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On the differences between polynomial values and perfect powers

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Dedicated to the memory of Professor Péter Kiss

Abstract. Let $F(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$, and let x, b, y, m be non-zero integers with $m \geq 2$, $|y| \geq 2$ and $F(x) \neq by^m$. Under some natural assumptions on F, we give explicit lower bounds for $|F(x) - by^m|$, depending only on n, m, b, $H(F)$ and n, b, $F(x)$, $H(F)$, respectively. These results generalize Theorems 1 and 2 of Bugeaud and Hajdu [8]. To prove our results, we slightly improve and make completely explicit the upper bound obtained in [3] for the unknown exponent m in the superelliptic equation (1).

1. Introduction

Let a, b, x, y, n, m be non-zero integers with $n, m \geq 2$, $|y| \geq 2$ and $ax^n \neq by^m$. The first effective lower bound for $|ax^n - by^m|$ which is independent of x and y was proved by Turk [16], in case of $a = b = 1$. A result of similar strength valid for arbitrary a and b, however not completely explicit, can also be deduced from the work of Shorey [14]. Later,

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Bugeaud [5] considerably sharpened Turk's estimate for $|x^n - y^m|$. Recently, thanks to some refined arguments, Bugeaud and Hajdu [8] improved and extended Bugeaud's result to arbitrary a and b. The purpose of this paper is to generalize the results of Bugeaud and Hajdu [8] to differences of the form $|F(x) - by^m|$, where $F(X) \in \mathbb{Z}[X]$ is a polynomial of degree $n \geq 2$.

Under certain assumptions on F, we derive explicit lower bounds for $|F(x)-by^m|$ (cf. Theorem 2) from our Theorem 1 which provides an explicit upper bound for the exponent m in the equation

$$f(x) = by^m \text{ in } x, y, m \in \mathbb{Z}, \quad \text{with } |y| \geq 2,\ m \geq 1, \tag{1}$$

in terms of b and the height of $f \in \mathbb{Z}[X]$. The first results proving that m is bounded were given by Tijdeman [15] and Schinzel and Tijdeman [13]. Later, several effective but not completely explicit upper bounds were obtained for m; see [2], [4], [3] and the references given there. Our Theorem 1 slightly improves and makes explicit in each parameter the previously best known bound (cf. [3]) on m. In our proof we will follow the approach of Brindza, Evertse and Győry [4]. They gave an estimate for m from above in terms of the discriminant of f.

2. New results

Throughout the paper, we use the following notation. For every positive real number s, we put $\log_* s = \max\{1, \log s\}$. Let

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = a_0 \prod_{i=1}^{n} (x - \alpha_i), \quad a_0 \neq 0,$$

be a polynomial with integer coefficients. We write

$$H(f) = \max_{0 \leq i \leq n} |a_i| \quad \text{and} \quad M(f) = |a_0| \prod_{i=1}^{n} \max(1, |\alpha_i|)$$

for the "classical" height and the Mahler-height of f, respectively.

Theorem 1. *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$ and b a non-zero integer. If f has at least two distinct roots, then equation* (1) *with $x, y, m \in \mathbb{Z}$ and $|y| \geq 2,\ m \geq 1$ implies*

$$m \leq 2^{24n+56} n^{7n+17} M(f)^{3n-3} (\log_* M(f))^{3n} (\log_* |b|)^{\frac{5}{2}}.$$

As was mentioned above, our Theorem 1 slightly improves and makes completely explicit the previously best known result of this type, estabilished in [3]. In the special case $f(x) = ax^n + c$, a similar result was proved in [8]. Our Theorem 1 is also related to Theorem 5 of Brindza, Evertse and Győry [4], where it is assumed that $b = 1$ and f is irreducible and monic, but the bound given for m depends only on n and the discriminant of f.

In the proof of Theorem 1 we will follow the approach of [4]. We obtain the following result as a consequence of Theorem 1.

Theorem 2. *Let $F(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$, and let b, x, y, m be integers with $b \neq 0$, $m \geq 1$, $|y| \geq 2$. Suppose that $F(x) \neq by^m$, and if $F(X)$ is of the special form $F(X) = t_1(X - t_2)^n + t_3$ with $t_1, t_2, t_3 \in \mathbb{Z}$, then also assume that $F(x) \neq by^m + t_3$. Then we have*

$$|F(x) - by^m| \geq m^{\frac{1}{3n}} 2^{-8-\frac{56}{3n}} n^{-\frac{23}{6}-\frac{17}{3n}} \left(H(F) \log_*^{\frac{5}{6n}} |b| \right)^{-1}. \qquad (2)$$

We note that to give a lower bound for $|F(x) - by^m|$, we need to use the classical height instead of the Mahler-height. The reason is that for every $k \in \mathbb{Z}$, plainly $H(F - k) \leq H(F) + |k|$, but $M(f)$ does not have a similar nice property. However, the use of the classical height already in Theorem 1 would result in a worse estimate for $|F(x) - by^m|$.

