Publ. Math. Debrecen
63/3 (2003), 421–429

On the continuous solutions of a generalization of the Gołąb–Schinzel equation

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Abstract. Let J be a real nontrivial interval, $0 \in J$, $F : \mathbb{R}^2 \to \mathbb{R}$ be symmetric, $0 \in F(\mathbb{R}^2)$, $M : J \to \mathbb{R}$ be continuous, and $M(0) = 0$. We determine the continuous solutions $f : \mathbb{R} \to J$ of the functional equation

$$f(x + M(f(x))y) = F(x, y).$$

The functional equation

$$f(x + M(f(x))y) = F(x, y), \tag{1}$$

where $f, M : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$, is a generalization of the well known Gołąb–Schinzel equation

$$f(x + f(x)y) = f(x)f(y). \tag{2}$$

For the details concerning functional equation (2) and applications of it we refer e.g. to [1], [2], [4], [7], [11], [12], [15], [16], [19] and for its generalizations to [3], [5], [6], [8]–[10], [13], [14], [17], [18].

If we consider (1) as an equation of three unknown functions f, M, and F, then it is very easy to describe the general solution of it. Namely, given arbitrary functions $f, M : \mathbb{R} \to \mathbb{R}$, the function $F : \mathbb{R}^2 \to \mathbb{R}$ is uniquely

Mathematics Subject Classification: 39B22.
Key words and phrases: functional equation, Gołąb–Schinzel functional equation, continuous solution.

determined by (1). Moreover, with $y = 0$, from (1) we get $f(x) = F(x, 0)$ for $x \in \mathbb{R}$.

In the case where we solve the equation with respect to f (assuming that F and M are given) it is clear that the last equality does not need to be a sufficient condition for f to satisfy (1). In other words, for some F and M there may be no solutions $f : \mathbb{R} \to \mathbb{R}$ of (1).

In this paper we determine those pairs of functions $M : J \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$, which admit a continuous solution $f : \mathbb{R} \to J$ of (1), under the additional assumptions that F is symmetric, $0 \in F(\mathbb{R}^2)$, $M : J \to \mathbb{R}$ is continuous, and $M(0) = 0$, where J is a nontrivial real interval with 0.

We begin with the following

Lemma 1. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g(0) \neq 0$, $0 \in g((0, +\infty))$ and*

$$g(x + g(x)y) = g(y + g(y)x) \quad \text{for } x, y \in \mathbb{R}. \tag{3}$$

Then there exist $b, d \in (0, +\infty)$ with

$$D := g^{-1}((-\infty, 0)) \cap [0, +\infty) \in \{\emptyset, (d, +\infty)\}, \tag{4}$$

$$B := g^{-1}((0, +\infty)) \cap [0, +\infty) = [0, b). \tag{5}$$

Proof. For the proof of (4) by contradiction suppose that there are $a, w \in (0, +\infty)$ with $a < w$, $g(a) < 0$ and $g(w) = 0$. Put $i_0 = \inf\{x \in (a, +\infty) : g(x) = 0\}$. Then, by the continuity of g, $w \geq i_0 > a$, $g(i_0) = 0$ and $g((a, i_0)) \subset (-\infty, 0)$. Take $c \in (\frac{1}{2}(a + i_0), i_0)$ with

$$g(c) > \frac{a - i_0}{2i_0}.$$

Clearly

$$a = \frac{a + i_0}{2} + \frac{a - i_0}{2} < c + g(c)i_0 < c < i_0.$$

Thus, on account of (3),

$$0 > g(c + g(c)i_0) = g(i_0 + g(i_0)c) = g(i_0) = 0.$$

This is a contradiction, which completes the proof of (4).

