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On the central series of the adjoint group of a nilpotent p-algebra

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Abstract. Let R be a finite nilpotent algebra over a field of characteristic p and G its adjoint group. If the product of the orders of all factors $\gamma_i(G)/\gamma_{i+1}(G)$ of the lower central series $\gamma_j(G)$ with $|\gamma_i(G)/\gamma_{i+1}(G)| \geq p^3$ is bounded by some positive integer k, then the order of G is also bounded in terms of p and k.

1. Introduction

An associative nilpotent algebra R over a field F of prime characteristic p is called a *p-algebra*. Every associative nilpotent algebra R forms a group under the "circle multiplication" $a \circ b = ab + a + b$ for each pair of elements in R. This nilpotent group is called the *adjoint group* of R and is denoted by R°.

Several results are known about the structure of the adjoint group of a nilpotent algebra (see for instance [3], [4], [5], [6]). For example, Kruse showed in [5] that the nilpotency class of the adjoint group of a nilpotent algebra R does not exceed $(\dim R + 1)/2$. On the other hand, it was proved in [1], that the adjoint group G of any nilpotent p-algebra R of dimension at least 6 has at least 3 generators. Hence in almost all

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cases we have $|G/G'| \geq p^3$. In the following we obtain further information about the structure of finite p-groups that occur as the adjoint group of some nilpotent p-algebra.

If G is a finite p-group, we consider the natural numbers $\mu_i = \mu_i(G)$ determined by $|\gamma_i(G)/\gamma_{i+1}(G)| = p^{\mu_i}$; here $\gamma_i(G)$ denotes the i-th term of the lower central series of G. It follows from the above result in [5] that only p-groups G for which the mean value of μ_i is at least 2 can occur as the adjoint group of some finite nilpotent p-algebra of dimension at least 4. But what can be said about groups G for which $|G : G'| < k$ and the other μ_i are bounded by 2? The following theorem shows that in this situation the order of G is bounded.

Theorem 1.1. *Let G be a finite p-group such that the numbers μ_i satisfies $\sum_{\mu_i \geq 3} \mu_i \leq k$. If G is the adjoint group of some nilpotent p-algebra R, then $|G| \leq f(p,k)$ for some function f depending only on p and k.*

As an immediate consequence of this theorem we deduce the following

Corollary 1.2. *Let G be a finite p-group with two generators. If G occurs as the adjoint group of a nilpotent p-algebra, then the order of G is bounded.*

It was proved in [1], that in this case $|G| \leq p^5$.

The notation is as follows. The n-th power of an algebra R is the subalgebra R^n of R generated by the set of elements of the form $x_1 x_2 \dots x_k$ with $k \geq n$, where $x_1, x_2, \dots, x_k \in R$. The algebra R is called *nilpotent* if $R^m = 0$ for some positive integer m. The largest natural number n such that $R^n \neq 0$ is the *nilpotency class* of R. The subalgebra of an algebra R generated by the set of elements $x_1, x_2, \dots, x_s$ will be denoted by $\ll x_1, x_2, \dots, x_s \gg$ whereas the subspace of the algebra R generated by these elements is $\langle x_1, x_2, \dots, x_s \rangle$. If R is a nilpotent algebra over the field F, then $\widehat{R} = R \oplus F \cdot 1$ is its unital hull, i.e. the algebra obtained from R by the adjoining a unity. The *annihilator* of R is $\mathrm{Ann}(R) = \{x \in R \mid xy = yx = 0 \text{ for all } y \in R\}$ and the center of R is $Z(R) = \{x \in R \mid yx = xy \text{ for all } y \in R\}$. Furthermore ${}^l\mathrm{Ann}(S) = \{x \in R \mid xy = 0 \text{ for all } y \in S\}$, ${}^r\mathrm{Ann}(S) = \{x \in R \mid yx = 0 \text{ for all } y \in S\}$. The minimal number of generators of an algebra R will be denoted by $d(R)$. Similarly $d(G)$

denotes the minimal number of generators of the adjoint group $R^\circ = G$ of R. Note that $d(R) \le d(G)$. We will write $d_i = \dim R^i/R^{i+1}$.

If the dimension (the order) of some algebra R (of some group G) is bounded in terms of parameters $a, b, \dots, c$, we will say that the dimension of R (the order of G) is $(a, b, \dots c)$-*bounded*.

