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Continuity of functions which are convex with respect to means

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Abstract. Some generalization of Bernstein–Doetsch and Sierpiński theorems from the theory of Jensen convex functions are presented.

1. Introduction

Throughout the paper let I and J be open intervals such that $J \subset I \subset \mathbb{R}$.

We say that a function $f : I \longrightarrow \mathbb{R}$ is *Jensen convex* iff

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \quad x, y \in I.$$

In 1915 Bernstein and Doetsch proved that every Jensen convex function $f : I \longrightarrow \mathbb{R}$ locally bounded above at a point $x_0 \in I$ is continuous [3]. Moreover, in 1920 Sierpiński proved that every Lebesgue measurable Jensen convex real function is continuous [12].

In this paper we transfer these results on a class of functions which are convex with respect to means satisfying some additional conditions. Generally the idea of proof is derived from [8] and [6]. Similar problems are considered also in [11].

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The next two Sections introduce the notions of means and functions convex with respect to a mean to the reader. The last Sections 4 and 5, contain main results and their applications.

2. Means

A function $M : I^2 \longrightarrow I$ is called a *mean* iff

$$\min\{x,y\} \le M(x,y) \le \max\{x,y\}, \quad x,y \in I.$$

If, moreover, for all $x \neq y$, $x,y \in I$ these inequalities are sharp, then M is said to be a *strict* mean.

If $M : I^2 \longrightarrow I$ is a mean, then

$$M(x,x) = x, \quad x \in I$$

and

$$M(K,K) = K$$

for every subinterval $K \subset I$.

We call a mean $M : I^2 \longrightarrow I$ *symmetric* iff

$$M(x,y) = M(y,x), \quad x,y \in I.$$

In the sequel the family of weighted quasi-arithmetic means and the logarithmic mean play a crucial role.

Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous and strictly monotonic function. Let $t \in (0,1)$ be fixed. A *weighted quasi-arithmetic mean* $M_{\varphi,t} : I^2 \longrightarrow I$ is defined by the formula

$$M_{\varphi,t}(x,y) = \varphi^{-1}(t\varphi(x) + (1-t)\varphi(y)), \quad x,y \in I.$$

The particular cases of this mean are among others: the*weighted arithmetic mean* $(\varphi(x) = x,\ x \in I,\ t \in (0,1))$ $M_t(x,y) = tx + (1-t)y$, $x,y \in I$; the *arithmetic mean* $\left(\varphi(x) = x,\ x \in I;\ t = \frac{1}{2}\right)$ $A(x,y) = \frac{x+y}{2}$, $x,y \in I$; the *geometric mean* $\left(\varphi(x) = \ln x,\ x \in I \cap (0,\infty);\ t = \frac{1}{2}\right)$ $G(x,y) = \sqrt{xy}$, $x,y \in I \cap (0,\infty)$ and the *harmonic mean* $\left(\varphi(x) = \frac{1}{x},\ x \in I \cap (0,\infty);\ t = \frac{1}{2}\right)$ $H(x,y) = \frac{2xy}{x+y}$, $x,y \in I \cap (0,\infty)$.

The weighted quasi-arithmetic mean is continuous and strictly increasing with respect to both variables ([1], as well as [2], [4]).

A *logarithmic mean* is called function $L : (0, \infty)^2 \longrightarrow (0, \infty)$ defined by formula

$$L(x,y) = \begin{cases} \dfrac{x-y}{\ln x - \ln y}, & x \neq y \\ x, & x = y \end{cases}, \quad x, y \in (0, \infty).$$

The logarithmic mean is not a quasi-arithmetic mean ([5]). It has, among others, the following properties ([10], [5], as well as [4]): L is a strict symmetric mean; for every $x > 0$ a function $L(x, \cdot\,)$ is an increasing homeomorphism of interval $(0, \infty)$ on itself and for every $x, y > 0$ the inequalities

$$\sqrt{xy} \leq L(x,y) \leq \frac{x+y}{2},$$

are fulfilled, besides the equalities hold iff $x = y$.

3. Functions convex with respect to a mean

Let $M : I^2 \longrightarrow I$ be a mean.

We say that a function $w : J \longrightarrow I$ is

(i) *M-convex* (or *convex with respect to the mean M*) iff

$$w(M(x,y)) \leq M(w(x), w(y)), \quad x, y \in J;$$

(ii) *M-concave* (or *concave with respect to the mean M*) iff

$$w(M(x,y)) \geq M(w(x), w(y)), \quad x, y \in J;$$

(iii) *M-affine* (or *affine with respect to the mean M*) iff

$$w(M(x,y)) = M(w(x), w(y)), \quad x, y \in J.$$

Notice that if $M(x,y) = \frac{x+y}{2}$, $x, y \in I$, then the above conditions determine Jensen convex (concave, affine) functions, and if $M_t(x,y) = tx + (1-t)y$, $x, y \in I$, $t \in (0,1)$, then fulfilling the above conditions for every $t \in (0,1)$ determines convex (concave, affine) functions.