As in [8], by combining Theorem 2 with an estimate for the size of the solutions of superelliptic equations, we derive a lower bound for $|F(x) - by^m|$ in terms of $|F(x)|$.

Theorem 3. *Let $F(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$, and let b, x, y, m be integers with $b \neq 0$, $m \geq 3, |y| \geq 2$. Suppose that $F(x) \neq by^m$, and if $F(X)$ is of the special form $F(X) = t_1(X - t_2)^n + t_3$*

with $t_1, t_2, t_3 \in \mathbb{Z}$, then also assume that $F(x) \neq by^m + t_3$. Then

$$|F(x) - by^m| \geq c_1 n^{-\frac{23}{6}} H(F)^{-1} (\log_* |b|)^{-\frac{4}{3n+1}} (\log_* \log_* |F(x)|)^{\frac{1}{3n+1}}, \quad (3)$$

where c_1 denotes an effectively computable absolute constant.

Theorem 2 generalizes the estimate

$$|ax^n - by^m| \geq m^{2/5n}(20n)^{-2-11/n} \left(|a| \log_*^{\frac{1}{n}} |b|\right)^{-1}$$

of Bugeaud and Hajdu [8]. Similarly, our Theorem 3 is an extension of Theorem 2 of [8]. Observe that our bound in (2) in the special case $F(x) = ax^n$ yields an estimate of similar strength as in [8], up to the exponent of m. This difference comes from the fact that $\Delta(ax^n + k) \leq c_2|k|^n$, while in general we only have $\Delta(F(x)+k) \leq c_3|k|^{2n}$. Here $\Delta(g(x))$ denotes the discriminant of $g(x) \in \mathbb{Z}[x]$ and c_2, c_3 are constants depending on a, n and F, respectively.

3. Some lemmas

For a non-zero algebraic number α of degree l over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{i=1}^{l}(X - \alpha_i)$, let

$$h(\alpha) = \frac{1}{l}\left(\log |a| + \sum_{i=1}^{l} \log \max(1, |\alpha_i|)\right)$$

denote the absolute logarithmic height of α. Let $\mathbb{K}$ be a number field of degree $d_{\mathbb{K}}$, with unit rank $r_{\mathbb{K}}$ and regulator $R_{\mathbb{K}}$. In the course of our proof, we use an independent system of units in $\mathbb{K}$ with small height, provided by the following lemma.

Lemma 1. *There exists an independent system $\varepsilon_1, \ldots, \varepsilon_{r_{\mathbb{K}}}$ of units in $\mathbb{K}$ satisfying*

$$\prod_{i=1}^{r_{\mathbb{K}}} h(\varepsilon_i) \leq {d_{\mathbb{K}}}^{-r_{\mathbb{K}}} r_{\mathbb{K}}! R_{\mathbb{K}} \quad (4)$$

and

$$h(\varepsilon_i) \leq r_{\mathbb{K}}! {d_{\mathbb{K}}}^{-1} (9(\log 3d_{\mathbb{K}})^3/8)^{r_{\mathbb{K}}-1} R_{\mathbb{K}}, \quad i = 1, \ldots, r_{\mathbb{K}}. \quad (5)$$

Moreover, for every non-zero algebraic integer $\alpha \in \mathbb{K}$*, there exists a unit* ε *in the multiplicative subgroup generated by* $\varepsilon_1, \ldots, \varepsilon_{r_\mathbb{K}}$ *such that*

$$h(\varepsilon\alpha) \le (\log N_{\mathbb{K}/\mathbb{Q}}(\alpha))/(2d_\mathbb{K}) + (r_\mathbb{K}+1)^{r_\mathbb{K}+1} \log^{3r_\mathbb{K}+3}(3d_\mathbb{K}) R_\mathbb{K}. \tag{6}$$

Proof. This is a reformulation of Lemme 1 and Lemme 2 of [6]. □

Our proof ultimately depends on Baker's estimate for linear forms in logarithms. We use the following version due to Matveev [12], which is a sharpening of an estimate given by Baker and Wüstholz [1].

