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On the limit sets of orbits with some kinds of stabilities

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Abstract. In this paper, we discuss the omega limit sets of orbits that are asymptotically Liapunov stable and asymptotically Zhukovskij stable respectively. We strengthen all the results in [6] and moreover point out several crucial mistakes in [6].

1. Introduction

There are several recent papers ([6], [7]) concerning the omega limit sets of some kind of stability. In [6] the author considered the differential equations:

$$\frac{dx}{dt} = f(x), \quad x \in G \subset R^n, \tag{E}$$

where G is a closed bounded domain in R^n and $f \in C^r(G)$ $(r \geq 1)$. He stated the following results:

(a) If the solution $x(t, x_0)$ of (E) is asymptotically Liapunov stable, then its omega limit set $\omega(x_0)$ consists of fixed points.

(b) If the solution $x(t, x_0)$ of (E) is uniformly asymptotically Zhukovskij stable, then its omega limit set is a closed orbit or a fixed point.

The goal of this paper is to strengthen the both results of (a) and (b). Actually, following the ideas of [6], in Section 2 we prove that the result of (a) still holds for a flow defined on a metric space. In Section 3 we see

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that the proof of (b) in [6] is unbelievable, but the conclusion of (b) is also true for a flow defined on a locally compact metric space. Finally, we present an example to illustrate that the last remark in [6, p. 1999, line 16] is wrong. This example also shows that the assertion in [7, p. 86, line 12] is not true, i.e., that an orbit does not tend to a fixed point does not imply that there is no fixed points in its omega limit set of the orbit.

2. The omega limit set of an asymptotically Liapunov stable orbit

At first we fix some notations. Let (X, d) be a metric space with metric d, on which there is a flow $f : X \times R \to X$. Write $x \cdot t = f(x, t)$ and let $A \cdot J = \{x \cdot t \mid x \in A, t \in J\}$ for $A \subset X$ and $J \subset R$. So $x \cdot R$ and $x \cdot R^+$ are the orbit and the positive semi-orbit respectively of a point $x \in X$. The omega limit set of x is the set $\omega(x) = \{y \in X \mid \text{there is a sequence } t_n \in R^+ \text{ such that } t_n \to +\infty \text{ and } x \cdot t_n \to y\}$.

Definition 2.1 ([5, p. 108]). The orbit $x \cdot R$ of a point $x \in X$ is Liapunov stable for the flow f provided that given any $\epsilon > 0$, there is a $\delta > 0$ such that if $d(x, p) < \delta$, then $d(x \cdot t, p \cdot t) < \epsilon$ for all $t \geq 0$. The orbit $x \cdot R$ is asymptotically Liapunov stable provided it is Liapunov stable and there is a $\tau > 0$ such that if $d(x, p) < \tau$, then $d(x \cdot t, p \cdot t)$ goes to zero as t goes to infinity.

Theorem 2.2. *If the orbit $x \cdot R$ is asymptotically Liapunov stable and its omega limit set $\omega(x)$ is nonempty, then $\omega(x)$ consists of fixed points.*

Proof. Let $q \in \omega(x)$ and $t_n \to +\infty$ such that $x \cdot t_n \to q$. Since $x \cdot R$ is asymptotically Liapunov stable, there is a $\tau > 0$ such that if $d(x, p) < \tau$, then $d(x \cdot t, p \cdot t)$ goes to zero as t goes to infinity. Now choose a $\lambda > 0$ such that for any $\Delta t \in [0, \lambda]$ we have $d(x, x \cdot \Delta t) < \tau$. Letting $x_n = x \cdot t_n$, from the continuity of the flow, it follows that $x_n \cdot \Delta t \to q \cdot \Delta t$ and so $x \cdot (t_n + \Delta t) \to q \cdot \Delta t$. Thus $d(q, q \cdot \Delta t) \leq d(q, x \cdot t_n) + d(x \cdot t_n, x \cdot (t_n + \Delta t)) + d(x \cdot (t_n + \Delta t), q \cdot \Delta t) \to 0$ as $t_n \to +\infty$. That is $q = q \cdot \Delta t$ for any $\Delta t \in [0, \lambda]$, of course it implies that q is a fixed point (see [1, Ch. 2, Th. 2.2]). This completes the proof. □

The following example is presented by the referee, which shows that the ω-limit set can be more than one fixed point.

