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On the geometry of generalized metric spaces III. Spaces with special forms of curvature tensors

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Dedicated to Professor Lajos Tamássy on his 70th birthday

§0. Introduction

Let M be an n-dimensional differentiable manifold and $T(M)$ its tangent bundle. We consider the bundle M_T which does not contain zero vectors of $T(M)$, that is, $M_T := T(M) - \{0\}$. A generalized metric space M_n is a pair $(M_T, g_{ij}(x,y))$, where $g_{ij}(x,y)$ is a metric tensor satisfying the following assumptions:

(a) $g_{ij}(x,y)$ is positively homogeneous of degree 0 in y^i,

(b) g_{ij} is positive definite,

(c) ${g^*}_{ij} := \dot{\partial}_i \dot{\partial}_j F^2/2$ is regular, where $\dot{\partial}_i := \partial/\partial y^i$ and $F^2 := g_{ij} y^i y^j$.

In the previous papers ([1],[2]), we introduced three types of connections in M_n: (a) metrical connection $C\Gamma(N) = ({F_j}^i{}_k, {C_j}^i{}_k)$, (b) h-metrical connection $R\Gamma(N) = ({F_j}^i{}_k, 0)$ and (c) non-metrical connection $B\Gamma(G) = ({G_j}^i{}_k, 0)$ (see [1]).

Two metric tensors g_{ij}, ${g^*}_{ij}$ are related as

$$ {g^*}_{ij} = g_{ij} + C_{ij}, \quad C_{ij} := y^h \dot{\partial}_j g_{ih} \quad ([1], (2.8)), $$

which satisfies $C_{ij} = {C_i}^0{}_j = C_{ji}$ and $C_{i0} = 0$, where the index 0 denotes transvection by y.

From the assumption that geodesics introduced from the Finsler metric ${g^*}_{ij}$ are coincident with those from the generalized metric g_{ij}, that is,

$2G^i = N^i_0$, we have

$$N^i_k = G^i_k - P^i{}_k, \quad G^i_k := \dot{\partial}_k G^i,$$
$$2G^i := g^{*ih}(y^j \dot{\partial}_h \partial_j F^2 - \partial_h F^2)/2,$$

where $\partial_j := \partial/\partial x^j$ and the tensor $P^i{}_j$ is arbitrary but $P^i{}_0 = 0$.

In [2], we defined the curvature tensors R, K, H and the generalized metric spaces of R–, K– and H–isotropic (sectional) curvature and obtained the following results.

Theorem A. ([2]) *A generalized metric space of R–isotropic curvature is characterized by*

$$\begin{aligned} 6\{R_{hijk} - &R(g_{hj}g_{ik} - g_{hk}g_{ij})\} \\ &= \{(C_{ikr} + 2C_{kir})R^r{}_{hj} + (C_{hjr} + 2C_{jhr})R^r{}_{ik} - j|k\} \\ &\quad + 2(C_{hir} - C_{ihr})R^r{}_{jk} - (C_{jkr} - C_{kjr})R^r{}_{hi}, \end{aligned} \tag{0.1}$$

where $j|k$ means the interchange of indices j, k in the foregoing terms.

Theorem B. ([2]) *In a generalized metric space of R–isotropic curvature, if the relation*

$$R^i{}_{jk} = R(y_j \delta^i_k - y_k \delta^i_j) \tag{0.2}$$

is satisfied, then the following equation holds:

$$R_h{}^i{}_{jk} = R(g_{hj}\delta^i_k - g_{hk}\delta^i_j). \tag{0.3}$$

Theorem C. ([2]) *A generalized metric space of K–isotropic curvature is a Riemannian space of constant curvature, that is, $C_{ijk} = 0$ and the following equation holds*:

$$K_h{}^i{}_{jk} = K(g_{hj}\delta^i_k - g_{hk}\delta^i_j). \tag{0.4}$$

Theorem D. ([2]) *A generalized metric space of H–isotropic curvature is a Finsler space of constant curvature, that is, $g^*{}_{ij} = g_{ij}$ ($C_{ij} = 0$) and the following equation holds*:

$$H_h{}^i{}_{jk} = H(g_{hj}\delta^i_k - g_{hk}\delta^i_j). \tag{0.5}$$

The purpose of the present paper is to consider the inverse problems of the above results. That is, when a generalized metric space has the special forms of curvature tensors: (0.3), (0.4) and (0.5), respectively, we investigate the corresponding properties of the space. These are expressed in Theorems 1.2, 2.1 and 2.5.

We raise or lower the indices by means of g_{ij} only.

Notations and terminologies are those of [1] and [2].

§1. Curvature tensor $H_h{}^i{}_{jk}$

First we shall show

Theorem 1.1. *If a generalized metric space M_n $(n > 2)$ satisfies the relation*

$$H^i{}_{jk} = Hy_j\delta^i_k - j|k, \tag{1.1}$$

then the scalar H is a constant, and (1.1) is equivalent to

$$H_h{}^i{}_{jk} = H(g_{hj} + C_{hj})\delta^i_k - j|k. \tag{1.2}$$

Proof. From (1.1), we have

$$(a) \quad H^i{}_{jk} = Hy_j h^i_k - j|k, \qquad (b) \quad H^i{}_k = F^2 H h^i_k, \tag{1.3}$$

where $h^i_k := \delta^i_k - l^i l_k$, $l^i := y^i/F$ and $H^i{}_k := H^i{}_{0k}$.