Lemma 2. *Let* $\mathbb{K}$ *be an algebraic number field of degree* D *over* $\mathbb{Q}$*. Let* $\alpha_1, \ldots, \alpha_n \in \mathbb{K}^*$ *with absolute logarithmic heights* $h(\alpha_j)$ $(1 \le j \le n)$*, and* $\log \alpha_1, \ldots, \log \alpha_n$ *arbitrary fixed non-zero values of the logarithms. Suppose that*

$$A_j \ge \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad (1 \le j \le n).$$

Consider the linear form

$$\Lambda = b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n,$$

with $b_1, \ldots, b_n \in \mathbb{Z}$ *and put* $B = \max\{|b_1|, \ldots, |b_n|\}$*. If* $\Lambda \ne 0$*, then*

$$\log |\Lambda| > -C(n) \log(eD) \log(eB) D^2 \Omega,$$

where $\Omega = A_1 \cdots A_n$ *and* $C(n) = 2^{6n+20}$*.*

Proof. This is a reformulation of Corollary 2.3 of Matveev [12]. □

We deduce Theorem 3 from Theorem 2 by using an explicit upper bound for the size of the solutions of superelliptic equations.

Lemma 3. *Let* a *and* m *be non-zero integers with* $m \ge 3$ *and* $Q(X) = \prod_{i=1}^r (X-\alpha_i)^{e_i} \in \mathbb{Z}[X]$ *be a monic polynomial of degree* $n \ge 2$ *with distinct roots* $\alpha_1, \ldots, \alpha_r$*. Put* $\Delta(Q) = \prod_{i \ne j}(\alpha_i - \alpha_j)$ *and let* $m_i = m/\gcd(m, e_i)$ *for* $i = 1, \ldots, r$*. Suppose that for some* i*,* j *with* $1 \le i \ne j \le r$*, we have* $\gcd(m_i, m_j) \ge 3$*. Then all the solutions* $(x, y) \in \mathbb{Z}^2$ *of*

$$Q(x) = ay^m \tag{7}$$

satisfy

$$|x| \le H(Q)^{m+1} \exp\left\{(c_4 nm)^{c_5 n^2 m} |\Delta(Q)|^{5nm} |a|^{n^2 m} (\log_* |a\Delta(Q)|)^{2n^2 m}\right\},$$

where c_4 *and* c_5 *are effectively computable absolute constants.*

PROOF. This easily follows from the Proposition of BUGEAUD [7]. □

4. Proof of the theorems

We follow the method of the proofs in [4] and [3], but with explicit constants.

PROOF of Theorem 1. We have two cases to distinguish. First we assume that f has an irreducible factor $P \in \mathbb{Z}[X]$ of degree ≥ 2. Let δ be a root of P, moreover, let $R_{\mathbb{K}}$, $h_{\mathbb{K}}$, $D_{\mathbb{K}}$ and $r_{\mathbb{K}}$ be the regulator, class number, discriminant and unit rank of the field $\mathbb{K} = \mathbb{Q}(\delta)$, respectively. Combining the inequality

$$d_{\mathbb{K}} \leq \frac{2}{\log 3} \log |D_{\mathbb{K}}|,$$

due to GYŐRY [9] with a result of LENSTRA [10], we have

$$h_{\mathbb{K}} R_{\mathbb{K}} \leq \frac{1}{(d_{\mathbb{K}}-1)!} |D_{\mathbb{K}}|^{\frac{1}{2}} \log^{d_{\mathbb{K}}-1} |D_{\mathbb{K}}|. \tag{8}$$

By an estimate of MAHLER [11] on the discriminant of P, we get

$$|D_{\mathbb{K}}| \leq d_{\mathbb{K}}^{d_{\mathbb{K}}} M(P)^{2d_{\mathbb{K}}-2}.$$

Since $P|f$ implies $M(P) \leq M(f)$, this yields

$$|D_{\mathbb{K}}| \leq d_{\mathbb{K}}^{d_{\mathbb{K}}} M(f)^{2d_{\mathbb{K}}-2}.$$

Combining the last inequality with (8) we obtain

$$h_{\mathbb{K}} R_{\mathbb{K}} \leq \frac{1}{\sqrt{2\pi}} d_{\mathbb{K}}^{d_{\mathbb{K}}-1} e^{2d_{\mathbb{K}}-1} M(f)^{d_{\mathbb{K}}-1} (\log_* M(f))^{d_{\mathbb{K}}-1}. \tag{9}$$