Next observe that, in view of (4) and the hypotheses, $g(0) > 0$, whence $B \neq \emptyset$. So for the proof of (5) by contradiction suppose that $g(a) > 0$ and

$g(w)=0$ for some $a, w \in (0,+\infty)$, $w<a$. Let $s_0 = \sup\{x \in (0,a) : g(x)=0\}$. It is easy to see that $w \le s_0 < a$, $g(s_0) = 0$ and $g((s_0,a)) \subset (0,+\infty)$. Take $c \in (s_0, \frac{1}{2}(a+s_0))$ with $2s_0 g(c) < a - s_0$. Then

$$a = \frac{a+s_0}{2} + \frac{a-s_0}{2} > c + g(c)s_0 > c > s_0.$$

This brings a contradiction, because

$$0 < g(c+g(c)s_0) = g(s_0 + g(s_0)c) = g(s_0) = 0.$$

Thus we have completed the proof of Lemma 1. □

Lemma 2. *Let g be as in Lemma 1. Then there exists $c \in (-\infty, 0)$ such that one of the following two conditions holds:*

$$g(x) = cx + 1 \quad \textit{for } x \in \mathbb{R}; \tag{6}$$

$$g(x) = \max\{cx+1, 0\} \quad \textit{for } x \in \mathbb{R}. \tag{7}$$

Proof. On account of Lemma 1 there exist $b, d \in (0,+\infty)$ such that (4) and (5) are valid. Put $A = \{y + g(y)b : y \in \mathbb{R}\}$. Since $g(b) = 0$ and, by (3), $g(y+g(y)b) = g(b+g(b)y) = g(b) = 0$ for $y \in \mathbb{R}$, we have

$$g(A) = \{0\}. \tag{8}$$

Next A is connected and $b = b + g(b)b \in A$. Hence $A \subset [b,+\infty)$, which yields

$$g(y) \ge 1 - \frac{y}{b} \quad \text{for } y \in \mathbb{R}. \tag{9}$$

First consider the case $D \ne \emptyset$. Clearly $g(d) = 0$ and $d \ge b$. Suppose $d > b$. Then $1 - \frac{d}{b} < 0$ and, by (3),

$$0 = g(d) = g(d + g(d)x) = g(x + g(x)d) \quad \text{for } x \in \mathbb{R}.$$

This brings a contradiction, because, in view of (9),

$$x + g(x)\,d \ge x + \left(1 - \frac{x}{b}\right) d = x\left(1 - \frac{d}{b}\right) + d > d \quad \text{for } x < 0.$$

Thus we have proved that $b = d$. Hence from (8) we get $y + g(y)b = b$ for $y \in \mathbb{R}$, which implies (6) with $c = -\frac{1}{b}$.

Now assume $D = \emptyset$. Suppose that $g(z) \neq 1 - \frac{z}{b}$ for some $z < 0$. Then, by (9), $g(z) > 1 - \frac{z}{b}$, whence $z + g(z)b > b$ and consequently there is $e \in (0, b)$ with

$$z + g(z)y > b \quad \text{for } y \in (b - e, b). \tag{10}$$

Take $y_0 \in (b - e, b)$ with $g(y_0) < -\frac{y_0}{z}$, which means that $y_0 + g(y_0)z > 0$. Next $g(y_0) > 0$, $0 < y_0 < b$ and $z < 0$, so we have $y_0 + g(y_0)z \in (0, b)$. This brings a contradiction, because, by (10),

$$g(y_0 + g(y_0)z) = g(z + g(z)y_0) = 0.$$

In this way we have shown that

$$g(y) = 1 - \frac{y}{b} \quad \text{for } y \leq 0. \tag{11}$$

Take $y \in (0, b)$. Then $g(y) > 0$ and $y + g(y)x \to -\infty$ if $x \to -\infty$. Hence there exists $x < -\frac{by}{b-y}$ with $y + g(y)x < 0$. Note that $x(b - y) < -by$, which implies $x + y - \frac{1}{b}xy < 0$. Whence, by (11),

$$x + g(x)y = x + \left(-\frac{1}{b}x + 1\right) y = x + y - \frac{1}{b}xy < 0$$

and consequently

$$\begin{aligned} -\frac{1}{b}\left(y + g\left(y\right)x\right) + 1 = g\left(y + g\left(y\right)x\right) &= g\left(x + g\left(x\right)y\right) \\ &= -\frac{1}{b}\left(x + y - \frac{1}{b}xy\right) + 1. \end{aligned}$$

