Publ. Math. Debrecen
63/3 (2003), 445–459

The variation of an additive function on a Boolean algebra

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Abstract. The possibility of representing a positive additive function on a Boolean algebra A as the variation of an arbitrary additive function or a bounded additive function on A with values in an Abelian normed group G is investigated. In particular, it is shown that if such a representation exists, then we can find one with $G = \mathbb{R}$ or $G = \ell_\infty(\Gamma)$ for some Γ, respectively.

1. Introduction

Let A be a Boolean algebra, let G be an Abelian normed group, and let $\varphi : A \to G$ be additive. The variation $|\varphi|$ of φ takes values in $[0, \infty]$, is additive and $|\varphi|(0) = 0$ holds, i.e., $|\varphi|$ is a quasi-measure, in our terminology. No other properties of $|\varphi|$ seem to have been recorded in the literature, even in the case where φ is bounded and G is a normed space. The purpose of this paper is to exhibit two such properties, called (G) and (F) in the sequel, and to prove that every quasi-measure ν on A which has property (G) [resp., properties (G) and (F)] can be represented as the variation of an additive [resp., bounded additive] function on A with values in $\mathbb{R}$ [resp., $l_\infty(\Gamma)$ for some Γ]; see Theorem 1 of Section 4 and Theorem 2 of Section 5.

A similar problem for a positive measure ν on a σ-algebra of sets was solved in [11]. The methods applied there are, however, quite different

Mathematics Subject Classification: 28A10, 28A12, 28B10.
Key words and phrases: Boolean algebra, nonatomic, quasi-measure, semifinite, decomposition, Abelian normed group, additive function, variation.

from the present ones. A point in common is the idea of decomposing the function ν to be represented in the required form (see [11, Lemma 1] and Proposition 1 of Section 2 below). Moreover, the atomic structure of ν plays a role in both [11] and this paper.

The notation and terminology we need are explained in Sections 2 and 3, which also contain some auxiliary results. The remaining material is divided into Sections 4 and 5. The former deals with the additive case and the latter with the bounded additive case. Our approach to both cases is unified to some extent and is based on Propositions 1 and 2.

2. Preliminaries on quasi-measures

Throughout the paper A stands for a Boolean algebra with the operations of join, meet and difference denoted by $\vee$, $\wedge$ and $\smallsetminus$, respectively. The natural ordering of A is denoted by $\leq$ and its minimal and maximal elements by 0 and 1, respectively. For every $a \in A$ we denote by C_a the ideal in A generated by a, i.e.,

$$C_a = \{b \in A : b \leq a\}.$$

We say that A is *nonatomic* or *atomless* if for every nonzero $a \in A$ there are nonzero disjoint a_1, $a_2 \in A$ with $a_1 \vee a_2 = a$.

We call a function $\nu : A \to [0, \infty]$ a *quasi-measure* if it is additive and $\nu(0) = 0$ holds. For the quasi-measure ν we set

$$I_\nu = \{a \in A : \nu(a) < \infty\}.$$

Clearly, I_ν is an ideal in A. We say that ν is *semifinite* provided for every $a \in A$ we have

$$\nu(a) = \sup\{\nu(b) : b \in I_\nu \cap C_a\}.$$

Two properties (G) and (F) the quasi-measure ν may have will be basic for our purposes. The former is defined as follows:

(G) *Given* $a \in A \setminus I_\nu$ *and* $\eta > 0$, *there are disjoint* a_1, $a_2 \in A$ *with* $\nu(a_1)$, $\nu(a_2) > \eta$ *and* $a_1 \vee a_2 = a$.

We note that (G) holds if ν is semifinite. In the case where $\nu(A) \subset \{0, \infty\}$, (G) is, clearly, equivalent to the following stronger property which has already been considered by the author (see [10, the definition of a generous quasi-measure on p. 300]):

(G)$'$ *Given* $a \in A \setminus I_\nu$, *there are disjoint* $a_1, a_2 \in A \setminus I_\nu$ *with* $a_1 \vee a_2 = a$.

As is easily seen, (G)$'$ means that the quotient Boolean algebra A/I_ν is nonatomic.