Example 2.3. Consider the system in R^2 defined by differential equations in polar coordinates:

$$\dot{r} = -(r-1)^3, \qquad \dot{\theta} = r - 1.$$

The solutions are

$$r(t) = 1 + \frac{r_0 - 1}{[2t(r_0-1)^2+1]^{1/2}},$$

$$\theta(t) = \theta_0 + \frac{[2t(r_0-1)^2+1]^{1/2} - 1}{r_0 - 1}.$$

It is easy to verify that every orbit $p \cdot R$ outside the unit circle is asymptotically Liapunov stable but has $\omega(p)$ the whole unit circle.

3. The omega limit set of a uniformly asymptotically Zhukovskij stable orbit

In this section we suppose that the metric space X is locally compact. A set Y is invariant under the flow f if Y is a subset in X with $Y \cdot R = Y$, and an invariant set Y is a minimal set provided (i) Y is a closed, nonempty set and (ii) if Z is a closed, nonempty, invariant subset of Y, then $Z = Y$. In addition, in the following we let $B_r(x) = \{y \in X \mid d(x,y) < r\}$ and $S_r(x) = \{y \in X \mid d(x,y) \leq r\}$ be the open ball and the closed ball respectively with center x and radius $r > 0$. For $p \in X$ and $A \subset X$, let $d(p, A) = \inf\{d(p,z) | z \in A\}$, and then we define $N_r(A) = \{z \in X \mid d(z, A) < r\}$ for $r > 0$, it is called the generalized open r-ball about A of radius r.

Zhukovskij Stability ([3], [8]). The orbit $x \cdot R$ of a point x in X is Zhukovskij stable provided that given any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that for any $p \in B_\delta(x)$, then one can find a time parameterization τ_p such that $d(x \cdot t, p \cdot \tau_p(t)) < \epsilon$ holds for $t \geq 0$, where τ_p is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_p(0) = 0$. Moreover, if $d(x \cdot t, p \cdot \tau_p(t)) \to 0$ as

$t \to +\infty$ also holds, the orbit $x \cdot R$ is said to be asymptotically Zhukovskij stable.

Now we introduce the concept of uniformly asymptotically Zhukovskij stability, it is a simpler version that is equivalent to that in [6, Def. 4.1].

Definition 3.1. The orbit $x \cdot R$ of a point x in X is uniformly asymptotically Zhukovskij stable provided that given any $\epsilon > 0$, there is a $\delta > 0$ such that for each $t' \geq 0$ and $p \in B_\delta(x \cdot t')$, then one can find a time parameterization τ_p such that $d(x \cdot (t + t'), p \cdot \tau_p(t)) < \epsilon$ holds for $t \geq 0$, and also

$$d(x \cdot (t + t'), p \cdot \tau_p(t)) \to 0 \quad \text{as} \quad t \to +\infty, \tag{1}$$

where τ_p is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_p(0) = 0$.

Geometrically, the semi-orbit $p \cdot R^+$ will stay in a long tube of $x \cdot R^+$ with different time scales, and the tube is getting thinner and thinner as time tends to infinity.

In [6] the author proved the result of (b) in Section 1. However, the proof of [6, Theorem 5.1] has some errors in it. For example, in his proof the Poincaré map ϕ_t should not be defined by the same t for all $y \in D_\sigma(x_0)$. Otherwise, $\operatorname{diam}(\phi_t(D_\sigma(x))) \to 0$ does not hold as $t \to +\infty$. Also the point ω in the omega limit set of $x(t, x_0)$ should be excluded from being a fixed point, since it is impossible to define $D_\sigma(\omega)$ for a fixed point ω as in his proof. Now we shall give a strengthened conclusion. In the following, we always consider the case $\omega(x) \neq \emptyset$ for a point $x \in X$.