Thus $g(y) = -\frac{1}{b}y + 1$. This and (11) imply (7) (with $c = -\frac{1}{b}$), which completes the proof. □

Proposition 1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous solution of* (3) *such that $0 \in g(\mathbb{R})$. Then either $g(\mathbb{R}) = \{0\}$ or there exists $c \in \mathbb{R} \setminus \{0\}$ such that* (6) *or* (7) *holds.*

Proof. Suppose that $g(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. Then

$$0 \neq g(x_0) = g(x_0 + g(x_0)0) = g(0 + g(0)x_0) = g(g(0)x_0).$$

Hence $g(0) \neq 0$. Further, according to the hypothesis, there is $z_0 \in \mathbb{R}$ with $g(z_0) = 0$. If $z_0 > 0$, we derive the statement from Lemma 2. If $z_0 < 0$, we define $g_0 : \mathbb{R} \to \mathbb{R}$ by $g_0(x) = g(-x)$. Then

$$\begin{aligned} g_0(x + g_0(x)y) &= g(-x + g(-x)(-y)) = g(-y + g(-y)(-x)) \\ &= g_0(y + g_0(y)x) \end{aligned}$$

for $x, y \in \mathbb{R}$, $g_0(0) = g(0) \neq 0$ and $g_0(-z_0) = g(z_0) = 0$. Thus, on account of Lemma 2, there is $c_0 \in (-\infty, 0)$ such that $g_0(x) = c_0 x + 1$ for $x \in \mathbb{R}$ or $g_0(x) = \max\{c_0 x + 1, 0\}$ for $x \in \mathbb{R}$. Consequently (6) or (7) holds with $c = -c_0$, which completes the proof. □

Finally we have the following

Theorem 1. *Assume that J is a real nontrivial interval, $0 \in J$, $M : J \to \mathbb{R}$ is continuous, $M(0) = 0$, $F : \mathbb{R}^2 \to \mathbb{R}$ is symmetric, and $0 \in F(\mathbb{R}^2)$. Then a continuous function $f : \mathbb{R} \to J$ is a solution of equation* (1) *if and only if one of the following three conditions holds.*

(i) $f(\mathbb{R}) = \{0\} = F(\mathbb{R}^2)$.

(ii) *M is bijective and there exists $c \in \mathbb{R} \setminus \{0\}$ such that*

$$F(x, y) = M^{-1}((cx + 1)(cy + 1)) \quad \text{for } x, y \in \mathbb{R},$$

$$f(x) = M^{-1}(cx + 1) \quad \text{for } x \in \mathbb{R}.$$

(iii) *There exist a continuous one-to-one function $h : [0, +\infty) \to J$ and $c \in \mathbb{R} \setminus \{0\}$ such that $h(0) = 0$,*

$$M(y) = h^{-1}(y) \quad \text{for } y \in h([0, +\infty)),$$

$$F(x, y) = h(s(cx + 1)s(cy + 1)) \quad \text{for } x, y \in \mathbb{R},$$

$$f(x) = h(s(cx + 1)) \quad \text{for } x \in \mathbb{R},$$

where $s(x) = \max\{x, 0\}$.

Proof. It is easy to check that, in either of the cases described in (i)–(iii), f satisfies (1). So now assume that $f : \mathbb{R} \to J$ is a continuous solution of equation (1).