2. A special class of nilpotent algebras

In this section we consider a nilpotent algebra R which can be written as a sum of two subspaces in the form $R = L + \widehat{L}y$, where $L = \langle\langle x \rangle\rangle$ is a one-generator subalgebra of dimension n and $y \in R$. In this case $R = \langle\langle x, y\rangle\rangle$ with relations $yx = \phi(x) + \psi(x)y$, $y^2 = \alpha(x) + \beta(x)y$ for some polynomials $\phi, \psi, \alpha, \beta \in F[x]x$. Clearly, we may regard $\phi(x)$, $\alpha(x)$ as elements of L^2 and $\beta(x)$, $\psi(x)$ as elements in L, so that $\phi(x) \equiv \alpha(x) \equiv 0 \pmod{x^2}$. These notations will remain fixed untill the end of this section.

Lemma 2.1. *The algebra R has a basis of the form*

$$\{x, x^2, \dots, x^n, y, xy, \dots, x^m y\},$$

for some natural numbers $n \ge m$, such that $x^{n+1} = yx^{m+1} = 0$. If the minimal number of generators $d(R^\circ)$ does not exceed k, then $n - m - 1 \le kp/(p-1)$.

Proof. It follows from the above relations that $R^2 = \langle xy, x^2\rangle + R^3$. Therefore $R^i = xR^{i-1} + R^{i+1}$ and $d_i \le 2$ for each $i \ge 2$. If $d_2 = 1$, then R has a one-generator subalgebra L_1 of codimension 1 and we may replace L by L_1. It is easy to see that R has a basis of the required form. If there exists a natural number $j > 2$ such that $d_{j-1} = 2$ and $d_j = 1$, then either $x^{j-1}y = \lambda x^{j-1}$ for some $\lambda \in L$ or $x^j \in R^{j+1}$. In the first case we have $x^{j-1}(y - \lambda) = 0$ and we may consider $y - \lambda$ instead of y which gives the required assertion about the basis. If $x^j \in R^{j+1}$, then $xR^j \subseteq R^{j+2}$ and $R^{j+1} = 0$. In this case we are also done.

Now we consider a basis with the above properties and natural numbers n, m as above. Let $S = {}^r\mathrm{Ann}(x^{m+1})$. Clearly $\widehat{L}y \subseteq S$ and $R = L + S$, so that $S = S \cap L + \widehat{L}y$. Obviously $\dim(L \cap S) = m + 1$. It is clear that S is a right ideal of R. Since $R = L + S$ and for each $h \in L$

we have $x^i hS = hx^i S$, it follows that $hS \subseteq S$ for each $h \in L$ and S is also a right ideal of R. Now R/S is a one-generator algebra with $\dim R/S = n - m - 1$. It is easy to prove (see for instance [2]) that we have $d((R/S)^\circ) = r((R/S)^\circ) \geq (p-1)(n-m-1)/p$. Since $d(R^\circ) \leq k$ the lemma is proved. $\square$

Lemma 2.2. *Let R be as in the previous lemma and m, n as above. If $m = n$, then the algebra R is isomorphic with the subalgebra of the matrix algebra $M_2(\widehat{L})$, generated by the matrices*

$$u = \begin{pmatrix} x & 0 \\ \phi(x) & \psi(x) \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 \\ \alpha(x) & \beta(x) \end{pmatrix},$$

where x is a generator of L. If $G = R^\circ$ is the adjoint group of R and $|G : G'| \leq k$, then $\dim R \leq 4kp$.

Proof. To establish an isomorphism between R and the required subalgebra of the algebra $M_2(\widehat{L})$ it is enough to use the regular representation of this algebra regarded as an algebra over the ring $\widehat{L}$. Indeed, we have $\widehat{R} = \widehat{L} \cdot 1 \oplus \widehat{L} y$. Clearly, each element of R can be represented in a form $h = a_{11} + a_{12} y$ for some $a_{11} \in L, a_{12} \in \widehat{L}$. In this case we have also that $yh = a_{21} + a_{22} y$ with $a_{21}, a_{22} \in L$. So we may attach to each $h \in R$ the matrix

$$[h] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where the coefficients a_{ij}, $i, j \leq 2$ are uniquely determined by the above arguments. It is straightforward to see that for each $g, h \in R$, $\lambda \in F$ we have $[g + h] = [g] + [h]$, $[gh] = [g][h]$ and $[\lambda g] = \lambda[g]$. Thus the mapping $g \rightarrow [g]$ from R to $M_2(\widehat{L})$ is an algebra isomorphism.