Let a_0 denote the leading coefficient of f, and let $\beta_1, \ldots, \beta_n$ be the zeros of $g(x) = a_0^{n-1} f(\frac{x}{a_0})$. Set

$$\Delta(g) = \prod_{\beta_i \neq \beta_j} (\beta_i - \beta_j),$$

and write g in the form $g(x) = P_1^{k_1}(x) P_2(x)$ where $P_1(x) = a_0^{d_{\mathbb{K}}} P(\frac{x}{a_0})$ and P_2 are relatively prime polynomials in $\mathbb{Z}[X]$. Let $\beta_1, \ldots, \beta_{d_{\mathbb{K}}}$ be the zeros of P_1 with $\beta_1 = \delta$ and (x, y) be an arbitrary, however, fixed solution to (1). The greatest common divisor of the principal ideals $\langle a_0 x - \beta_1 \rangle$ and $\langle g(a_0 x)(a_0 x - \beta_1)^{-k_1} \rangle$ divides $\Delta^n(g)$. Therefore there are integral ideals A, B, C in $\mathbb{K}$ such that

$$A\langle a_0 x - \beta_1 \rangle = BC^w \tag{10}$$

where

$$w = \frac{m}{\gcd(m, k_1)}.$$

Further,

$$\max\{N_{\mathbb{K}/\mathbb{Q}}(A), N_{\mathbb{K}/\mathbb{Q}}(B)\} \leq |a_0 \cdot b \cdot \Delta(g)|^{n^2}.$$

Hence using Lemma 1, (6) and (9), by a simple calculation we obtain that the ideals $A^{h_{\mathbb{K}}}$ and $B^{h_{\mathbb{K}}}$ have generators α and β, respectively, with

$$\max\{h(\alpha), h(\beta)\} \leq c_6. \tag{11}$$

Here

$$c_6 = 0.12 n^3 (n-1) d_{\mathbb{K}}^{d_{\mathbb{K}}-1} e^{2d_{\mathbb{K}}-1} (r_{\mathbb{K}}+1)^{r_{\mathbb{K}}+1} \times (\log 3d_{\mathbb{K}})^{3r_{\mathbb{K}}+3} M(f)^{d_{\mathbb{K}}-1} (\log_* M(f))^{d_{\mathbb{K}}} \log_* |b|.$$

Equation (10) can be rewritten as

$$\alpha (a_0 x - \beta_1)^{h_{\mathbb{K}}} = \varepsilon \beta \gamma^w, \tag{12}$$

where γ is a generator of $C^{h_{\mathbb{K}}}$ and ε is a unit in $\mathbb{K}$. Let $\varepsilon_1, \ldots, \varepsilon_{r_{\mathbb{K}}}$ be an independent system of units with the properties specified in Lemma 1. Then we can express ε as $\varepsilon = \varepsilon' \varepsilon_1^{l_1} \ldots \varepsilon r_{\mathbb{K}}^{l_{r_{\mathbb{K}}}}$, where ε' is a unit with

$$h(\varepsilon') \leq (r_{\mathbb{K}}+1)^{r_{\mathbb{K}}+1} (\log(3d_{\mathbb{K}}))^{3r_{\mathbb{K}}+3} R_{\mathbb{K}}.$$

Modifying γ if necessary, we may assume that $\max_{1 \leq i \leq r_{\mathbb{K}}} |l_i| < w$.

If $|a_0 x| \leq M(g) + 1$ then

$$2^m \leq |y|^m \leq (2M(g)+1)^n,$$

and Theorem 1 is proved.

Otherwise, from $|a_0x| > M(g)+1$ it follows that $|a_0x-\beta_i|>1$ for $i=1,\dots,d_{\mathbb{K}}$. Thus we have

$$|a_0^{n-1}by^m|^{h_{\mathbb{K}}} \geq \max_{1\leq i\leq d_{\mathbb{K}}} |a_0x-\beta_i|^{h_{\mathbb{K}}}$$

$$\geq |a_0x-\beta_j|^{h_{\mathbb{K}}} \geq |\varepsilon'^{(j)}| \prod_{i=1}^{r_{\mathbb{K}}} |\varepsilon_i^{(j)}| |\alpha^{(j)}|^{-1} |\beta^{(j)}|\,|\gamma^{(j)}|$$

$$\geq \overline{|\varepsilon'|}^{-d_{\mathbb{K}}+1}\overline{|\varepsilon_1|}^{-w}\dots\overline{|\varepsilon_{r_{\mathbb{K}}}|}^{-w}\overline{|\alpha|}^{-1}\overline{|\beta|}^{-d_{\mathbb{K}}+1}\overline{|\gamma|}^{w}.$$

