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Distance functions based on neighbourhood sequences

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Abstract. P. P. Das *et al.* [2] characterized the periodic neighbourhood sequences which provide a metric and they gave an algorithm which provides a shortest path between two arbitrary points with a given neighbourhood sequence in the n-dimensional digital plane. In [4] Fazekas and his co-authors introduced the concept of generalized neighbourhood sequences, and investigated their structure in the finite and also in the ∞-dimensional plane. The purpose of this paper is to extend the algorithm of [2] to generalized neighbourhood sequences, both in case of finite and infinite dimension and prove its correctness. Moreover we prove necessary and sufficient conditions, when distance functions based on generalized neighbourhood sequences define a metric in arbitrary dimensional digital space.

1. Introduction

In digital geometry we use a discrete space, i.e., points can have only integer co-ordinates. We say that two different points in $\mathbb{Z}^m$ are k-neighbours ($k, m \in \mathbb{N},\ k \leq m$), if their corresponding coordinate values are equal up to at most k exceptions, and the difference of the exceptional values are at most 1. After fixing k, we may define the distance of two points as the number of steps of the shortest path between these points, where a step means moving from a point to one of its k-neighbours. It

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is easy to check that by this definition we get a metric on $\mathbb{Z}^m$, for each $k \in \{1, 2, \ldots, m\}$, and that these metrics are different.

To obtain these metrics we fixed k in the beginning, in other words, we used the same k in each step for walking from a point p to a point q in $\mathbb{Z}^m$. The situation is more complicated if we may change the value of k after every step. A sequence $(b_i)_{i=1}^{\infty}$ is called a neighbourhood sequence in $\mathbb{Z}^m$, if $b_i \in \{1, \ldots, m\}$ $(i \in \mathbb{N})$. The sequence is periodic, if there is an $l \in \mathbb{N}$, such that $b_{i+l} = b_i$ for every $i \in \mathbb{N}$. The concept of periodic neighbourhood sequences was introduced in [2], while the general notion in [4]. (We mention that the sequences in [2] were called "neighbourhood sequences" while in [4] "generalized neighbourhood sequences", but for simplicity we use the above definition.) By the help of a neighbourhood sequence $(b_i)_{i=1}^{\infty}$ we may define the distance of $p, q \in \mathbb{Z}^m$ in the following way. We take the length of a shortest path from p to q, but at the i-th step now we may move from a point to another if and only if they are b_i-neighbours. Certainly this notion is a generalization of the original one, as we may choose $b_i = k$ for each $i \in \mathbb{N}$, with any $k \in \{1, \ldots, m\}$.

In [2] the authors gave an algorithm which generates a shortest path between any $p, q \in \mathbb{Z}^m$, in case of periodic neighborhood sequences, but they did not prove that their algorithm works properly. In this paper we provide an extension of this algorithm, which determines a shortest path between any $p, q \in \mathbb{Z}^m$, and if such a path exists, then it works also in $\mathbb{Z}^{\infty}$, with respect to arbitrary neighbourhood sequences. We prove that our algorithm is correct.

As we mentioned, the neighbourhood sequence $(b_i)_{i=1}^{\infty}$ with $b_i = k$ $(i \in \mathbb{N})$ generates a metric on $\mathbb{Z}^m$ $(m \in \mathbb{N})$ for any $1 \leq k \leq m$. However, it is easy to find neighbourhood sequences, even periodic ones, such that the distances with respect to these sequences do not provide metrics on $\mathbb{Z}^m$. In [2] the authors gave a nice characterization of the periodic neighbourhood sequences, for which the above defined distance functions provide a metric on $\mathbb{Z}^m$. In this paper we extend this result to arbitrary neighbourhood sequences in $\mathbb{Z}^m$. It turns out that in fact the same criterion can be used for the characterization, as in [2]. However, to prove this we use a different method, which yields a considerable simplification in the formulation and proof of the assertion. We generalize our result also to the infinite dimensional digital plane $\mathbb{Z}^{\infty}$.