Define a homomorphism $\theta : G \rightarrow L^\circ$ by the rule:

$$\theta(a) = \det([a] + I) - 1,$$

where det is the determinant and I is the unit matrix for each matrix a. Clearly this map is defined on the set of all matrices of the form

$$\begin{pmatrix} L & \widehat{L} \\ L & L \end{pmatrix}$$

It is enough to check the property $\theta(a \circ b) = \theta(a) \circ \theta(b)$ for each pair of matrices from this set. Indeed, we have

$$\begin{aligned}\theta(a \circ b) &= \det([a \circ b] + I) - 1 = \det(([a] + I)([b] + I)) - 1 \\ &= \det([a] + I)\det([b] + I) - 1 = (\theta(a) + 1)(\theta(b) + 1) - 1 \\ &= \theta(a)\theta(b) + \theta(a) + \theta(b) = \theta(a) \circ \theta(b).\end{aligned}$$

Prove now that if $|G : G'| \leq k$, then $\dim R \leq 4kp$. We will determine the image of an element $[x] = u$ under the map θ. It is obvious that $\theta(u) = x\psi(x)+x+\psi(x)$. If $x+\psi(x) \not\equiv 0 \pmod{x^2}$, then $(x\psi(x)+x+\psi(x))^{p^m} = 0$ only when $p^m > n$. Clearly, if $p^m \leq n < p^{m+1}$, then there exists an element of order p^m in G/G'. However $|G/G'| \leq k$. Hence $k \geq p^m$. This implies $n < kp$.

Suppose that $x + \psi(x) \equiv 0 \pmod{x^2}$. In this case

$$\theta(u^2) = x^2 + (\psi(x))^2 + (x\psi(x))^2 \equiv 2x^2 \pmod{x^3}.$$

Hence if $2p^m \leq n$ and $p > 2$, then $k \geq |G/G'| \geq p^m$. Choose m such that $2p^m \leq n < 2p^{m+1}$. If $p > 2$, then by the above considerations $n \leq 2kp$.

Now let $p = 2$. As before we obtain that $\psi(x) \equiv x + x^2 \pmod{x^3}$. In this case

$$\theta(u^3) = x^3(\psi(x))^3 + x^3 + (\psi(x))^3 \equiv x^3 + (\psi(x))^3 \pmod{x^5}.$$

But $(x + x^2)^3 \equiv x^3 + x^4 \pmod{x^5}$, so that $n < 4kp$. The lemma is proved. □

Lemma 2.3. *Let $R = L + \widehat{L}y$ be a finite nilpotent algebra over a field of characteristic p with adjoint group $G = R^\circ$. If $|G : G'| \leq k$, then $\dim R \leq 4kp + 2k + 1$.*

Proof. By Lemma 2.1 R has a basis $\{x, x^2, \ldots, x^n, y, xy, \ldots, x^m y\}$. Suppose that $\dim L = n$. Then $x^m y \neq 0 = x^{m+1}y$. Since $|G : G'| \leq k$ we have $d(G) \leq k$. By Lemma 2.1 this implies that $n - m - 1 \leq kp/(p-1)$. Hence $n - m \leq 2k + 1$. It is easy to see that $R^{m+1} = \langle x^{m+1}, \ldots, x^n \rangle$ is an ideal of R with dimension $n - m$. Hence the adjoint group G_1 of the algebra R/R^{m+1} is a homomorphic image of G and so $|G_1 : G_1'| \leq k$. By Lemma 2.2 we have $\dim R/R^{m+1} \leq 4kp$. Therefore $\dim R \leq 4kp + 2k + 1$, and the lemma is proved. □

3. Some general lemmas

The following lemma generalizes a result of Stack [7] for commutative nilpotent algebras.

Lemma 3.1. *Let R be a nilpotent algebra over an arbitrary field F. If $d_i \leq 2$ for some $i > 1$, then $d_j \leq d_i$ for each $j \geq i$ and there is an element $x \in R$ such that $xR^{i-1} + R^{i+1} = R^i$ or $R^{i-1}x + R^{i+1} = R^i$.*

Proof. If $d_i = 1$, then $xR^{i-1} + R^{i+1} = R^i$ for some $x \in R$. Thus the lemma is obvious in this case.