If $F(\mathbb{R}^2) = \{0\}$, then, by (1), we have

$$f(x) = f(x + M(f(x))0) = F(x, 0) = 0 \quad \text{for } x \in \mathbb{R},$$

which means that (i) holds. Therefore it remains to consider the case where $F(x_1, y_1) \neq 0$ for some $x_1, y_1 \in \mathbb{R}$.

According to the hypothesis there exist $x_0, y_0 \in \mathbb{R}$ such that $F(x_0, y_0) = 0$. Put $z_0 = x_0 + M(f(x_0))y_0$, $z_1 = x_1 + M(f(x_1))y_1$, and $g = M \circ f$. Suppose that $g(0) = 0$. Then (1) yields

$$\begin{aligned} 0 \neq F(x_1, y_1) &= f(z_1) = f(z_1 + g(z_1)0) = F(z_1, 0) = F(0, z_1) \\ &= f(0 + g(0)z_1) = f(0) = f(0 + g(0)z_0) = F(0, z_0) \\ &= F(z_0, 0) = f(z_0 + g(z_0)0) = f(z_0) = F(x_0, y_0) = 0, \end{aligned}$$

a contradiction. Thus $g(0) \neq 0$. Further $g(z_0) = M(f(z_0)) = M(F(x_0, y_0)) = 0$ and, for $x, y \in \mathbb{R}$,

$$\begin{aligned} g(x + g(x)y) &= M(f(x + M(f(x))y)) = M(F(x, y)) \\ &= M(F(y, x)) = g(y + g(y)x), \end{aligned}$$

whence, on account of Proposition 1, (6) or (7) holds with some $c \in \mathbb{R} \setminus \{0\}$.

Suppose first that g is of the form (6). Then $g(\mathbb{R}) = \mathbb{R}$ and

$$M(f(y_1)) = g(y_1) = cy_1 + 1 \neq cy_2 + 1 = g(y_2) = M(f(y_2)) \tag{12}$$

for every $y_1, y_2 \in \mathbb{R}, y_1 \neq y_2$, which means that the function $f(\mathbb{R}) \ni y \to M(y) \in \mathbb{R}$ is a bijection. Hence $f(\mathbb{R}) = J$, because M is continuous. Next

$$\begin{aligned} M(F(x, y)) &= g(x + g(x)y) \\ &= c(x + (cx + 1)y) + 1 = (cx + 1)(cy + 1) \end{aligned} \tag{13}$$

for $x, y \in \mathbb{R}$. Consequently (ii) holds.

Now assume (7). Let $P = \{x \in \mathbb{R} : cx + 1 \geq 0\}$ and $P_0 = P \setminus \{-\frac{1}{c}\}$. Then $g(\mathbb{R} \setminus P_0) = \{0\}$, $M(f(P)) = g(P) = [0, +\infty)$, and (12) holds for every $y_1, y_2 \in P, y_1 \neq y_2$. Thus the function $M_0 : f(P) \ni y \to M(y) \in [0, +\infty)$ is bijective. Put $h = M_0^{-1}$. It is easily seen that $f(x) = h(cx + 1)$ for $x \in P$ and, for $x \in \mathbb{R} \setminus P_0$,

$$f(x) = f(x + g(x)z_0) = F(x, z_0) = F(z_0, x) = f(z_0) = 0,$$

whence $h(0) = f\left(-\frac{1}{c}\right) = 0$ and $f(y) = h(s(cy+1))$ for $y \in \mathbb{R}$. Further

$$\begin{aligned} F(y,x) &= F(x,y) = f(x + M(f(x))y) \\ &= f(x) = 0 \quad \text{for } x \in \mathbb{R} \setminus P_0,\ y \in \mathbb{R}. \end{aligned}$$

Finally observe that, for $x, y \in P_0$, $c(x+g(x)y)+1 = (cx+1)(cy+1) > 0$, which means that $x + g(x)y \in P_0$ and consequently (13) holds. This completes the proof. $\square$

The following corollary generalizes to some extent Corollary 4 in [8].