Substitution of (1.3)(b) into the identity

$$3H^i{}_{jk} = H^i{}_{k(j)} - H^i{}_{j(k)} \qquad ([1], (3.6)(b)) \tag{1.4}$$

gives

$$H^i{}_{jk} = (Hy_j + \frac{1}{3}F^2 H_{(j)})h^i_k - j|k. \tag{1.5}$$

Comparing (1.5) with (1.3)(a), we get $H_{(j)}h^i_k - j|k = 0$, from which we have $(n-2)H_{(j)} = 0$. Hence, H is independent of y^i.

Next, if we apply the Bianchi identity

$$H^i{}_{jk/\!/l} + j|k|l = 0 \qquad ([1], (3.10)(b)),$$

we have

$$H_{/\!/l}(y_j h^i_k - j|k) + H_{/\!/j}(y_k h^i_l - k|l) + H_{/\!/k}(y_l h^i_j - j|l) = 0.$$

Contracting i with l, we get

$$y_j H_{/\!/i} h^i_k - y_k H_{/\!/i} h^i_j = 0.$$

Transvection of this equation by y^j yields $H_{/\!/i} h^i_k = 0$. Since H is independent of y^i, the last equation means

$$H_{,k} - (H_{,i} l^i) l_k = 0, \qquad H_{,k} := \partial_k H. \tag{1.6}$$

Differentiating (1.6) by y^j and using $Fl^i{}_{(j)} = h^i_j$, $Fl_{i(j)} = h_{ij} + C_{ij}$, we have

$$(H_{,i} l^i)(g^*{}_{jk} - l_j l_k) = 0. \tag{1.7}$$

Transvecting (1.7) by g^{*jk} and noting $g^{*jk}l_k = l^j$, we obtain $(n-1)(H_{,i}l^i) = 0$. Making use of this result, we see $H_{,k} = 0$ from (1.6). Hence H is a constant. The last assertion of the theorem is easily derived from $y_{j(h)} = g_{hj} + C_{hj}$. Q.E.D.

Theorem 1.2. *If a generalized metric space M_n $(n > 2)$ satisfies the relation*

$$H_h{}^i{}_{jk} = H(g_{hj}\delta^i_k - g_{hk}\delta^i_j), \tag{1.8}$$

then the space is a Finsler space of constant curvature H.

Proof. It is sufficient to prove $C_{jk} = 0$, which means that the space is a Finsler space. Transvecting (1.8) by y^h and using $H_0{}^i{}_{jk} = H^i{}_{jk}$, we get (1.1). Hence, from Theorem 1.1, we have (1.2). Comparing (1.2) with (1.8), we have $C_{hj}\delta^i_k - j|k = 0$, from which $(n-1)C_{jk} = 0$ is derived. Q.E.D.

§2. Curvature tensors $R_h{}^i{}_{jk}$ and $K_h{}^i{}_{jk}$

Theorem 2.1. *In a generalized metric space M_n $(n > 2)$, if the relation*

$$R_h{}^i{}_{jk} = R(g_{hj}\delta^i_k - g_{hk}\delta^i_j) \tag{2.1}$$

is satisfied, then the space is one of R–isotropic curvature.

Proof. (2.1) gives

$$R^i{}_{jk} = R(y_j\delta^i_k - y_k\delta^i_j). \tag{2.2}$$

It is proved that (2.1) means vanishing of the left-hand side of (0.1) in Theorem A and using (2.2), the right-hand side of (0.1) vanishes. Q.E.D.

It is not yet proved that the scalar R of (2.1) is a constant. Now we shall prepare the following

Proposition 2.2. *In a generalized metric space, the following relations are valid*:

$$E^i{}_{k(j)} - E^i{}_{j(k)} = 3E^i{}_{jk} - J_j{}^i{}_k, \quad E^i{}_j := E^i{}_{0j}; \tag{2.3}$$

$$R^i{}_{k(j)} - R^i{}_{j(k)} = 3R^i{}_{jk} + J_j{}^i{}_k, \quad R^i{}_j := R^i{}_{0j}, \tag{2.4}$$

where

$$(a) \quad J_j{}^i{}_k := P^i{}_{jk/0} + 2(P^i{}_{j/k} + P^i{}_{kr}P^r{}_j + P^i{}_rP^r{}_{jk}) - j|k. \tag{2.5}$$

PROOF. After some calculations, (2.3) follows from

$$
(2.6)\quad \begin{aligned} E^i{}_{jk(h)} &= E_h{}^i{}_{jk} - (P^i{}_{jh/k} + P^r{}_{jh}P^i{}_{kr} - j|k) \qquad ([1],(3.9)(c)),\\ &E_h{}^i{}_{jk} + E_j{}^i{}_{kh} + E_k{}^i{}_{hj} = 0 \qquad ([1],(3.10)(a)). \end{aligned}
$$

Next, (2.4) follows from (2.3), (1.4) and

$$
(2.7)\qquad H^i{}_{jk} = R^i{}_{jk} + E^i{}_{jk}, \quad H^i{}_k = R^i{}_k + E^i{}_k \quad ([2],(1.9)(c),(d)).
$$

Q.E.D.