(G) and (G)$'$ are also equivalent provided A is σ-complete. Indeed, it follows from (G) that given $a \in A \setminus I_\nu$, we can find pairwise disjoint $b_1, b_2, \dots$ in A with $\nu(b_i) > 1$ for each i. Setting $a_1 = \bigvee_{i=1}^\infty b_{2i}$ and $a_2 = a \smallsetminus a_1$, we see that (G)$'$ holds. If, in addition, ν is σ-additive, then (G) is further equivalent to the condition that $\nu(d) < \infty$ for every atom d of ν (see [10, Proposition 1; cf. also Example 2 therein]).

In general, (G) is strictly weaker than (G)$'$, as the example of the counting quasi-measure on the algebra of finite and cofinite subsets of $\mathbb{N}$ shows.

The latter basic property ν may have is defined as follows:

(F) *There exists a constant* M *such that given* $a \in I_\nu$, *we can find* $a_1, \dots, a_n$ *in* A *with* $\nu(a_i) \le M$ *for each* i *and* $\bigvee_{i=1}^n = a$.

Roughly speaking, this means that ν is uniformly bounded on the family of its atoms which have finite ν-quasi-measure.

We shall give two simple examples to show that there is no logical dependence between properties (G) and (F), in general.

Example 1. Let $A = 2^{\mathbb{N}}$ and let $\nu = \sum_{n=1}^\infty n\delta_n$, where δ_n denotes the Dirac measure on $2^{\mathbb{N}}$ concentrated at $\{n\}$. Clearly, ν has property (G) (in fact, even (G)$'$). On the other hand, (F) fails, since $\nu(\{n\}) \to \infty$.

Example 2. Let $A = 2^\Gamma$, where Γ is a nonempty set, and let $\nu = \infty \cdot \delta_\gamma$ for some $\gamma \in \Gamma$. Then (G), clearly, fails, while (F) holds.

The first part of the following result is contained in [9, Propositions 3.1.8 and 3.1.9], and we use the argument of [9] below.

Proposition 1. *Let* ν *be a quasi-measure on* A. *Then there exist quasi-measures* ν_1 *and* ν_2 *on* A *such that*

(a) ν_1 *is semifinite;*

(b) $\nu_2(A) \subset \{0, \infty\}$*;*

(c) $\nu = \nu_1 + \nu_2$.

If, moreover, ν has property (F) *or* (G) *or both, then ν_1 and ν_2 can be chosen with those properties.*

Proof. Set

$$\nu_1(a) = \sup\{\nu(b) : b \in I_\nu \cap C_a\}$$

for all $a \in A$. As is easily seen, ν_1 is a quasi-measure on A and (a) holds. Suppose ν has property (F) with some constant M. We claim that ν_1 has property (F) with the same constant M. Indeed, fix $a \in I_{\nu_1}$, and take $b \in I_\nu$ with $b \le a$ and

$$\nu_1(a) - \nu(b) \le M.$$

By assumption, we can find $b_1, \dots, b_n$ in A with $\nu(b_i) \le M$ for each i and $\bigvee_{i=1}^n b_i = b$. Thus, (F) holds for ν_1, M, a and $a \smallsetminus b$, $b_1, \dots, b_n$.

Set

$$J = \{a \in A : \nu(b) = \nu_1(b) \text{ for every } b \in C_a\}.$$

Clearly, J is an ideal in A with $I_\nu \subset J$. Set

$$\nu_2(a) = \begin{cases} 0 & \text{if } a \in J, \\ \infty & \text{if } a \in A \setminus J. \end{cases}$$

Then ν_2 is a quasi-measure on a A and (b) and (c) hold. Suppose ν has property (G). Fix $a \in A \setminus I_{\nu_2}$, and take $b \in C_a$ with $\nu(b) > \nu_1(b)$. Since $\nu(b) = \infty$, there are disjoint $b_1, b_2 \in A$ with

$$\nu(b_1),\ \nu(b_2) > \nu_1(b) \quad \text{and} \quad b = b_1 \vee b_2.$$

In particular, $\nu(b_i) > \nu_1(b_i)$, whence $\nu_2(b_i) = \infty$ for $i = 1, 2$. Consequently, ν_2 has property (G). $\square$

Remark 1. Conditions (a)–(c) of Proposition 1 do not determine ν_1 and ν_2 uniquely. Indeed, if ν is infinite and semifinite, the proof above yields $\nu_1 = \nu$ and $\nu_2 = 0$. Alternatively, we could then take for ν_2 the quasi-measure which equals 0 on I_ν and ∞ otherwise. On the other hand, if $\nu(A) = \{0, \infty\}$, the proof above yields $\nu_2 = \nu$ and $\nu_1 = 0$, but we could then take for ν_1 an arbitrary semifinite quasi-measure majorized by ν.