Here $\overline{|\nu|}$ denotes the house of the algebraic number ν, i.e. the maximum of the absolute values of its conjugates, and j is the appropriate index for which $|\gamma^{(j)}| = \overline{|\gamma|}$. Supposing $m \geq n+1$ (otherwise Theorem 1 follows), the last inequality yields

$$h(\gamma) \leq 2.182 c_6 d_{\mathbb{K}}^2 \log_* |y|,$$

with the same c_6 as above. We may assume that $|a_0x| \geq \frac{1}{2}|y|^{\frac{m}{n}}$, or else we obtain

$$|a_0x| + M(g) \geq |y|^{\frac{m}{n}},$$

and Theorem 1 follows. Thus we get

$$|a_0x-\beta_i| \geq \frac{1}{4}|y|^{\frac{m}{n}} \quad (1\leq i\leq d_{\mathbb{K}}). \tag{13}$$

We may suppose that

$$\frac{|\beta_i-\beta_j|}{|a_0x-\beta_i|} \geq \frac{|\beta_2-\beta_1|}{|a_0x-\beta_2|}, \quad 1\leq i,j\leq d_{\mathbb{K}},\ i\neq j.$$

Hence we have

$$\prod_{\substack{1\leq i,j\leq d_{\mathbb{K}}\\ \beta_i\neq\beta_j}} \frac{|\beta_i-\beta_j|}{|a_0x-\beta_i|} \leq \frac{4^{d_{\mathbb{K}}(d_{\mathbb{K}}-1)}|\Delta(g)|}{|y|^{\frac{m d_{\mathbb{K}}(d_{\mathbb{K}}-1)}{n}}}. \tag{14}$$

If $\left(\frac{a_0x-\beta_1}{a_0x-\beta_2}\right)^{h_{\mathbb{K}}} = 1$, then $\frac{\beta_1-\beta_2}{a_0x-\beta_1}$ is an algebraic integer. Thus

$$\left|N_{\mathbb{L}/\mathbb{Q}}\left(\frac{\beta_1-\beta_2}{a_0x-\beta_1}\right)\right| = \left|\frac{N_{\mathbb{L}/\mathbb{Q}}(\beta_1-\beta_2)}{\left(N_{\mathbb{K}/\mathbb{Q}}(a_0x-\beta_1)\right)^s}\right| \geq 1,$$

with $\mathbb{L} = \mathbb{Q}(\beta_1, \beta_2)$ and $[\mathbb{L} : \mathbb{K}] = s$. Combining this last inequality with (13), by $s \le d_{\mathbb{K}}$ we obtain

$$\begin{aligned}|\Delta(g)|^{n^2} &\ge |N_{\mathbb{L}/\mathbb{Q}}(\beta_1 - \beta_2)| \ge |N_{\mathbb{K}/\mathbb{Q}}(a_0 x - \beta_1)|^s \\ &\ge \left| \left(\frac{1}{4} |y|^{m/n} \right)^{d_{\mathbb{K}}} \right|^s \ge 2^{d_{\mathbb{K}} s (m/n - 2)} \ge 2^{(2m/n) - 2n^2},\end{aligned}$$

which implies Theorem 1.

If $(\frac{a_0 x - \beta_1}{a_0 x - \beta_2})^{h_{\mathbb{K}}} \ne 1$, then we may assume that $|y|^{\frac{m}{2n}} \ge 2|\Delta(g)| h_{\mathbb{K}}$ (otherwise we would obtain a much better estimate for m). So by (14) we get

$$\begin{aligned}&\log \left| \left(\frac{a_0 x - \beta_1}{a_0 x - \beta_2} \right)^{h_{\mathbb{K}}} - 1 \right| \\ &\le \log \left(h_{\mathbb{K}} \left| \frac{a_0 x - \beta_1}{a_0 x - \beta_2} - 1 \right| \right) \le -\frac{m}{2n} \log_* |y|.\end{aligned} \tag{15}$$