Now let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous and strictly monotonic functions. Fix a $t \in (0,1)$. Function $w : J \longrightarrow I$ is called *$M_{\varphi,t}$-convex* iff

$$w(M_{\varphi,t}(x,y)) \leq M_{\varphi,t}(w(x),w(y)), \quad x,y \in J.$$

Function $w : J \longrightarrow I$ is called *M_{φ}-convex* iff w is $M_{\varphi,t}$-convex for every $t \in (0,1)$. We define *$M_{\varphi,t}$-concave*, *$M_{\varphi,t}$-affine*, *M_{φ}-concave* and *M_{φ}-affine* functions in a similar way.

Notice that if $\varphi = \mathrm{id}$, then M_{φ}-convexity determines classical convexity and hence M_{φ}-convexity is the generalization of convexity.

4. Main results

Let $M : I^2 \longrightarrow I$ be a mean.

In the sequel we will consider means M satisfying some of the following conditions:

(i) for every $x \in I$ functions $M(\,\cdot\,,x)$ and $M(x,\,\cdot\,)$ project open sets onto open sets;

(ii) there exists a $\lambda \in (0,1)$ such that for every $x,y \in I$

$$M(x,y) \leq \lambda \max\{x,y\} + (1-\lambda)\min\{x,y\};$$

(iii) for every $x,y \in I,\ x \neq y$

$$\min\{x,y\} < M(x,y);$$

(iv) for every $x \in J$ and for every $[c,d] \subset J$ such that $x \in (c,d)$ there exists a $\delta > 0$ such that for every $u \in (x-\delta, x+\delta)$ there exists $v \in (c,d)$ such that $x = M(v,u)$;

(v) for every $x \in I$ function $M(\,\cdot\,,x)$ is continuous;

(vi) for every positive Lebesgue measurable set $T \subset I$ the condition $\operatorname{int} M(T,T) \neq \emptyset$ is fulfilled;

(vii) for every $x \in I$ functions $M(\,\cdot\,,x)$ and $M(x,\,\cdot\,)$ are homeomorphisms of interval I in itself.

Note that if condition (vii) is satisfied, then $M(\,\cdot\,,x)$ and $M(x,\,\cdot\,)$ are strictly increasing functions. We start with useful two remarks which proofs will be omitted.

Remark 4.1. If $M : I^2 \longrightarrow I$ satisfies (ii) and (iii), then M is a strict mean.

Remark 4.2. If $M : I^2 \longrightarrow I$ satisfies (vii), then M also satisfies (i), (iv) and (v).

Before proving theorems joining local boundedness above of M-convex functions with continuity we prove the following

Lemma 4.3. *Let $M : I^2 \longrightarrow I$ be a mean satisfying* (i)–(v). *Let $f : J \longrightarrow I$ be M-convex function. If f is locally bounded above at $x_0 \in J$, then f is locally bounded above at every point of J.*

Proof. Let U_0 be a neighbourhood of x_0 for which there exists $k_0 \in \mathbb{R}$ such that $f(x) \leq k_0$ for $x \in U_0$. Fix arbitrarily $z_0 \in J$ and put $x_{n+1} = M(x_n, z_0)$, $n \in \mathbb{N} \cup \{0\}$. The sequence $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ is bounded and monotonic and hence converges. Let $\bar{z} = \lim_{n\to\infty} x_n$. From the definition of the sequence x_n and (v) we obtain $\bar{z} = M(\bar{z}, z_0)$. Applying Remark 4.1 we get $\bar{z} = z_0$. Put $U_{n+1} = M(z_0, U_n)$, $n \in \mathbb{N} \cup \{0\}$. By virtue of (i), $(U_n)_{n \in \mathbb{N} \cup \{0\}}$ is a sequence of open sets. We prove the existence of constants k_n, $n \in \mathbb{N} \cup \{0\}$ such that $f|_{U_n} \leq k_n$. For $n = 0$ it follows from the assumption. Now suppose that $f|_{U_n} \leq k_n$ and let $x \in U_{n+1}$. Then $x = M(z_0, u)$ for some $u \in U_n$. Hence, by M-convexity of f, (ii) and boundedness above of f on U_n we get