Remark 2. The second part of Proposition 1 can be reversed as follows. If quasi-measures ν_1 and ν_2 on A have property (F) [resp., property (G)], then so does $\nu_1 + \nu_2$.

3. Preliminaries on group-valued additive functions

Let G be an *Abelian normed group*. This means, in particular, that G is equipped with a map $\|\cdot\| : G \to [0,\infty)$, the norm of G, with the following three properties: $\|x\| = 0$ if and only if $x = 0$, $\|x\| = \|-x\|$ and $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in G$. Standard examples are $\mathbb{R}$, the additive group of real numbers, and its subgroup $\mathbb{Q}$ of rational numbers, both equipped with the usual absolute value.

Every real or complex normed space is a normed group. Of special importance for our purposes is the Banach space $l_\infty(\Gamma)$ of bounded scalar functions on a (nonempty) set Γ, with the supremum norm $\|\cdot\|_\infty$.

Let $\varphi : A \to G$ be additive (i.e., a group-valued charge, as some authors say). We denote by $|\varphi|$ the *variation of* φ, i.e., the map from A into $[0,\infty]$ whose value at $a \in A$ equals the supremum of the sums $\sum_{i=1}^n \|\varphi(a_i)\|$, where $a_1, \ldots, a_n$ are pairwise disjoint elements of A with $\bigvee_{i=1}^n a_i = a$ (see [4, Definition I.1.4] for the case where G is a Banach space). As is easily seen, $|\varphi|$ is a quasi-measure on A.

For $\varphi : A \to G$ we set

$$\|\varphi\| = \sup\{\|\varphi(a)\| : a \in A\}.$$

We say that φ is *bounded* if $\|\varphi\| < \infty$.

We denote by $a(A,G)$ the set of all additive $\varphi : A \to G$ and we put

$$ba(A,G) = \{\varphi \in a(A,G) : \|\varphi\| < \infty\}.$$

Equipped with the pointwise addition, $a(A,G)$ is an Abelian group, and $ba(A,G)$ is a subgroup of $a(A,G)$. Moreover, $\|\cdot\|$ defined above is a group norm in $ba(A,G)$.

The following result will be applied jointly with Proposition 1 in Sections 4 and 5.

Proposition 2. *Let $\varphi_1, \varphi_2 \in a(A,G)$, let $|\varphi_1|$ be semifinite and let $|\varphi_2|(A) \subset \{0,\infty\}$. We then have*

$$|\varphi_1 + \varphi_2| = |\varphi_1| + |\varphi_2|.$$

Proof. We only need to show that $|\varphi_1|(a) + |\varphi_2|(a) \le |\varphi_1 + \varphi_2|(a)$ for all $a \in A$. This is clear if $|\varphi_2|(a) = 0$. Suppose $|\varphi_2|(a) = \infty$ and consider two cases.

1. $|\varphi_1|(a)<\infty$. Given pairwise disjoint $a_1, \ldots, a_n$ in A with $\bigvee_{i=1}^n a_i=a$, we have

$$|\varphi_1+\varphi_2|(a) \geq \sum_{i=1}^n \|\varphi_1(a_i)+\varphi_2(a_i)\|$$

$$\geq \sum_{i=1}^n \|\varphi_2(a_i)\| - \sum_{i=1}^n \|\varphi_1(a_i)\| \geq \sum_{i=1}^n \|\varphi_2(a_i)\| - |\varphi_1|(a),$$

whence $|\varphi_1+\varphi_2|(a)=\infty$.