2. Notation and definitions

Throughout the paper N will denote an arbitrary element of the set $\mathbb{N}\cup\{\infty\}$. Let $\mathbb{Z}^N$ be the N-dimensional digital plane, i.e., $\mathbb{Z}^N=\{(z(i))_{i=1}^N : z(i)\in\mathbb{Z}\}$. We shall refer to the elements of $\mathbb{Z}^N$ as points.

Definition 2.1. A function $d:\mathbb{Z}^N\times\mathbb{Z}^N\to\mathbb{R}\cup\{\infty\}$ is called a metric on $\mathbb{Z}^N$, if it satisfies the following conditions:

a) $\forall p,q\in\mathbb{Z}^N : d(p,q)\geq 0$, and $d(p,q)=0$ if and only if $p=q$,

b) $\forall p,q\in\mathbb{Z}^N : d(p,q)=d(q,p)$,

c) $\forall p,q,r\in\mathbb{Z}^N : d(p,q)+d(q,r)\geq d(p,r)$ (triangle inequality).

We adopt some definitions from [2] and [4].

Definition 2.2. Let p and q be two points in $\mathbb{Z}^N$. The i-th coordinate of the point p is indicated by $p(i)$. Let k be an integer with $1\leq k\leq N$. The points p and q are k-neighbours, if the following two conditions hold:

- $|p(i)-q(i)|\leq 1$ for all i,
- $\sum_{i=1}^N |p(i)-q(i)|\leq k$.

And the points $p,q\in\mathbb{Z}^\infty$ are ∞-neighbours, if the first condition holds.

Definition 2.3. In the N-dimensional space the infinite sequence $B=(b_i)_{i=1}^\infty$ $(1\leq b_i\leq N)$ is called an ND-neighbourhood sequence (or shortly ND-n.s.). If for some $l\in\mathbb{N}$, $b_i=b_{i+l}$ holds for every $i\in\mathbb{N}$, then B is called periodic with period l, or simply l-periodic. An ND-n.s. is periodic, if it is l-periodic with some $l\in\mathbb{N}$.

Definition 2.4. Let p and q be two points in $\mathbb{Z}^N$ and $B=(b_i)_{i=1}^\infty$ an ND-n.s. A finite point sequence $\Pi(p,q;B)$ of the form $p=p_0,p_1,\ldots,p_m=q$, where $p_{i-1},p_i\in\mathbb{Z}^N$ are b_i-neighbours for $1\leq i\leq m$, is called a B-path from p to q. We write $m=|\Pi(p,q;B)|$ for the length of the path.

Note that the length of the path $p=p_0=q$ is zero independently of the n.s. B.

Remark 2.5. In case of finite dimension, there is a B-path between any two points, with any neighbourhood sequence B. However, in $\mathbb{Z}^\infty$ it is possible that there is no B-path between two points. For example if the set $\{|p(i) - q(i)| : i \in \mathbb{N}\}$ is unbounded, then there is no neighbourhood sequence B, for which a B-path would exist between the points $p = (p(i))_{i=1}^{\infty}$ and $q = (q(i))_{i=1}^{\infty}$.

Definition 2.6. Let p and q be two points in $\mathbb{Z}^N$ and B an ND-n.s. If there is no B-path between these points, then we put $d(p, q; B) = \infty$. Otherwise denote by $\Pi^*(p, q; B)$a shortest path from p to q, and set $d(p, q; B) = |\Pi^*(p, q; B)|$. We call $d(p, q; B)$ the B-distance of p and q.

Definition 2.7. Let B_1 and B_2 be two neighbourhood sequences in $\mathbb{Z}^N$. We say that B_1 is faster than B_2, if

$$d(p, q; B_1) \leq d(p, q; B_2) \quad \text{for all } p, q \in \mathbb{Z}^N.$$

We denote this relation by $B_1 \sqsupseteq^* B_2$.

The relation $\sqsupseteq^*$ was introduced by Das [3] in the two dimensional space, and by Fazekas *et al.* in [4] for higher dimensions.

For later use we need to introduce some further notations.

Notation 2.8. Let p and q be two points in $\mathbb{Z}^N$. Put $w(i) = |p(i) - q(i)|$ for all i, and $w = (w(i))_{i=1}^{N}$. The point w is called the absolute difference of p and q.