Suppose now that $d_i = 2$. It is clear that R^i is generated by monomials of the form $x_1x_2 \dots x_j$, where $x_1, x_2, \dots, x_j \in R$ and $j \geq i$. Therefore there are two monomials $e_1 = x_1x_2 \dots x_i$ and $e_2 = y_1y_2 \dots y_i$ such that $\langle e_1, e_2\rangle \oplus R^{i+1} = R^i$. Let $x = x_i$, $y = y_i$ and $u_1 = x_1x_2 \dots x_{i-1}$, $u_2 = y_1y_2 \dots y_{i-1}$. Thus $e_1 = u_1x$, $e_2 = u_2y$.

Suppose that u_1 and u_2 are linearly independent and $R^i \neq R^{i-1}x + R^{i+1}$, $R^i \neq R^{i-1}y + R^{i+1}$. Then $u_1y = \mu e_2 \bmod R^{i+1}$ and $u_2x = \lambda e_1 \bmod R^{i+1}$ with $\mu, \lambda \in F$. If the elements $e_1 + \mu e_2$ and $\lambda e_1 + e_2$ are linearly independent modulo R^{i+1}, then $u_1(x + y)$ and $u_2(x + y)$ form a basis of R^i/R^{i+1}, so that $R^i = R^{i-1}(x + y) + R^{i+1}$. Hence these elements are linearly dependent which implies $\lambda\mu = 1$. Now we may assume that $u_1x = e_1$, $u_1y = e_2$. In this case we have $u_1x = x_1(x_2 \dots x_{i-1}x)$ and $u_1y = x_1(x_2 \dots x_{i-1}y)$, which proves that $R^i = x_1R^{i-1} + R^{i+1}$, as claimed.

Show that $d_{i+1} \leq d_i$. By the above considerations we may assume that $R^i = R^{i-1}x + R^{i+1}$ for some $x \in R$. Then $R^{i+1} = RR^i = R(R^{i-1}x + R^{i+1}) = R^ix + R^{i+2}$. Therefore

$$R^{i+1} \subseteq (\langle e_1, e_2\rangle + R^{i+1})x + R^{i+2} \subseteq \langle e_1x, e_2x\rangle + R^{i+2}.$$

This shows that $d_{i+1} = \dim R^{i+1}/R^{i+2} \leq 2 = d_i$. □

Lemma 3.2. *Let R be a nilpotent algebra such that $d_i = d_{i+1} = d \leq 2$ for some $1 < i < n$ and $R^i = xR^{i-1} + R^{i+1}$ for some $x \in R$. Then $R^i = \langle x^i, x^{i+1}, \dots, x^n, x^{i-1}y, \dots, x^my\rangle$ for some $y \in R$, $m \leq n$, $x^{n+1} = x^{m+1}y = 0$. If $d_i = 1$, then $R^i = \langle x^i, x^{i+1}, \dots, x^n\rangle$. Moreover $R = D + S$ where $S = {}^r\mathrm{Ann}(x^{i-1})$ and $D = \langle\langle x, y\rangle\rangle$.*

Proof. If $d = 1$ then clearly $R^i = L^i = x^{i-1}L$ for some subalgebra $L = \langle\langle x\rangle\rangle$. The lemma is evident in this case.

Suppose that $d = 2$. We have $R^i = xR^{i-1} + R^{i+1}$ for some $x \in R$. Since $d_{i+1} = 2$ it follows that $R^i = \langle e_1, e_2\rangle + R^{i+1}$ for some $e_1, e_2 \in R^i$ and $R^{i+1} = \langle xe_1, xe_2\rangle + R^{i+2}$. On the other hand, there are elements $v_1, v_2 \in R^{i-1}$ such that $xv_1 = e_1$, $xv_2 = e_2$. Thus $w_1 = xe_1 = x^2v_1$, $w_2 = xe_2 = x^2v_2$. Choose an element e_1 such that $e_1 = x^jba$ with $x^jb \in R^{i-1}$ and j is the largest possible such exponent. Then we have $w_1 = x^{j+1}ba$ and $x^{j+1}b \in R^i$. If $x^{j+1}b \in R^{i+1}$, then $w_1 = x^{j+1}ba \in R^{i+2}$, which is not the case. Therefore we may replace e_1 or e_2 by $x^{j+1}b$, which gives a contradiction. Therefore, we may assume that $e_1 = x^i$ and $w_1 = x^{i+1}$. Using the same arguments, we can easily prove that e_2 can be chosen in the form $x^{i-1}y$ for some $y \in R$. Let $D = \langle\langle x, y\rangle\rangle$. It is clear that $R^i = x^{i-1}D + R^{i+1}$. It follows by induction that $R^j = x^{j-1}D + R^{j+1}$ for each $j \geq i$. In particular, it is easy to see that $R^i \subseteq x^{i-1}D = D^i$. In this case $R^i = \langle x^i, \dots, x^n, x^{i-1}y, \dots, x^my\rangle$ for some integers $m \leq n$ such that $x^{n+1} = 0 = x^{m+1}y$. Suppose that $h \in R$. Then we have $x^{i-1}h \in R^i$. It is obvious that $x^{i-1}h = x^{i-1}l$ for some $l \in D$, since $R^i \subseteq D^i = x^{i-1}D$. Therefore $x^{i-1}(h - l) = 0$, i.e. $h - l \in {}^r\mathrm{Ann}(x^{i-1}) = S$. In this case $R = D + S$, and the lemma is proved. □