2. $|\varphi_1|(a)=\infty$. If there exists $b \in C_a$ with $|\varphi_1|(b)<\infty$ and $|\varphi_2|(b)=\infty$, then, by Case 1,

$$|\varphi_1+\varphi_2|(a) \geq |\varphi_1+\varphi_2|(b)=\infty.$$

Otherwise, $|\varphi_2|(b)=0$ for all $b \in C_a$ with $|\varphi_1|(b)<\infty$, and so

$$|\varphi_1+\varphi_2|(a) \geq |\varphi_1+\varphi_2|(b)=|\varphi_1|(b).$$

The semifiniteness of $|\varphi_1|$ yields $|\varphi_1+\varphi_2|(a)=\infty$. □

4. The variation of an arbitrary additive function

The following essentially known result will be applied in the proofs of Lemmas 2 and 3.

Lemma 1. *Let A_0 be a subring of A and let $\varphi_0 : A_0 \to H$, where $H=\mathbb{Q}$ or $\mathbb{R}$, be additive. Then there exists $\varphi \in a(A,H)$ which extends φ_0.*

Proof. In view of the Stone representation theorem, we may assume that A is an algebra of subsets of some set Ω. Let S [resp., S_0] stand for the $\mathbb{Q}$-linear space of A-simple [resp., A_0-simple] functions over Ω with values in $\mathbb{Q}$. As is well known (cf. [2, Corollary 3.1.8]), there exists a unique $\mathbb{Q}$-linear operator $\Phi_0 : S_0 \to H$ with

$$\Phi_0(1_a)=\varphi_0(a) \quad \text{for all } a \in A_0.$$

By a standard transfinite argument, Φ_0 can be extended to a $\mathbb{Q}$-linear operator $\Phi : S \to H$. Set $\varphi(a)=\Phi(1_a)$ for all $a \in A$. Clearly, φ is as desired. □

Remark 3. In fact, Lemma 1 holds for an arbitrary Abelian group H; cf. [3, Theorem 4 and Remark 1], where groups of simple functions over Ω with values in $\mathbb{Z}$ are considered and the argument is based on a theorem of G. Nöbeling.

Lemma 2. *If ν is a semifinite quasi-measure on A, then there exists $\varphi \in a(A, \mathbb{R})$ with $|\varphi| = \nu$.*

Proof. Apply Lemma 1 to $H = \mathbb{R}$, $A_0 = I_\nu$ and $\varphi_0 = \nu | I_\nu$. Since I_ν is an ideal in A, the resulting φ satisfies $|\varphi|(a) = \nu(a)$ for all $a \in I_\nu$. The assertion now follows by the semifiniteness of ν. $\square$

Remark 4. It is straightforward that in Lemma 2, and so in Theorem 1 below, the group $\mathbb{R}$ cannot be replaced by any of its proper subgroups. Indeed, choose for A and ν the Lebesgue σ-algebra of $[0,1]$ and the Lebesgue measure over $[0,1]$, respectively. Take $\varphi \in a(A, \mathbb{R})$ with $|\varphi| = \nu$. By the Hahn decomposition theorem, $\varphi(A)$ includes one of the intervals $[0, 1/2]$ or $[-1/2, 0]$, and so is not included in a proper subgroup of $\mathbb{R}$.

Lemma 3. *If A is nonatomic, then there exists $\varphi \in a(A, \mathbb{Q})$ with $|\varphi|(a) = \infty$ for every nonzero $a \in A$.*

Proof. We argue in two steps.

Step I. We assume that A has the additional property thay $|C_a| = |A|$ for every nonzero $a \in A$, i.e., A is cardinality-homogeneous in the terminology of [8, pp. 198–199]. In view of the Stone representation theorem, we may assume that A is an algebra of subsets of some set Ω. Let S stand for the $\mathbb{Q}$-linear space of A-measurable simple functions over Ω with values in $\mathbb{Q}$.

Set $\alpha = |A|$. Arrange the nonzero elements of A into a transfinite sequence $\{a_\beta : \beta < \alpha\}$. By the additional property of A, the dimension of

$$\mathrm{lin}_{\mathbb{Q}}\{1_b : b \in C_a\}$$

equals α for every nonzero $a \in A$. This allows us to define, by transfinite induction, elements b_β of A such that for all $\beta < \alpha$ we have

(1) $b_\beta \in C_{a_\beta}$;

(2) $1_{b_\beta} \notin \mathrm{lin}_{\mathbb{Q}}\{1_{b_\gamma} : \gamma < \beta\}$.