$$f(x) \leq M(f(z_0), f(u)) \leq \max(f(z_0), f(u)) \leq \max(f(z_0), k_n) = k_{n+1},$$

which shows that f is bounded above on each set U_n, $n \in \mathbb{N} \cup \{0\}$. Fix $\rho > 0$ such that $[z_0 - \rho, z_0 + \rho] \subset J$. From (iv) (for $x = z_0$, $c = z_0 - \rho$, $d = z_0 + \rho$) there exists a $\delta > 0$ such that $\rho \geq \delta$ and for each $u \in (z_0 - \delta, z_0 + \delta)$ there exists a $v \in (z_0 - \rho, z_0 + \rho)$ such that $z_0 = M(v, u)$. Let $N \in \mathbb{N}$ be choosen so that $x_N \in (z_0 - \delta, z_0 + \delta)$. Therefore there exists $w_0 \in (z_0 - \rho, z_0 + \rho)$ such that $z_0 = M(w_0, x_N)$. Notice that $w_0 \in J$. Now we put $W_0 = M(w_0, U_N)$. Since $x_N \in U_N$ and $z_0 = M(w_0, x_N)$, then W_0 is a neigbourhood of z_0. For every $w \in W_0$ we have $w = M(w_0, u)$ for some $u \in U_N$. Hence we get

$$f(w) \leq M(f(w_0), f(u)) \leq \max(f(w_0), f(u)) \leq \max(f(w_0), k_N) = k,$$

which shows that f is locally bounded above at the point z_0. This ends the proof. $\square$

Theorem 4.4. *Let $M : I^2 \longrightarrow I$ be a mean satisfying* (i), (ii) *and* (iv). *Let $f : J \longrightarrow I$ be M-convex function. If f is locally bounded above at $x_0 \in J$, then f is continuous at x_0.*

Proof. Let U_0 be a neighbourhood of x_0 for which there exists $k_1 \in \mathbb{R}$ such that $f(x) < k_1$ for $x \in U_0$. Put $k = k_1 - f(x_0)$ and fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$0 < \max\left\{\frac{k\lambda^{n+1}}{1-\lambda}, k\lambda^n\right\} < \varepsilon.$$

Let $U_{n+1} = M(U_n, x_0)$, $n \in \mathbb{N} \cup \{0\}$. It follows from (i) that $(U_n)_{n\in\mathbb{N}\cup\{0\}}$ is a sequence of open sets containing x_0. Without loss of generality we may assume that U_n are intervals for $n \in \mathbb{N} \cup \{0\}$. Now we show that for $n \in \mathbb{N} \cup \{0\}$ the following implication

$$x \in U_n \Longrightarrow f(x) - f(x_0) < k\lambda^n \tag{1}$$

is fulfilled. Let $n = 0$. Since $x \in U_0$, then $f(x) < k_1$. Hence $f(x) - f(x_0) < k_1 - f(x_0) = k\lambda^0$. Now we assume that (1) is fulfilled for some $n \in \mathbb{N} \cup \{0\}$. If $x \in U_{n+1}$, then $x \in M(U_n, x_0)$. Therefore there exists an $u \in U_n$ such that $x = M(u, x_0)$. By M-convexity of f, (ii) and the inductivity assumptionwe have

$$\begin{aligned}
f(x) &= f(M(u, x_0)) \leq M(f(u), f(x_0)) \\
&\leq \lambda \max\{f(x_0), f(u)\} + (1-\lambda)\min\{f(x_0), f(u)\} \\
&\leq \lambda \max\{f(x_0), f(x_0) + k\lambda^n\} + (1-\lambda)\min\{f(x_0), f(x_0) + k\lambda^n\} \\
&= \lambda(f(x_0) + k\lambda^n) + (1-\lambda)f(x_0) = f(x_0) + k\lambda^{n+1},
\end{aligned}$$

which completes the proof of (1). From (iv) (for $x = x_0$, $(c, d) = U_n$) arises the existence of a $\delta_n > 0$ such that for every $u \in (x_0 - \delta_n, x_0 + \delta_n)$ there exists a $v \in U_n$ such that $x_0 = M(v, u)$. Put

$$V_n = (x_0 - \delta_n, x_0 + \delta_n) \cap U_n.$$

Note that $(V_n)_{n\in\mathbb{N}\cup\{0\}}$ is a sequence of neighbourhoods of point x_0 and $V_n \subset U_n$ for $n \in \mathbb{N} \cup \{0\}$. We show that the following implication

$$x \in V_n \Longrightarrow f(x_0) - f(x) < \max\left\{\frac{k\lambda^{n+1}}{1-\lambda}, k\lambda^n\right\} \tag{2}$$

is fulfilled. Take $x \in V_n$. The choice of δ_n guarantees the existence $v \in U_n$ such that $x_0 = M(v, x)$. According to the M-convexity of f and (ii) we get

$$\begin{aligned} f(x_0) &= f(M(v,x)) \le M(f(v), f(x)) \\ &\le \lambda \max\{f(v), f(x)\} + (1-\lambda)\min\{f(v), f(x)\}. \end{aligned} \tag{3}$$