In the case $\left| \left(\frac{a_0 x - \beta_1}{a_0 x - \beta_2} \right)^{h_{\mathbb{K}}} - 1 \right| > \frac{1}{3}$ one can obtain a very good bound for m by (15). Otherwise, using Lemma 2, (4) and (9), we get

$$\begin{aligned}&\left| \left(\frac{a_0 x - \beta_1}{a_0 x - \beta_2} \right)^{h_{\mathbb{K}}} - 1 \right| \\ &= \left| \left(\frac{\varepsilon_1}{\varepsilon_1^{(2)}} \right)^{l_1} \cdots \left(\frac{\varepsilon_{r_{\mathbb{K}}}}{\varepsilon_{r_{\mathbb{K}}}^{(2)}} \right)^{l_{r_{\mathbb{K}}}} \frac{\varepsilon' \beta / \alpha}{\varepsilon'^{(2)} \beta^{(2)} / \alpha^{(2)}} \left(\frac{\gamma}{\gamma^{(2)}} \right)^w - 1 \right| \\ &\ge \frac{1}{2} \left| b_0 \log(-1) + \sum_{i=1}^{r_{\mathbb{K}}} l_i \log \frac{\varepsilon_i}{\varepsilon_i^{(2)}} + \log \frac{\varepsilon' \beta / \alpha}{\varepsilon'^{(2)} \beta^{(2)} / \alpha^{(2)}} + w \log \frac{\gamma}{\gamma^{(2)}} \right| \\ &\ge \exp \left\{ -c_7(n) M(f)^{3n-3} (\log_* M(f))^{3n-1} \log_*^2 |b| \log_* |y| \log_* m \right\},\end{aligned}$$

where b_0 is an integer with $|b_0| \le w(r_{\mathbb{K}} + 1)$ and

$$c_7(n) = 2^{23.1n + 48.418} n^{7n+14} \log n.$$

Here the superscript $^{(2)}$ denotes the image under the isomorphism $\mathbb{Q}(\beta_1) \to \mathbb{Q}(\beta_2)$. The comparison of this lower bound with (15) completes the proof in the first case.

In the second case f has only rational roots. Hence, all the zeros of g are integral. Let β_1 and β_2 be two distinct roots of g, of multiplicities k_1 and k_2, respectively. Repeating the argument used in the first case one gets

$$u_i(a_0x - \beta_i) = v_i y_i^w, \quad i = 1, 2 \tag{16}$$

where $w = \frac{m}{(m,k_1k_2)}$, $u_i, v_i, y_i \in \mathbb{Z}$, $|y_i| \geq 2$ and $|u_i| \leq |\Delta(g)^n|$, $|v_i| \leq |a_0^{n-1}b|$ $(i = 1, 2)$. We may suppose that $|y_2| \geq |y_1|$. Set $\Lambda_1 = \log \frac{v_1u_2}{v_2u_1} + w \log \left(\frac{y_1}{y_2}\right)$. From (16) we deduce

$$\left|\frac{u_2(\beta_2 - \beta_1)}{v_2 y_2^w}\right| = \left|\frac{v_1u_2}{v_2u_1}\left(\frac{y_1}{y_2}\right)^w - 1\right| \geq \frac{1}{2}|\Lambda_1|.$$

Using Lemma 2 again we have

$$\frac{m}{\log m} < 2^{41} n^5 \log_* M(f) \log_* |b|,$$

and Theorem 1 is proved. □

Proof of Theorem 2. Let $k = F(x) - by^m$. We apply Theorem 1 with $f(x) = F(x) - k$. Combining

$$M(f) \leq \sqrt{(n+1)}H(f)$$

with

$$H(f) \leq H(F) + |k| \leq 2H(F)|k|$$

and expressing $|k|$, we obtain the lower bound for $|F(x) - by^m|$ stated in the theorem. □

Proof of Theorem 3. Set $k = F(x) - by^m$ and let a_0 be the leading coefficient of F. By applying Lemma 3 to the equation

$$Q(a_0x) = a_0^{n-1}by^m$$

with $Q(x) = a_0^{n-1}(F(\frac{x}{a_0}) - k)$ and $a = a_0^{n-1}b$, and using the inequalities

$$|a_0| \leq H(F - k) \leq 2H(F)|k|,$$

we obtain a bound for $|x|$, hence for $|F(x)|$, in terms of $H(F)$, b, n, k and m. Namely, we get

$$\log_* \log_* |F(x)| \leq c_8 n^3 m \log_* m \log_* H(F) \log_* |b| \log_* |k|,$$

where c_8 is an effectively computable absolute constant. Further, from Theorem 2 we have

$$m^{\frac{1}{3n}} \le 2^{8+\frac{56}{3n}} n^{\frac{23}{6}+\frac{17}{3n}} H(F) \log_*^{\frac{5}{6n}} |b|\,|k|.$$

Combining these estimates, Theorem 3 easily follows. □

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