In view of (2), we can find $T \subset S \setminus \{1_{b_\beta} : \beta < \alpha\}$ so that the set

$$\{1_{b_\beta} : \beta < \alpha\} \cup T$$

is a Hamel basis for S. In consequence, there exists a (unique) $\mathbb{Q}$-linear functional Φ on S with

$$\Phi(1_{b_\beta}) = 1 \text{ for all } \beta < \alpha \text{ and } \Phi(t) = 0 \text{ for all } t \in T.$$

Set $\varphi(a) = \Phi(1_a)$ for all $a \in A$. Clearly, $\varphi \in a(A, \mathbb{Q})$. For every nonzero $a \in A$ there exist nonzero pairwise disjoint $c_1, c_2, \dots$ in C_a. By (1), we can find ordinals $\beta_1, \beta_2, \dots < \alpha$ with $b_{\beta_i} \in C_{c_i}$ for each i. We then have

$$\sum_{i=1}^{n} |\varphi(b_{\beta_i})| = n,$$

whence $|\varphi|(a) = \infty$.

Step II. A is an arbitrary nonatomic Boolean algebra. Let D be a subset of A of nonzero pairwise disjoint elements with $\sup D = 1$ and C_d cardinality-homogeneous for each $d \in D$ (see [8, Lemma 13.12], and note that, in view of [8, Lemma 4.9], its proof does not require the assumption that A be complete). By Step I, there exists $\varphi_d \in a(C_d, \mathbb{Q})$ with $|\varphi_d|(a) = \infty$ whenever $d \in D$ and $a \in C_d$ is nonzero. Denote by J the ideal in A generated by D. For every $a \in J$ set

$$\xi(a) = \sum_{d \in D} \varphi_d(a \wedge d).$$

Since $\{d \in D : a \wedge d \neq 0\}$ is finite, this definition is correct. Moreover, $\xi : J \to \mathbb{Q}$ is additive. By Lemma 1, there exists $\varphi \in a(A, \mathbb{Q})$ which extends ξ. Now, given nonzero $a \in A$, we have $a \wedge d \neq 0$ for some $d \in D$, so that

$$|\varphi|(a) \geq |\varphi|(a \wedge d) = |\varphi_d|(a \wedge d) = \infty.$$

This completes the proof. □

Remark 5. In the case where A is the quotient Boolean algebra of $2^{\mathbb{N}}$ by the ideal of finite subsets of $\mathbb{N}$, Lemma 3 is related to the following result due to Godefroy and Talagrand ([7, Proposition 5]; see also [6, Proposition 3]): *There exists $\varphi \in a(2^{\mathbb{N}}, \mathbb{R})$ such that $\varphi(M) = 0$ if and only if $M \subset \mathbb{N}$ is finite.* Then $|\varphi|(M) = 0$ or ∞ according as M is finite or infinite. Indeed, suppose M is infinite, and take an uncountable family $\mathfrak{N}$ of (infinite) subsets of M with $N \cap N'$ finite whenever $N, N' \in \mathfrak{N}$ and $N \neq N'$ (see, e.g., [8, p. 80]). Then there exist different $N_1, N_2, \dots$ in $\mathfrak{N}$ with

$$\inf \bigl\{ |\varphi(N_i)| : i = 1, 2, \dots \bigr\} > 0.$$

Set $P_1 = N_1$, and $P_{i+1} = N_{i+1} \setminus \bigcup_{j=1}^{i} N_j$ for $i = 1, 2, \dots$. Clearly,

$$\sup \Biggl\{ \sum_{i=1}^{n} |\varphi(P_i)| : n = 1, 2, \dots \Biggr\} = \infty.$$

Hence $|\varphi|(M) = \infty$.

Note, however, that the range of φ of the Godefroy–Talagrand result has cardinality $2^{\aleph_0}$. Indeed, let $\mathfrak{C}$ be a family of subsets of $\mathbb{N}$ such that $|\mathfrak{C}| = 2^{\aleph_0}$ and $N \subset N'$ or $N' \subset N$ and $N \vartriangle N'$ is infinite whenever N, $N' \in \mathfrak{C}$ are different. Clearly, φ is injective when restricted to $\mathfrak{C}$. (To define $\mathfrak{C}$, consider the sets $\{q \in \mathbb{Q} : q < \eta\}$, where $\eta \in \mathbb{R}$, and use the equipotency of $\mathbb{Q}$ and $\mathbb{N}$; cf. also [12, the passage preceding Theorem 5.4].)