Lemma 3.2. *If the orbit $x \cdot R$ of a point x is uniformly asymptotically Zhukovskij stable with nonempty omega limit set, then its omega limit set $\omega(x)$ is minimal.*

Proof. Otherwise, $\omega(x)$ has a proper closed invariant subset $A \subset \omega(x)$ with $A \neq \emptyset$. Choose a point $p \in \omega(x) \setminus A$, then $\lambda = d(p, A) > 0$. Now for a sufficiently large t', we can find a point $q \in A$ satisfying $d(x \cdot t', q) < \delta$ (with δ the number defined as in the Definition 3.1). Also there exists a sequence $t_i \geq t'$ such that $t_i \to +\infty$ and $x \cdot t_i \to p$. Since A is invariant, it follows $q \cdot R \subset A$. However, for large t_i we have $d(x \cdot t_i, p) < \lambda/2$, it follows $d(x \cdot t_i, q \cdot R) \geq d(p, A) - d(x \cdot t_i, p) \geq \lambda/2$ for large t_i. It is contradictory to (1) in the Definition 3.1, since $d(x \cdot t', q) < \delta$ holds. Thus $\omega(x)$ is minimal. $\square$

Corollary 1. *Assume that x is uniformly asymptotically Zhukovskij stable. If there is a fixed point in $\omega(x)$ then $\omega(x) = \{p\}$. Also if there is a closed orbit γ in $\omega(x)$ then $\omega(x) = \gamma$.*

From the proof of the Lemma 3.2, it is easy to conclude:

Corollary 2. *Any closed nonempty invariant set A must be δ-apart from a uniformly asymptotically Zhukovskij stable semi-orbit $x \cdot R^+$ if $A \cap \omega(x) = \emptyset$, where δ is the number defined as in the Definition 3.1.*

Lemma 3.3 ([2, p. 414])**.** *Let X be a Hausdorff topological space and $F : X \to X$ be continuous. If for each open covering $\{W_\alpha\}$ of X there is at least one $x \in X$ such that both x and $F(x)$ belong to a common W_α, then F has a fixed point.*

Theorem 3.4. *If an orbit $x \cdot R$ is uniformly asymptotically Zhukovskij stable with nonempty omega limit set, then its omega limit set $\omega(x)$ is a fixed point or a closed orbit.*

Proof. Assume that $\omega(x)$ is not a singleton, we shall show that $\omega(x)$ is a closed orbit. Choose a point $p \in \omega(x)$ and it is a regular point from the Corollary 1. Now let a sequence $\{t_i\}_{i=1}^{\infty} \subset R^+$ such that $t_i \to +\infty$ and $x \cdot t_i \to p$. Thus there is a positive σ ($\sigma < \delta$) such that the closed ball $S_\sigma(p)$ lies in the open ball $B_\delta(x \cdot t_k)$ for some $t_k \in \{t_i\}_{i=1}^{\infty}$ and so does the set $S_\sigma(p) \cdot [-\theta, \theta]$ for a sufficiently small $\theta > 0$. From the local compactness of X, we may also suppose that $S_\sigma(p)$ is compact. Since p is a regular point, by the tubular flow theorem [4, Ch. 5, Section 2], there is a transversal $\Sigma \subset S_\sigma(p) \cdot [-\theta, \theta]$ with $p \in \Sigma$ such that for each $y \in S_\sigma(p) \cdot [-\theta, \theta]$, the arc of $y \cdot R$ in $S_\sigma(p) \cdot [-\theta, \theta]$ crosses Σ at a unique $t = \phi(y)$, where $\phi(y)$ is continuous on $y \in S_\sigma(p) \cdot [-\theta, \theta]$. Because of $S_\sigma(p) \cdot [-\theta, \theta] \subset B_\delta(x \cdot t_k)$, it follows from (1) in the Definition 3.1 that for each $y \in S_\sigma(p) \cdot [-\theta, \theta]$ there is a $T(y) > 0$ such that $d(x \cdot (t + t_k), y \cdot \tau_y(t)) < \sigma/2$ for $t \geq T(y)$. Thus from the compactness of $S_\sigma(p) \cdot [-\theta, \theta]$ and the continuity of the flow f, one can find a positive $M = \sup\{T(y) \mid y \in S_\delta(p) \cdot [-\theta, \theta]\} < +\infty$ such that for each $y \in S_\sigma(p) \cdot [-\theta, \theta]$, $d(x \cdot (t + t_k), y \cdot \tau_y(t)) < \sigma/2$ holds for $t \geq M$. Fix a $t_l > t_k$ and $t_l - t_k \geq M$ with $d(x \cdot t_l, p) < \sigma/2$. Now we define a Poincaré map $F : \Sigma \to \Sigma$ as follows. If $y \in \Sigma$ ($\subset S_\sigma(p) \cdot [-\theta, \theta]$), then we have $d(x \cdot (t + t_k), y \cdot \tau_y(t)) < \sigma/2$ for $t \geq M$, it implies $d(p, y \cdot \tau_y(t_l - t_k)) \leq d(p, x \cdot t_l) + d(x \cdot t_l, y \cdot \tau_y(t_l - t_k)) < \sigma/2 + \sigma/2 = \sigma$. So it follows

