) = O\left(\frac{1}{n} p^{-\frac{t}{2}\sqrt{n}}\right) \to 0$$

for any $t > 0$. Thus

$$\frac{v_p(q_{n+1}) - \frac{p}{p-1}\frac{(n+1)(n+2)}{2}}{\sqrt{\mathbb{V}(v_p(q_{n+1}))}}$$

has asymptotically normal distribution and the proof is completed.

## 5. Proof of Theorem 3

We first notice that by Lemma 1 the random variables $|\frac{a_{r+1}(x)}{a_r(x)}|_p \equiv p^{\Delta_r}$ are independent and identically distributed with infinite expectation. We write $s = \log_p y$ iff $y = p^s$ and use the truncation method of FELLER [2, Chapter 10, § 2], applied to the random variables $U_r, V_r (r \le n)$ defined by

$$U_r(x) = \left|a_{r+1}/a_r(x)\right|_p, \ V_r(x) = 0 \text{ if } \left|a_{r+1}/a_r(x)\right|_p \le n \log_p n,$$

$$U_r(x) = 0, \ V_r(x) = \left|a_{r+1}/a_r(x)\right|_p \text{ if } \left|a_{r+1}/a_r(x)\right|_p > n \log_p n.$$

Then

$$\mathbb{P}\left[x \in I : \frac{1}{n\log_p n}\left|\sum_{r=1}^{n}\left|\frac{a_{r+1}(x)}{a_r(x)}\right|_p - (p-1)\right| > \varepsilon\right] \tag{5.1}$$

$$\le \mathbb{P}\left[\,\left|U_1 + \cdots + U_n - (p-1)n\log_p n\right| > \varepsilon n \log_p n\right] \tag{5.2}$$

$$+\, \mathbb{P}\left[V_1 + \cdots + V_n \ne 0\right], \tag{5.3}$$

and using Lemma 1,

$$\mathbb{P}\left[V_1 + \cdots + V_n \ne 0\right] \le n\mathbb{P}\left[\left|\frac{a_2(x)}{a_1(x)}\right|_p > n \log_p n\right] \tag{5.4}$$

$$= n \sum_{\substack{k \\ p^k > n\log_p n}} (p-1)p^{-k} \ll \frac{1}{\log_p n} = o(1). \tag{5.5}$$

Now note that

$$\mathbb{E}(U_1 + \cdots + U_n) = n\mathbb{E}(U_1), \ \mathbb{V}(U_1 + \cdots + U_n) = n\mathbb{V}(U_1),$$

where

$$\mathbb{E}(U_1) = \sum_{|\frac{a_2(x)}{a_1(x)}|_p \le n \log_p n} p^k \mathbb{P}[\Delta_1 = k] = \sum_{p^k \le n \log_p n} p^{-k}(p-1)p^k \tag{5.6}$$

$$= (p-1)\log_p\bigl([n \log_p n]\bigr), \tag{5.7}$$

and

$$\mathbb{V}(U_1) < \mathbb{E}(U_1^2) = \sum_{p^k \le n \log_p n} (p-1)p^k < qn \log_p n.$$

Chebyshev's inequality then yields

$$\mathbb{P}\Big[\big|U_1 + \cdots + U_n - n\mathbb{E}(U_1)\big| > \varepsilon n \mathbb{E}(U_1)\Big] \tag{5.8}$$

$$\le \frac{n\mathbb{V}(U_1)}{\bigl(\varepsilon n \mathbb{E}(U_1)\bigr)^2} < \frac{pn^2 \log_p n}{\Bigl(\varepsilon(p-1)n \log\bigl([n \log_p n]\bigr)\Bigr)^2} = o(1). \tag{5.9}$$

Since $\mathbb{E}(U_1) \sim (p-1)\log_p n$ as $n \to \infty$, Theorem 2 follows.

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