The implication (iii) $\Longrightarrow$ (i) of Theorem 1 below generalizes [10, Proposition 6]. This is due to the equivalence of properties (G) and (G)$'$ under the assumption that A be σ-complete (see Section 2 above).

Theorem 1. *For a quasi-measure ν on A the following three conditions are equivalent:*

(i) *ν has property* (G)*;*

(ii) *There exist an Abelian normed group G and $\psi \in a(A, G)$ with $|\psi| = \nu$;*

(iii) *There exists $\varphi \in a(A, \mathbb{R})$ with $|\varphi| = \nu$.*

Proof. Clearly, (iii) implies (ii), and so it is enough to show that (i) implies (iii) and (ii) implies (i).

Suppose (i) holds. We first consider the special case where $\nu(A) = \{0, \infty\}$. Due to property (G), the quotient Boolean algebra A/I_ν is nonatomic. Denote by h the canonical homomorphism of A onto A/I_ν. In view of Lemma 3, there exists $\tilde{\varphi} \in a(A/I_\nu, \mathbb{Q})$ with $|\tilde{\varphi}|(h(a)) = \infty$ for every a in $A \setminus I_\nu$. Setting $\varphi = \tilde{\varphi} \circ h$, we have $|\varphi| = |\tilde{\varphi}| \circ h$, which yields (iii). The general case follows from the special case with the help of Propositions 1 and 2 and Lemma 2.

Let ψ be as in (ii). We shall show that $|\psi|$ has property (G). Fix $a \in A$ with $|\psi|(a) = \infty$ and $\eta > 0$. Set

$$\vartheta = \sup\{\|\psi(b)\| : b \in C_a\}.$$

We consider two cases.

1. $\vartheta < \infty$. Then there exist pairwise disjoint $b_1, \ldots, b_n$ in A with

$$\bigvee_{i=1}^{n} b_i = a \quad \text{and} \quad \sum_{i=1}^{n} \|\psi(b_i)\| > \vartheta + \eta.$$

Consequently,

$$\sum_{i \neq j} \|\psi(b_i)\| > \eta, \quad \text{and so} \quad |\psi|\Big(\bigvee_{i \neq j} b_i\Big) > \eta$$

whenever $1 \leq j \leq n$. On the other hand, $|\psi|(b_j) = \infty$ for some j.

2. $\vartheta = \infty$. Then there exists $a_1 \in C_a$ with

$$\|\psi(a_1)\| > \|\psi(a)\| + \eta.$$

It follows that

$$\|\psi(a \setminus a_1)\| = \|\psi(a) - \psi(a_1)\| \geq \|\psi(a_1)\| - \|\psi(a)\| > \eta.$$

Hence $|\psi|(a_1)$, $|\psi|(a \setminus a_1) > \eta$. □

From Theorem 1 we get immediately the following corollary.

Corollary 1. *If G is an Abelian normed group and $\psi \in a(A, G)$, then there exists $\varphi \in a(A, \mathbb{R})$ with $|\varphi| = |\psi|$.*

5. The variation of a bounded additive function

We start with two lemmas which are analogues, in the bounded case, of Lemmas 2 and 3 of Section 4.

Lemma 4. *If a semifinite quasi-measure ν on A has property* (F), *then there exist $E \subset A$ and $\varphi \in ba(A, l_\infty(E))$ with $|\varphi| = \nu$.*

Proof. Set $E = \{a \in A : \nu(A) \leq M\}$, where M is given by property (F), and

$$\varphi(a)(e) = \nu(a \wedge e) \quad \text{for all } a \in A \text{ and } e \in E.$$

Clearly, $\varphi \in a(A, l_\infty(E))$ and $\|\varphi\| \leq M$. Moreover, $\|\varphi(a)\|_\infty \leq \nu(a)$ for all $a \in A$, whence $|\varphi| \leq \nu$. To prove the other inequality, fix $a \in I_\nu$. According to property (F), we choose pairwise disjoint $e_1, \ldots, e_n$ in E with $\bigvee_{i=1}^n e_i = a$. We then have

$$|\varphi|(a) \geq \sum_{i=1}^n \|\varphi(e_i)\|_\infty = \sum_{i=1}^n \nu(e_i) = \nu(a).$$

Since ν is semifinite, this implies $|\varphi| \geq \nu$, completing the proof. □

The following lemma will be given two proofs. The first is, in some sense, more explicit, while the second is more economical in the choice of Γ and slightly more elementary.