$y\cdot\tau_y(t_l-t_k) \in S_\sigma(p)$ and then $y\cdot(\tau_y(t_l-t_k)+\phi(y)) \in \Sigma$ for a $\phi(y) \in [-\theta,\theta]$. Thus we define $F(y) = y\cdot(\tau_y(t_l-t_k)+\phi(y))$. The continuity of F comes from the continuities of the flow f, τ_y and $\phi(y)$. Note that F may not be the first return map. Next, if $\{W_\alpha\}$ is an open covering of Σ for its subspace topology from X, let $p \in W_\alpha = \Sigma \cap U$, where U is an open set in X. Choose an $r > 0$ with $B_r(p) \subset U \cap S_\sigma(p)$ and let $x\cdot t_i \in B_{r/2}(p)$ for $t_i \geq T > t_l$. Thus from (1) in the Definition 3.1, we assert $d(F^n(p), x\cdot t_i) < r/2$ for $n \geq N$ and some $t_i \geq T' \geq T$, where $F^n(p)$ is the nth iterate of p. Hence, $d(F^N(p),p) \leq d(F^N(p), x\cdot t_m) + d(x\cdot t_m, p) < r/2+r/2 = r$ holds for some $t_m \geq T'$ and similarly $d(F^{N+1}(p),p) \leq d(F^{N+1}(p), x\cdot t_n) + d(x\cdot t_n, p) < r/2+r/2 = r$ for some $t_n \geq T'$. It follows that both $F^{N+1}(p)$ and $F^N(p)$ lie in $B_r(p)$. So we obtain that both $F(F^N(p))$ and $F^N(p)$ belong to W_α. By the Lemma 3.3 we conclude that $F : \Sigma \to \Sigma$ has a fixed point q. Obviously, $q\cdot R$ is a closed orbit, from the Corollaries 1 and 2, we immediately obtain $\omega(x) = q\cdot R$. This completes the proof. □

Example 3.5. Consider the system in R^2 defined by differential equations in polar coordinates:

$$\dot{r} = r(1-r), \qquad \dot{\theta} = 1-r. \tag{2}$$

The solutions are

$$r(t) = \frac{r_0 e^t}{1-r_0+r_0e^t}, \qquad \theta(t) = \theta_0 - \ln[r_0 + (1-r_0)e^{-t}].$$

It is easy to see that every orbit outside the unit circle is asymptotically Zhukovskij stable, but not uniformly asymptotically Zhukovskij stable. Obviously, the unit circle is composed of fixed points of the system (2), and it is the omega limit set of every point outside the unit circle. However it is not a closed orbit. This illuminates the last remark in [6, p. 1999, line 16] is not true, and also the conclusion of [7, p. 86, line 12] is wrong.

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