Definition 2.9. Let $n \in \mathbb{N}$ and $B = (b_i)_{i=1}^{\infty}$ an ND-n.s. Put $b_i^{(n)} = \min(b_i, n)$ and $B^{(n)} = \left(b_i^{(n)}\right)_{i=1}^{\infty}$. The sequence $B^{(n)}$ is called the n-dimensional limited sequence of B. Denote by $f_k(i)$ the i-th subsums of the k-dimensional limited sequence of B, i.e., put

$$f_k(i) = \begin{cases} \sum_{j=1}^{i} b_j^{(k)}, & \text{if } 1 \leq i, \\ 0, & \text{if } i = 0. \end{cases}$$

Definition 2.10. Let $B = (b_i)_{i=1}^{\infty}$ an ND-n.s. The sequence $B(j) = (b_i)_{i=j}^{\infty}$ is called the j-shifted sequence of B.

The following lemma is very useful if we would like to decide numerically whether an ND-n.s. is faster or not than another one.

Lemma 2.11. *Let B_1 and B_2 be two ND-n.s.-es. Then,*

$$d(p, q; B_1) \leq d(p, q; B_2), \quad \textit{for all } p, q \in \mathbb{Z}^N,$$

if and only if

$$f_k^{(1)}(i) \geq f_k^{(2)}(i), \quad \textit{for all } i,k \in \mathbb{N},\ 1 \leq k \leq N,$$

where $f_k^{(1)}(i)$ *and* $f_k^{(2)}(i)$ *correspond to* B_1 *and* B_2*, respectively.*

Proof. The proof of this result is in [4]. See Theorem 3.2 for finite N and Theorem 4.10 for the ∞-dimensional digital space. □

Remark 2.12. As a simple consequence of the previous lemma we obtain that if B_1 and B_2 are ∞D-n.s.-s with $B_1 \sqsupseteq^* B_2$, then for every $i \in \mathbb{N}$ among the first i elements of B_1 there are at least as many ∞ symbols as among the first i elements of B_2.

3. Minimal path and path length

In this section we give an algorithm which provides a shortest path between arbitrary two points in $\mathbb{Z}^N$, if such a path exists. As we mentioned in Remark 2.5 it is possible that there is not path between two given points with a given n.s. in infinite dimension. The following lemma provides a criterion for the existence of a path between two points.

Lemma 3.1. *Let* p *and* q *be given points in* $\mathbb{Z}^\infty$*, and* B *a* ∞*D-n.s. There is a* B*-path between* p *and* q *if and only if the following two conditions hold:*

- *the set* $\{w(i)\}$ *is finite, where* w *is the absolute difference of* p *and* q*,*
- *the sequence* B *contains the* ∞ *symbol at least* k *times, where* k *is the maximal value in* w*, which occurs infinitely many times.*

Proof. The statement is a simple consequence of Theorem 4.8. in [4]. □

Remark 3.2. If the sequence B is periodic, and it contains the element ∞, then B contains ∞ at infinitely many positions.

The following algorithm provides one of the shortest B-paths between two arbitrary points in $\mathbb{Z}^N$, if such a path exists. This algorithm is based on the algorithm in [2], which works in finite dimension with periodic sequences only. The main difference between the original algorithm and the following one is in step 4, where we may use non periodic and/or ∞D-n.s.

Algorithm 3.3. *Input: An ND-n.s.* $B = (b_i)_{i=1}^{\infty}$ *and* $p, q \in \mathbb{Z}^N$*, such that* $d(p, q; B) < \infty$*.*