Lemma 5. *If A is nonatomic, then there exist a set Γ and $\varphi \in ba(A, l_\infty(\gamma))$ with $|\varphi|(a) = \infty$ for all nonzero $a \in A$.*

Proof. It is enough to find Γ and $\varphi \in ba(A, l_\infty(\Gamma))$ with $\|\varphi(a)\|_\infty = 1$ for all nonzero $a \in A$. This will be done in two different ways.

1. By the Stone representation theorem, we may assume that A is an algebra of subsets of some set Γ. Put $\varphi(a) = 1_a$ for every $a \in A$.

2. Set $\Gamma = A \setminus \{0\}$ and choose for every $c \in \Gamma$ a probability quasi-measure ν_c on A with $\nu_c(c) = 1$. Put $\varphi(a)(c) = \nu(a \wedge c)$ for all $a \in A$ and $c \in \Gamma$. □

In connection with Proposition 3 below note that the variation of φ of Lemma 5 does not change if we replace the original norm of $l_\infty(\Gamma)$ by an equivalent one.

Lemma 6. *Let G be an Abelian normed group, let $\varphi \in ba(A, G)$ and let $|\varphi|(1) < \infty$. If μ is a two-valued quasi-measure on A with $\mu \leq |\varphi|$, then $\mu(1) \leq \|\varphi\|$.*

Proof. Fix $\varepsilon > 0$ and choose pairwise disjoint $a_1, \ldots, a_n$ in A with

$$\bigvee_{i=1}^{n} a_i = 1 \quad \text{and} \quad |\varphi|(1) \leq \sum_{i=1}^{n} \|\varphi(a_i)\| + \varepsilon.$$

For each $j = 1, \ldots, n$ we then have

$$|\varphi|(1) \leq \|\varphi(a_j)\| + \sum_{i \neq j} |\varphi|(a_i) + \varepsilon,$$

whence $|\varphi|(a_j) \leq \|\varphi(a_j)\| + \varepsilon$. Since $\mu(1) = \mu(a_j)$ for some j, it follows that $\mu(1) \leq \|\varphi\| + \varepsilon$. This yields the assertion. □

Theorem 2. *For a quasi-measure ν on A the following three conditions are equivalent:*

(i) *ν has properties* (F) *and* (G)*;*

(ii) *There exist an Abelian normed group G and $\psi \in ba(A, G)$ with $|\psi| = \nu$;*

(iii) *There exist a set Γ and $\varphi \in ba(A, l_\infty(\Gamma))$ with $|\varphi| = \nu$.*

Proof. Clearly, (iii) implies (ii). To see that (i) implies (iii), we only have to modify the proof of the corresponding implication of Theorem 1. The modification consists in appealing to Lemmas 4 and 5 in place of Lemmas 2 and 3. Moreover, we have to note that, given abstract sets Γ_1 and Γ_2 with $|\Gamma_1| \leq |\Gamma_2|$, we can treat $l_\infty(\Gamma_1)$ as a subspace of $l_\infty(\Gamma_2)$.

We shall complete the proof by showing that (ii) implies (i). To this end, let ψ be as in (ii). In view of Theorem 1, (ii) $\Longrightarrow$ (i), $|\psi|$ has property (G). We shall check that $|\psi|$ has property (F) with arbitrary $M > \|\psi\|$. We may restrict ourselves to the case where $|\psi|(1) < \infty$ and consider only $a = 1$.