Corollary 1. *Let J and M be as in Theorem 1. Then a continuous function $f : \mathbb{R} \to J$ satisfies the functional equation*

$$f(x + M(f(x))y) = f(x)f(y) \tag{14}$$

if and only if one of the following four conditions holds.

1° $f(\mathbb{R}) = \{0\}$.

2° $f(\mathbb{R}) = \{1\}$.

3° *$J = \mathbb{R}$ and there exist $a > 0$ and $c \in \mathbb{R} \setminus \{0\}$ such that $M(x) = |x|^{\frac{1}{a}}(\mathrm{sign}(x))$ and $f(x) = |cx+1|^a(\mathrm{sign}(cx+1))$ for $x \in \mathbb{R}$.*

4° *$[0, +\infty) \subset J$ and there exist $a > 0$ and $c \in \mathbb{R}\setminus\{0\}$ such that $M(x) = x^{\frac{1}{a}}$ for $x \in [0, +\infty)$ and $f(x) = (\max\{cx+1, 0\})^a$ for $x \in \mathbb{R}$.*

Proof. It is easy to check that if one of conditions 1°–4° holds, then f satisfies (14). So assume now that $f : \mathbb{R} \to J$ is a continuous solution of equation (14).

First consider the case where $0 \notin f(\mathbb{R})$. Suppose that $x \in \mathbb{R}$ and $M(f(x)) \neq 1$. Put

$$z = \frac{x}{1 - M(f(x))}.$$

Then $z = x + M(f(x))z$ and consequently

$$f(z) = f(x + M(f(x))z) = f(x)f(z),$$

whence $f(x) = 1$.

Thus we have proved that, for every $x \in \mathbb{R}$, $f(x) = 1$ or $M(f(x)) = 1$. Since M is continuous, this means that $f(\mathbb{R}) = \{1\}$ or $M(f(\mathbb{R})) = \{1\}$.

In the first case we clearly get 2° and in the latter one we have

$$f(x+y) = f(x + M(f(x))y) = f(x)f(y) \quad \text{for } x, y \in \mathbb{R}. \tag{15}$$

Next $f(\mathbb{R}) \neq \{1\}$ and (15) imply $f(x) = \exp cx$ for $x \in \mathbb{R}$ with some $c \in \mathbb{R} \setminus \{0\}$ (see e.g. [1]). But then $f(\mathbb{R}) = (0, +\infty)$ and consequently $M((0, +\infty)) = M(f(\mathbb{R})) = \{1\}$, which is impossible, because M is continuous and $M(0) = 0$.

Now assume that $0 \in f(\mathbb{R})$. Put $F(x, y) = f(x)f(y)$ for $x, y \in \mathbb{R}$. Then $0 \in F(\mathbb{R}^2)$ and f satisfies (1). Thus conditions (i)–(iii) of Theorem 1 are valid. It is easily seen that, in case (ii), M^{-1} and, in case (iii), h are multiplicative. This completes the proof (see [1], pp. 29–31). □

Below we give two simple examples showing that without the assumptions $M(0) = 0$ and $0 \in F(\mathbb{R}^2)$ in Theorem 1 the statement of the theorem is not valid.

Example 1. Let $f(x) = \sin x$ and $F(x, y) = \sin(x + y)$ for $x, y \in \mathbb{R}$, $M(x) = |x|$ for $|x| > 1$ and $M([-1, 1]) = \{1\}$. Then (1) holds, $0 \in F(\mathbb{R}^2)$, and $M(0) = 1$.

Example 2. Let $f(x) = 1 + \exp x$ and $F(x, y) = 1 + \exp(x + y)$ for $x, y \in \mathbb{R}$, $M(x) = x$ for $x < 1$ and $M([1, +\infty)) = \{1\}$. Then (1) holds, $0 \notin F(\mathbb{R}^2)$, and $M(0) = 0$.

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(Received May 13, 2002; revised October 16, 2002)