- *step 1. Let* $w^{(0)}$ *be the absolute difference of* p *and* q*,* $t(i) = \operatorname{sgn}(p(i) - q(i))$*, for all* i*, and put* $j = 0$ *and* $\Pi = (p)$*.*
- *step 2. If* $w^{(j)}(i) = 0$ *for every* i *then goto step 8, else set* $j = j + 1$*.*
- *step 3. Put* $w^{(j)} = w^{(j-1)}$*.*
- *step 4. If* b_j *is finite, then select the largest* b_j *entries of* $w^{(j)}$*. If* b_j *is infinite, then select all the entries of* $w^{(j)}$*.*
- *step 5. For each selected* $w^{(j)}(i)$ *with* $w^{(j)}(i) \neq 0$*, let* $w^{(j)}(i) = w^{(j-1)}(i) - 1$*.*
- *step 6. Append to the path* Π *the point* x_j *defined by* $x_j(i) = q(i) + w^{(j)}(i)t(i)$ *for all* i*.*
- *step 7. Goto 2.*
- *step 8. Output* Π *as a minimal* B*-path between* p *and* q*, and* j *as the length of this path.*

We illustrate our algorithm on the following simple example.

Example 3.4. Let $B = (3, \infty, 2, 2, 2, \dots)$ ($b_i = 2$ for $i \geq 3$) an ∞D-n.s., and $p = (3, 2, 2, 2, 1, 1, 1, \dots)$ ($p(i) = 1$ for $i \geq 5$), and $q = (0)_{i=1}^{\infty}$.

The algorithm provides the following path:

$$\begin{aligned}\Pi = (&(3,2,2,2,1,1,1,1,1,1,1,\dots) = p,\\ &(2,1,1,2,1,1,1,1,1,1,1,\dots),\\ &(1,0,0,1,0,0,0,0,0,0,0,\dots),\\ &(0,0,0,0,0,0,0,0,0,0,0,\dots) = q).\end{aligned}$$

Thus, the B-distance of p and q is $d(p, q; B) = 3$.

The following theorem is about the correctness of our algorithm. Since it is an extension of the algorithm in [2] this theorem proves that the original algorithm correct also. (In [2] there is not a proof for that.) We use the term step as step of the algorithm, however some steps are complex.

Theorem 3.5. *Algorithm 3.3 terminates after finitely many steps and provides a B-path with minimal length between the points p and q.*

Proof. Observe that for every $j \geq 0$, x_{j-1} and x_j are b_j-neighbours. Moreover, by its definition, $w^{(j)}$ is the absolute difference of x_j and q. Let $p = y_0, y_1, y_2, \ldots, y_j, \ldots$ be any point sequence, where y_{j-1} and y_j are b_j-neighbours for $j \geq 0$, and let $v^{(j)}$ be the absolute difference of y_j and q. We show that if $v^{(j)}$ is identically zero for some $j \geq 0$, then so is $w^{(j)}$. For this purpose, put for $t \geq 0$

$$\|v^{(j)}\|_t = \sum_{i=1}^{N} \max(v^{(j)}(i) - t, 0), \quad (j \geq 0),$$

and in particular

$$\|w^{(j)}\|_t = \sum_{i=1}^{N} \max(w^{(j)}(i) - t, 0), \quad (j \geq 0).$$

We even claim that for every $t \geq 0$ and $j \geq 0$ we have $\|w^{(j)}\|_t \leq \|v^{(j)}\|_t$. We proceed by induction. For $j = 0$, as $x_0 = y_0 = p$, we certainly have $w^{(0)} = v^{(0)}$, whence $\|w^{(0)}\|_t = \|v^{(0)}\|_t$ $(t \geq 0)$. Suppose, that for some $j \geq 0$ we have

$$\|w^{(j)}\|_t \leq \|v^{(j)}\|_t \quad (t \geq 0).$$

We should prove that the same is valid with $j+1$ in place of j. Suppose the contrary and choose an l such that

$$\|w^{(j+1)}\|_l > \|v^{(j+1)}\|_l.$$

Now we distinguish three cases.

First, suppose that $||w^{(j+1)}||_l = \infty$. Then we also have $\|w^{(j)}\|_l = \infty$. Hence, by $\|w^{(j)}\|_l \leq \|v^{(j)}\|_l$ we get $\|v^{(j)}\|_l = \infty$. Therefore by $\|v^{(j+1)}\|_l < \infty$, $b_j = \infty$ and $\|v^{(j)}\|_{l+1} < \infty$. Combining this with the induction hypothesis for $t = l+1$, we get $\|w^{(j)}\|_{l+1} < \infty$. However, as $\|w^{(j)}\|_{l+1} < \infty$ and $b_j = \infty$, in view of Step 5 of the Algorithm $\|w^{(j+1)}\|_l < \infty$. This is a contradiction, so this case cannot hold.