By the Sobczyk–Hammer decomposition theorem (see [2, Theorem 5.2.7]), there exist quasi-measures $\mu_0, \mu_1, \ldots$ on A such that

$$\sum_{i=0}^{\infty} \mu_i = |\psi|,$$

μ_0 is strongly continuous (i.e., given $\varepsilon > 0$, there exists $c_1, \dots, c_n$ in A with $\bigvee_{k=1}^n c_k = 1$ and $\mu_0(c_k) < \varepsilon$ for each k) while μ_i, $i = 1, 2, \dots$, takes at most two values and μ_i and μ_j are linearly independent whenever $i \neq j$ and μ_i, $\mu_j \neq 0$. Fix m with

$$\sum_{i=m+1}^{\infty} \mu_i(1) < M - \|\psi\|.$$

By [2, Proposition 5.2.2], there exist pairwise disjoint $b_1, \dots, b_m$ in A such that

$$\bigvee_{j=1}^{m} b_j = 1 \quad \text{and} \quad \mu_j(b_j) = \mu_j(1) \text{ for each } j.$$

In view of Lemma 6, it follows that

$$\sum_{i=1}^{\infty} \mu_i(b_j) < \mu_j(1) + M - \|\psi\| \leq M \text{ for each } j.$$

Set

$$\varepsilon = M - \max\left\{ \sum_{i=1}^{\infty} \mu_i(b_j) : j = 1, \dots, m \right\}.$$

Let $c_1, \dots, c_n$ be given according to the strong continuity of μ_0. We then have

$$\bigvee_{j=1}^{m} \bigvee_{k=1}^{n} b_j \wedge c_k = 1 \quad \text{and} \quad |\psi|(b_j \wedge c_k) < M \text{ for all } j,\ k.$$

Thus, $|\psi|$ has property (F). □

From Theorem 2 we get immediately the following corollary.

Corollary 2. *If G is an Abelian normed group and $\psi \in ba(A, G)$, then there exist a set Γ and $\varphi \in ba(A, l_\infty(\Gamma))$ with $|\varphi| = |\psi|$.*

Remark 6. Condition (iii) of Theorem 2 can be reformulated as follows:

(iii)′ *There exist a normed space X and $\varphi \in ba(A, X)$ with $|\varphi| = \nu$.*

Indeed, given a normed space X, there exist a set Γ and a linear isometric embedding of X into $l_\infty(\Gamma)$ (see, e.g., [1, Proposition II.1.3]).

The appearance of $l_\infty(\Gamma)$ in Lemma 4 and Theorem 2 is, to some extent, necessary. Moreover, no global restriction on the cardinality of Γ in condition (iii) of Theorem 2 is possible, which is in sharp contrast with both Theorem 1 above and Theorem 2 of [11]. This is seen from our final result.

Proposition 3. *Let X be a Banach space and let Γ be an infinite set. If there exists $\varphi \in ba(2^\Gamma, X)$ with $|\varphi|(\{\gamma\}) = 1$ for all $\gamma \in \Gamma$, then X contains an isomorphic copy of $l_\infty(\Gamma)$.*

Proof. There exists a (unique) bounded linear operator $\Phi : l_\infty(\Gamma) \to X$ such that

$$\Phi(1_M) = \varphi(M) \quad \text{for all } M \in 2^\Gamma$$

(see [4, pp. 5–6]). Clearly, we have

$$\|\Phi\left(1_{\{\gamma\}}\right)\| = \|\varphi(\{\gamma\})\| = |\varphi|(\{\gamma\}) = 1 \quad \text{for all } \gamma \in \Gamma.$$

A result of Rosenthal ([13, Proposition 1.2 and Remark 1 following it]; see also [5, théorème]) yields a subset Γ' of Γ such that $|\Gamma'| = |\Gamma|$ and Φ when restricted to the closed subspace

$$\{x \in l_\infty(\Gamma) : x(\gamma) = 0 \text{ for all } \gamma \in \Gamma \setminus \Gamma'\}$$

of $l_\infty(\Gamma)$ is an isomorphism. Thus, the assertion holds. □

Added in proof. A result closely related to Theorem 1, (ii) $\Longrightarrow$ (i), is contained in H. Weber [FN-toplogies and group-valued measures, in: Handbook of Measure Theory (E. Pap, ed.), Vol. 1, North-Holland, Amsterdam, 2002, 703–743, Proposition 2.12].

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(Received August 12, 2002; revised November 27, 2002)