Lemma 3.3. *Let R be a nilpotent algebra containing a subspace $M = L + \widehat{L}y$ for some $y \in R$ and $L = \langle\langle x\rangle\rangle$ such that $R^i \subseteq x^{i-1}M$ for some $1 < i$. Then $R = M + S$ where $S = {}^r\mathrm{Ann}(x^{i-1})$ and $\dim S$ is $(d(R), i)$-bounded.*

Proof. Suppose that $R^i \subseteq M$ and $h \in R$. Then $x^{i-1}h \in R^i \subseteq x^{i-1}M$ and $x^{i-1}h = x^{i-1}m$ for some $m \in M$. It follows that $x^{i-1}(h - m) = 0$, so that $h \in M + {}^r\mathrm{Ann}(x^{i-1})$. In this case $R^{i+1} \subseteq x^iM$ and R/R^{i+1} has $(d(R), i)$-bounded dimension. Hence $\dim(S + R^{i+1})/R^{i+1} = \dim S/(S \cap R^{i+1})$ is $(d(R), i)$-bounded. On the other hand, $S \cap R^{i+1} = S \cap x^iM$. By Lemma 3.2 we may assume that R^i has a basis of the form $\{x^i, x^{i+1}, \dots, x^n, x^{i-1}y, x^iy, \dots, x^my\}$ with $x^{n+1} = 0 = x^{m+1}y$. If $s \in S \cap R^{i+1}$, then $s = l_1 + l_2y$ with $l_1, l_2 \in L$. Since $x^{i-1}s = 0$ it is clear that $x^{i-1}l_1 = x^{i-1}l_2y = 0$. Hence $l_1 = x^{n-i+1}l'$, $l_2 = x^{m-i}l''$ for $m > i$, where $l', l'' \in L$. In each case $\dim(S \cap R^{i+1}) \leq 2i$. Therefore $\dim S$ is $(d(R)), i)$-bounded. The lemma is proved. □

Lemma 3.4. *Let R, M, S and i be as in the previous lemma. Then there exists an ideal T of R such that $R = M + T$ and $\dim T$ is $(d(R^\circ), i)$-bounded.*

Proof. By Lemma 3.3 we have $R = M + S$, where S is a right annihilator of x^{i-1}, whose dimension is $(d(R), i)$-bounded. Clearly, S is a right ideal of R. If $j \geq \dim S$, then $SR^j = 0$. Indeed, if this is not true, then $sz_1z_2 \dots z_j \neq 0$ for some $z_1, z_2, \dots z_j \in R$ and $s \in S$. By the well-known Frobenius lemma (see, for instance [5]) the elements

$$sz_1z_2 \dots z_j,\ sz_1z_2 \dots z_{j-1}, \dots,\ sz_1,\ s$$

are linearly independent and are contained in S. Since $\dim S \leq j$ this is a contradiction.

Now $S \subseteq {}^l\mathrm{Ann}(R^j) = T$ for some j, which is $(d(R), i)$-bounded. Since $R = M + S$ and $S \subseteq T$, we have $R = M + T$. It is obvious that T is a left ideal of R. Clearly, $ThR^j \subseteq TR^{j+1} = 0$ for every $h \in R$. Hence $Th \subseteq T$ for each $h \in R$, and so T is also a right ideal of R.