First, assume that $0 < \lambda \le \frac{1}{2}$. Hence, from the above inequality we get

$$f(x_0) \le \frac{f(v) + f(x)}{2},$$

which by virtue of (1) implies

$$f(x_0) - f(x) \le f(v) - f(x_0) < k\lambda^n.$$

Now we assume that $\frac{1}{2} < \lambda < 1$. If $f(x) \le f(v)$, then from (3) and (1) we get

$$\begin{aligned} f(x_0) &\le \lambda f(v) + (1-\lambda) f(x) \\ &= \lambda f(v) + f(x) - \lambda f(x) = \lambda(f(v) - f(x)) + f(x) \\ &< \lambda(k\lambda^n + f(x_0) - f(x)) + f(x). \end{aligned}$$

Therefore

$$f(x_0) - f(x) < k\lambda^{n+1} + \lambda(f(x_0) - f(x)),$$

or, equivalently,

$$f(x_0) - f(x) < \frac{k\lambda^{n+1}}{1-\lambda}.$$

If $f(x) > f(v)$, than again from (3) and (1) we get

$$\begin{aligned} f(x_0) &\le \lambda f(x) + (1-\lambda) f(v) < \lambda f(x) + (1-\lambda)(f(x_0) + k\lambda^n) \\ &= \lambda f(x) + (1-\lambda) f(x_0) + (1-\lambda) k\lambda^n. \end{aligned}$$

Hence

$$\lambda(f(x_0) - f(x)) < k\lambda^n(1-\lambda) \le k\lambda^{n+1},$$

and, consequently,

$$f(x_0) - f(x) < k\lambda^n,$$

which completes the proof of (2). Conditions (1) and (2) imply the continuity of f at point x_0. $\square$

The next theorem is a generalization of the Bernstein–Doetsch result.

Theorem 4.5. *Let $M : I^2 \longrightarrow I$ be a mean satisfying* (i)–(v). *If $f : J \longrightarrow I$ is M-convex functions locally bounded above at $x_0 \in J$, then f is continuous in J.*

Proof. It follows from Lemma 4.3 that function f is locally bounded above at every point of interval J, and hence on account of Theorem 4.4 it is continuous in J. $\square$

As a consequence we obtain an analogous result to Sierpiński.

Theorem 4.6. *Let $M : I^2 \longrightarrow I$ be a mean satisfying* (i)–(vi). *If $f : J \longrightarrow I$ is M-convex and Lebesgue measurable, then f is continuous.*

Proof. Put

$$T_n = f^{-1}((-\infty, n)) = \{x \in J : f(x) < n\}, \quad n \in \mathbb{N}.$$

$(T_n)_{n\in\mathbb{N}}$ is a sequence of Lebesgue measurable sets such that $T_n \subset T_{n+1}$, $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} T_n = J$. Let l denote the Lebesgue measure in $\mathbb{R}$. Notice that

$$0 < l(J) = l\left(\bigcup_{n\in\mathbb{N}} T_n\right) = \lim_{n\longrightarrow\infty} l(T_n).$$

Thus there exists an $N \in \mathbb{N}$ such that $l(T_N) > 0$. It follows from (vi) that $\operatorname{int} M(T_N, T_N) \neq \emptyset$. Let U be an open set contained in $\operatorname{int} M(T_N, T_N)$. Hence, if $u \in U$, then there exist $t_1, t_2 \in T_N$ such that $u = M(t_1, t_2)$. Therefore

$$\begin{aligned} f(u) &\leq M(f(t_1), f(t_2)) \\ &\leq \lambda \max\{f(t_1), f(t_2)\} + (1-\lambda)\min\{f(t_1), f(t_2)\} \\ &\leq \lambda N + (1-\lambda)N = N. \end{aligned}$$

Therefore the function f is locally bounded above at some point. Applying Theorem 4.5 finishes the proof. $\square$

5. Applications

Now we use the following

Lemma 5.1 ([7], [9]). *Let $A, B \subset \mathbb{R}$ be sets such that $l(A) > 0$ and $l(B) > 0$. Suppose that $D \subset \mathbb{R}^2$ is an open set containing $A \times B$ and that $f : D \longrightarrow \mathbb{R}$. If $a \in A$ and $b \in B$ are density points of sets A and B respectively, furthermore, function f and the partial derivatives f'_x, f'_y are continuous in the neighbourhood of point (a, b) and $f'_x(a, b) \neq 0$, $f'_y(a, b) \neq 0$, then $f(A \times B)$ contains an interval.*

Theorem 5.2. *The logarithmic mean satisfies condition* (i)–(vii).