Remark. Using the relation $D_j{}^i{}_k = P^i{}_{jk} + P^i{}_{j(k)} = D_k{}^i{}_j$ ([1],(3.2)(a)), we can rewrite (2.5) as

$$
(2.5)\qquad (b)\quad J_j{}^i{}_k = -P^i{}_{j(k)/0} + 2(P^i{}_{j/k} + P^i{}_{kr}P^r{}_j + P^i{}_rP^r{}_{jk}) - j|k.
$$

Theorem 2.3. *In a generalized metric space M_n $(n>2)$ with* (2.1), *if the tensor $J_j{}^i{}_k$ vanishes, then the scalar R is independent of y^i.*

PROOF. Substituting (2.1) and (2.2) into the relation

$$
H_h{}^i{}_{jk} = R_h{}^i{}_{jk} - C_h{}^i{}_rR^r{}_{jk} + E_h{}^i{}_{jk} \qquad ([2],\ (1.7)(a),\ (1.9)(b)),
$$

we have

$$
(2.8)\qquad H_h{}^i{}_{jk} = R(g_{hj}\delta^i_k - y_jC_h{}^i{}_k - j|k) + E_h{}^i{}_{jk}.
$$

Transvection of (2.8) by y^h gives

$$
H^i{}_{jk} = R(y_j\delta^i_k - j|k) + E^i{}_{jk}.
$$

Differentiating this equation by y^h, we have

$$
(2.9)\qquad H_h{}^i{}_{jk} = \{R_{(h)}y_j\delta^i_k + R(g_{jh} + C_{jh})\delta^i_k - j|k\} + E^i{}_{jk(h)}.
$$

From (2.8), (2.9) and the identities (2.6), we have

$$
R(y_jC_h{}^i{}_k + C_{hj}\delta^i_k) + R_{(h)}y_j\delta^i_k - P^i{}_{jh/k} - P^r{}_{jh}P^i{}_{kr} - j|k = 0.
$$

Transvection of this equation by y^j yields

$$
\begin{aligned} R(F^2C_h{}^i{}_k - C_{hk}y^i) + F^2R_{(h)}h^i_k - 2P^i{}_{h/k} + {}&\\ P^i{}_{kh/0} - 2P^r{}_hP^i{}_{kr} + 2P^r{}_{kh}P^i{}_r &= 0. \end{aligned}
$$

Making $-h|k$ in the above equation, we obtain

$$
F^2R_{(j)}h^i_k - j|k = J_j{}^i{}_k.
$$

Therefore, by our assumption, we get $R_{(j)}h^i_k - j|k = 0$. Contracting i and k, we have $(n-2)R_{(j)} = 0$. Hence R is independent of y^i. Q.E.D.

Theorem 2.4. *In a generalized metric space M_n $(n > 2)$ with* (2.1), *if the tensor $P^i{}_k$ vanishes, then the scalar R is a constant.*

Proof. From (2.5)(b), we see that if $P^i{}_k = 0$, then $J_j{}^i{}_k = 0$ holds good. On the other hand, by the definition

$$E^i{}_{jk} = E_0{}^i{}_{jk} = P^i{}_{j/k} + P^r{}_j D_r{}^i{}_k - j|k \quad ([1], (3.9)(a)),$$

we see that if $P^i{}_k = 0$, then we have $E^i{}_{jk} = 0$. Hence, from (2.7), we have $H^i{}_{jk} = R^i{}_{jk}$, which means $H^i{}_{jk} = Ry_j\delta^i_k - j|k$ from (2.2). Consequently, noting Theorem 1.1, we have that R is a constant. Q.E.D.

Theorem 2.5. *A generalized metric space M_n $(n > 2)$ with*

$$K_h{}^i{}_{jk} = K(g_{hj}\delta^i_k - g_{hk}\delta^i_j) \tag{2.10}$$

is a Riemannian space of constant curvature.

Proof. (2.10) is equivalent to $K_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij})$. Consequently, making use of the identity

$$K_{hijk} + K_{ihjk} = -g_{hi(r)}R^r{}_{jk} \qquad ([1], (3.14)(b)),$$

we have

$$g_{hi(r)}R^r{}_{jk} = 0. \tag{2.11}$$

On the other hand, from (2.10), we see

$$K_h{}^i{}_{jk}y^h = R^i{}_{jk} = K(y_j\delta^i_k - y_k\delta^i_j).$$

Substituting this equation into (2.11), we have $g_{hi(k)}y_j - j|k = 0$. Hence, transvection of this equation by y^j gives $g_{hi(k)} = 0$, which means that the space is a Riemannian space of constant curvature. Q.E.D.

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