Now, we assume that $\|w^{(j+1)}\|_l < \infty$, and also $\|w^{(j)}\|_l < \infty$. Then by $\|v^{(j+1)}\|_l < \infty$ we have $\|v^{(j)}\|_{l+1} < \infty$. If $\|v^{(j)}\|_l < \infty$ is also true, then combining this with the induction hypothesis for $t = l + 1$, we obtain

$$\|v^{(j)}\|_l - \|w^{(j)}\|_l \geq (\|v^{(j)}\|_l - \|v^{(j)}\|_{l+1}) - (\|w^{(j)}\|_l - \|w^{(j)}\|_{l+1}).$$

Here, the right hand side equals the difference of the numbers of entries in the sequences v and w, respectively, which are larger than l. In the $(j+1)$th step we can modify maximum these amounts of coordinate values in v and w, respectively. So, this inequality immediately implies $\|w^{(j+1)}\|_l \leq \|v^{(j+1)}\|_l$, which is a contradiction. On the other hand, if $\|v^{(j)}\|_l = \infty$, then by $\|v^{(j+1)}\|_l < \infty$ we get $b_j = \infty$. In this case, in view of Step 5 of the Algorithm, we have $\|w^{(j+1)}\|_l = \|w^{(j)}\|_{l+1}$. Certainly, $\|v^{(j+1)}\|_l \geq \|v^{(j)}\|_{l+1}$ is also valid. Hence, by the induction hypothesis we get a contradiction, and our statement is proved in this case.

Finally, assume that $\|w^{(j+1)}\|_l < \infty$, but $\|w^{(j)}\|_l = \infty$. Then again $b_j = \infty$, and by the previous argument we get $\|w^{(j+1)}\|_l = \|w^{(j)}\|_{l+1}$ and $\|v^{(j+1)}\|_l \geq \|v^{(j)}\|_{l+1}$. By using the induction hypothesis with $t = l+1$ we get $\|w^{(j+1)}\|_l \leq \|v^{(j+1)}\|_l$, which is a contradiction.

Thus, we proved that

$$\|w^{(j)}\|_t \leq \|v^{(j)}\|_t, \quad \text{for every } j \geq 0,\ t \geq 0,$$

which implies that if $v = (0)_{i=1}^N$ then so is w. By our assumption $d(p, q; B) < \infty$, there is a minimal B-path $p = y_0, y_1, \ldots, y_m = q$ between p and q. Hence, $p = x_0, x_1, \ldots, x_j = q$ is also a minimal path, with $j = m$. Thus, the algorithm terminates after finitely many steps, and outputs a minimal B-path between p and q. □

4. Condition for a distance to be a metric

The distance based on an arbitrary neighbourhood sequence in general does not satisfy the conditions of a metric. However, in geometry those distances are useful, which have this property. In this section we give a necessary and sufficient condition for a distance based on a neighbourhood sequence to be a metric.

Lemma 4.1. *Let p and q be arbitrary points in $\mathbb{Z}^\infty$, and let the ∞D-n.s. B_1 be faster than the ∞D-n.s. B_2 (i.e., $B_1 \sqsupseteq^* B_2$). If there is no B_1-path between p and q, then does not exist B_2-path between them, too.*

Proof. From Definition 2.7, if $d(p,q;B_1)=\infty$ and B_1 is faster than B_2, then $d(p,q;B_2)=\infty$. □

The next theorem is the extension of the result of [2], concerning periodic neighbourhood sequences in finite dimension, to the general case. To prove their result, the authors in [2] introduced relatively complicated geometric notions, such as wave-front of a neighbourhood sequence, etc. To formulate and prove our result we need only the simple concepts of the faster relation and the shifted sequence.