Proof. Conditions (i)–(v) follow from the properties of the logarithmic mean mentioned in the second Section and from the Remark 4.2. Now we prove (vi). To this end we show that the assumptions of Lemma 5.1 are fulfilled. We take an arbitrary set $T \in (0, \infty)$ such that $l(T) > 0$. Let $a, b \in T$ be the points of density of T such that $a \neq b$. Let $\delta > 0$ be choosen so that $[(a - \delta, a + \delta) \times (b - \delta, b + \delta)] \cap \{(x, y) \in \mathbb{R}^2 : x = y\} = \emptyset$. We put $T_1 = T \cap (a - \delta, a + \delta)$, $T_2 = T \cap (b - \delta, b + \delta)$. Notice that $l(T_1) > 0$, $l(T_2) > 0$ and $T_1 \times T_2 \subset T \times T \setminus \{(x, x) : x \in (0, \infty)\}$. Evidently $L : (0, \infty)^2 \longrightarrow (0, \infty)$ and L, L'_x, L'_y are continuous in the neighbourhood $(a - \delta, a + \delta) \times (b - \delta, b + \delta)$ of point (a, b) and $L'_x(a, b) \neq 0$, $L'_y(a, b) \neq 0$. Hence, in view of Lemma 5.1 int $L(T_1, T_2) \neq \emptyset$. All the more int $L(T, T) \neq \emptyset$. Therefore, the logarithmic mean fulfils condition (vi). $\square$

Corollary 5.3. *Assume that $(a, b) \subset (0, \infty)$. If $f : (a, b) \longrightarrow (0, \infty)$ is L-convex and locally bounded above at $x_0 \in (a, b)$ function, then f is continuous.*

Corollary 5.4. *Assume that $(a, b) \subset (0, \infty)$. If $f : (a, b) \longrightarrow (0, \infty)$ is L-convex and Lebesgue measurable function, then f is continuous.*

For the family of weighted quasi-arithmetic means we have the following

Theorem 5.5. *If $\varphi : I \longrightarrow \mathbb{R}$ is a strictly increasing and concave (strictly decreasing and convex) function and $t \in (0, 1)$ is fixed, then the weighted quasi-arithmetic mean $M_{\varphi,t} : I^2 \longrightarrow I$ satisfies* (i)–(v) *and* (vii).

Proof. Conditions (i), (iii), (iv), (v) and (vii) follow from the properties of weighted quasi-arithmetic mean mentioned in the second Section and from Remark 4.2. Now we prove (ii). Assume that φ is concave and strictly increasing. For a fixed $t \in (0,1)$ we have for every $x, y \in I$

$$\varphi^{-1}(t\varphi(x) + (1-t)\varphi(y)) \leq (\varphi^{-1} \circ \varphi)(tx + (1-t)y),$$

hence

$$M_{\varphi,t}(x,y) \leq tx + (1-t)y. \tag{4}$$

We consider a case $t \in \left[\frac{1}{2}, 1\right)$. Suppose that $\lambda = t$. If $x > y$, then from (4) we get

$$M_{\varphi,t}(x,y) \leq \lambda \max\{x,y\} + (1-\lambda) \min\{x,y\}.$$

If $x \leq y$, then $(1-t)(y-x) \leq t(y-x)$, which is equivalent to the following inequalities

$$tx + (1-t)y \leq ty + (1-t)x \leq \lambda \max\{x,y\} + (1-\lambda) \min\{x,y\}.$$

This together with (4) give (ii). If $t \in \left(0, \frac{1}{2}\right)$, then we take $\lambda = 1-t$. The proof in the case when φ is convex and strictly decreasing is analogous. $\square$

Corollary 5.6. *If $\varphi : I \longrightarrow \mathbb{R}$ is a strictly increasing and concave (strictly decreasing and convex) function and $t \in (0,1)$ is fixed, then every $M_{\varphi,t}$-convex function $f : J \longrightarrow I$ locally bounded above at $x_0 \in J$ is continuous.*

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