Next we show that $\dim T$ is $(d(R^\circ), i)$-bounded. Obviously $T \subseteq {}^l\mathrm{Ann}(x^j)$. Hence $\dim T \cap L \leq j$. By the isomorphisms theorem we have $R/T \simeq M/M \cap T$ and $M/M \cap T = L_1 + \widehat{L}_1 z$ for some $z \in M/M \cap T$, $L_1 \simeq L/(L \cap T)$. It is easy to see that $\dim L_1 = n - j$, where $n = \dim L$. By Lemma 2.1 we have $\dim R/T = n - j + m_1 + 1$ with $n - j - m_1 \leq 2d(R^\circ)$. On the other hand, $m_1 + n - j + 1 = \dim M/(M \cap T) = \dim M - \dim(M \cap T)$. Hence $\dim(M \cap T) = \dim M - (m_1 + n - j + 1) \leq n - m_1 + j$. Since $n - m_1 \leq j + 2d(R^\circ)$ it follows that $\dim(M \cap T)$ is $(d(R^\circ), i)$-bounded. Since $S \subseteq T$ and $R = M + S$, we have that $T = (T \cap M) + S$. Therefore $\dim T$ is $(d(R^\circ), i)$-bounded. The lemma is proved. □

4. Proof of the theorem

Let R be a finite nilpotent p-algebra and $G = R^\circ$ its adjoint group. It follows from the hypothesis of the theorem, that $|G : G'| \leq p^k$. Hence $d(R) \leq d(G) \leq k$. We show that there exists an integer i depending only on k such that $d_i = \dim R^i/R^{i+1} \leq 2$.

Let n_1, n_2, n_3 denote the number of $d_i = 1, d_i = 2$ or $d_i \geq 3$ respectively. It follows from Lemma 2.1 that if $d_i \leq 2$, then $d_{i+1} \leq d_i$.

Prove that $n_1 \leq k-1$. Denote by $n(R) = n$ the nilpotency class of R. Suppose that $d_{i-1} > 1$ and $d_i = 1$. It is obvious that $n_1 = n - i$. By Lemma 3.1 we have $R = L + S$, where $S = {}^l\mathrm{Ann}(x^{i-1})$ is a left ideal of R and $L = \langle\langle x\rangle\rangle$ for some $x \in R$. However this is also a right ideal since $Slx^{i-1} = Sx^{i-1}l = 0$ for each $l \in L$. Hence there exists a natural homomorphism $R \to L/(L\cap S)$ with kernel S. Note that $\dim(L\cap S) = i-1$ and $\dim L/(L\cap S) = n-i+1$. Since L is commutative, then $|G : G'| \geq p^{n-i+1}$. It follows from $|G : G'| \leq p^k$ that $n-i+1 \leq k$ and $n_1 = n-i \leq k-1$ as claimed.

Since $n = n(R) = n_1 + n_2 + n_3$ and $\dim R \geq n_1 + 2n_2 + 3n_3$, then we have

$$n_1 + 2n_2 + 3n_3 = 3n - n_2 - 2n_1 \leq \dim R \leq k + 2(m-t),$$

where t is the number of μ_i such that $\mu_i \geq 3$. Recall that $0 \subset R^n \subset \cdots \subset R^2 \subset R$ is a lower central series of R and $1 = \gamma_m(G) \subset \cdots \subset \gamma_2(G) \subset \gamma_1(G) = G = R^\circ$ is the lower central series of G. Hence $n \geq m$. Now we have

$$n + 2m - n_2 - 2n_1 \leq 3n - n_2 - 2n_1 \leq k + 2(m-1) \leq k + 2m.$$

Therefore $n_3 = n - n_2 - n_1 \leq k + n_1 \leq 2k-1$. It follows that for some $i \leq n_3 + 1 \leq 2k$ we have $d_i \leq 2$. By Lemma 3.4 there exists an ideal T of R with (k,p)-bounded dimension such that $R/T = L + \widehat{L}y$ for some one-generator subalgebra L of R/T and $y \in R/T$. By Lemma 2.3 the dimension of R/T is also (k,p)-bounded. The theorem is proved.

5. Proof of the corollary

Let the group G with $d(G) \leq 2$ be the adjoint group of a nilpotent p-algebra R. Then $\dim R/R^2 \leq 2$. Clearly the class of the adjoint group of R/R^3 is at most 2 and its commutator subgroup has order at most 2. Thus $\dim R^2/R^3 \leq 2$ and by Lemma 3.1 we have $d_i \leq 2$ for each $i \geq 2$. Hence the order of G is bounded by Theorem 1.1. The corollary is proved.

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