Theorem 4.2. *The distance function based on an ND-n.s. B is a metric on $\mathbb{Z}^N$, if and only if $B(i)$ is faster than B for all $i\in\mathbb{N}$.*

Proof. First, we prove sufficiency. The validity of properties a) and b) of Definition 2.1 is trivial; it can be seen, e.g., following Algorithm 3.3. Indeed, the distance $d(p,q;B)$ depends only on the absolute difference w of p and q, and on B. As the definition of w is symmetric in p and q, thus $d(p,q;B)=d(q,p;B)$. It is clear, that the distance is zero if and only if the absolute difference of the points has only zero elements, i.e., if the points are the same. Otherwise, the distance is a positive integer or infinite. Therefore all distances generated by a n.s. satisfy these two properties. Hence enough to deal with the triangle inequality.

Now we prove that property c) is true if and only if $B(i)$ is faster than B for all $i\in\mathbb{N}$. Let $p,q,r\in\mathbb{Z}^N$, such that their distances are finite. Then, we can find a B-path Π between p and r which is a concatenation of a minimal B-path between p and q, and a minimal $B(i)$-path between q and r, where $i=d(p,q;B)+1$, and $B(i)$ is the i-shifted sequence of B. Hence,

$$|\Pi|=d(p,q;B)+d(q,r;B(i)).$$

The assumption that $B(i)$ is faster than B means that

$$d(q,r;B(i))\le d(q,r;B).$$

Thus,

$$|\Pi|\le d(p,q;B)+d(q,r;B).$$

By the definition of the B-distance we have

$$d(p,r;B)\le|\Pi|,$$

hence

$$d(p, r; B) \leq d(p, q; B) + d(q, r; B).$$

Now, suppose that not all the distances are finite between p, q and r. If $d(p, q; B) = \infty$ or $d(q, r; B) = \infty$ then c) is trivially valid. Assume, that $d(p, r; B) = \infty$, but $d(p, q; B) = s < \infty$. If there would be a $B(s)$-path between q and r, then there would also be a B-path between p and r. (We could concatenate a shortest B-path between p and q, with length s, and a $B(s)$-path between q and r.) As the shifted sequence $B(s)$ is faster than B, by Lemma 4.1 there is no B-path between q and r. So, $d(q, r; B) = \infty$, and c) is valid in this case, too.

Now, we prove necessity. Assume that for some $j \in \mathbb{N}$, $B(j)$ is not faster than B, but $d(B)$ has property c). In this case, by Definition 2.7 there exist $p, q \in \mathbb{Z}^N$ such that $d(p, q; B(j)) = k$, and $d(p, q; B) < k$. The B-distance of two points depends only on their absolute difference, so we may assume that the coordinate values of p are non-negative, and q is the origin: $p(i) \geq 0$ and $q(i) = 0$, for all i. Define the point $r \in \mathbb{Z}^N$ in the following way:

$$r(i) = -|\{b_l : l \leq j \text{ and } b_l \geq i\}| \ (i \geq 1).$$

By our algorithm, it is easy to see that $d(q, r; B) = j$ and q is an element of one of the shortest paths between p and r. Then,

$$d(p, r; B) = d(q, r; B) + d(p, q; B(j)) = j + k,$$

as a shortest B-path between p and r can be obtained as a concatenation of a shortest $B(j)$-path from p to q and a shortest B-path from q to r. Thus,

$$d(p, q; B) + d(q, r; B) < k + j = d(p, r; B).$$

But we assumed that $d(p, q; B)$ has property c). This is a contradiction, and the proof is complete. □

Remark 4.3. By Lemma 2.11 one can decide, whether an ND-n.s. defines a metric or not.

5. Conclusions

In this paper we presented an extended algorithm of finding a shortest path which works with generalized neighbourhood sequences, both in case of finite and infinite dimension. Moreover a necessary and sufficient condition, when distance functions based on generalized neighbourhood sequences define a metric in arbitrary dimensional digital space was derived. Using the results of this paper one can measure distances in arbitrary dimensional digital space with arbitrary neighbourhood sequences. In the case when one is interested only on metrics, s/he can decide which neighbourhood sequences generate metrics by the help of our last theorem.

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