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On some inequalities for positively and negatively dependent random variables with applications

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Abstract. We present elementary proofs of some inequalities for the difference between a joint distribution or a density function and the product of their marginals. The inequalities obtained are applied to the consistency results of the kernel density estimator for positively and negatively dependent random variables.

1. Introduction

Positively and negatively dependent random variables (r.v.'s) found applications in statistical mechanics, reliability theory and mathematical statistics (cf. [3], [7], [8]). In this paper we will focus our attention on quadrant dependent and associated r.v.'s. Let us recall the notion of positive and negative quadrant dependence introduced by Lehmann [6]. The random variables X and Y are said to be positively quadrant dependent (PQD) if

$$P\,(X \leq x,\ Y \leq y) \geq P\,(X \leq x)\,P\,(Y \leq y) \tag{1}$$

and negatively quadrant dependent (NQD) if

$$P\,(X \leq x,\ Y \leq y) \leq P\,(X \leq x)\,P\,(Y \leq y) \tag{2}$$

for all $x, y \in \mathbb{R}$.

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These definitions were extended to the multivariate case by Esary *et al.* and Joag-Dev and Proschan (cf. [4], [5]) who introduced positive association (PA) and negative association (NA). The random variables $X_1, \dots, X_n$ are PA if

$$\operatorname{Cov}\left(f\left(X_1, \dots, X_n\right),\ g\left(X_1, \dots, X_n\right)\right) \geq 0$$

for any coordinatewise nondecreasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$ for which this covariance exists. The random variables $X_1, \dots, X_n$ are NA if for any nonempty and disjoint subsets $A, B \subset \{1, 2, \dots, n\}$

$$\operatorname{Cov}\left(f\left(X_i, i \in A\right),\ g\left(X_j, j \in B\right)\right) \leq 0$$

for any coordinatewise nondecreasing functions $f : R^A \to R$, $g : R^B \to R$, for which this covariance exists. An infinite collection $(X_n)_{n\in\mathbb{N}}$ of r.v.'s is PA (or NA) if every finite subcollection is PA (NA).

For any absolutely continuous r.v.'s X and Y let us denote by $F_{X,Y}(x,y)$ and $F_X(x)$, $F_Y(y)$ their joint distribution function (d.f.) and marginal d.f.'s; furthermore let us denote by $f_{X,Y}(x,y)$ and $f_X(x)$, $f_Y(y)$ the joint and marginal probability density functions (p.d.f.'s). Let us also introduce the following notation:

$$\begin{aligned} H_{X,Y}(x,y) &= P\left(X \leq x,\ Y \leq y\right) - P(X \leq x)P(Y \leq y) \\ &= F_{X,Y}(x,y) - F_X(x)F_Y(y), \\ \widetilde{f}_{X,Y}(x,y) &= f_{X,Y}(x,y) - f_X(x)f_Y(y). \end{aligned}$$

For associated random variables X and Y the upper bounds for $\sup_{x,y\in\mathbb{R}} |H_{X,Y}(x,y)|$ in terms of $\operatorname{Cov}^{1/3}(X,Y)$ were obtained in [1] and [9]. Bounds for $\sup_{x,y\in\mathbb{R}} \left|\widetilde{f}_{X,Y}(x,y)\right|$ in terms of $\operatorname{Cov}^{1/5}(X,Y)$ were proved in [10]. These inequalities were applied to problems of kernel estimation of the density and distribution function and were used in studying convergence of empirical function based on associated r.v.'s. Our goal is to obtain these inequalities by elementary methods not involving characteristic functions. As an application we investigate consistency of a kernel estimator (similar results were studied in [10] and [2]).

2. Inequalities

Let us denote by

$$\begin{aligned} \mathrm{Cov}_H(X,Y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H_{X,Y}(x,y)dxdy \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(P\left(X\le x,\ Y\le y\right) - P(X\le x)P(Y\le y)\right)dxdy \end{aligned} \tag{3}$$

the so called HOEFFDING covariance (cf. [6]) and note that if $\mathrm{Cov}(X,Y) = EXY - EXEY$ exists, then $\mathrm{Cov}(X,Y) = \mathrm{Cov}_H(X,Y)$. The Hoeffding covariance may be finite while the usual covariance is not defined, furthermore for PQD or NQD r.v.'s $\mathrm{Cov}_H(X,Y)$ always exists, although it may be infinite (as is further explained in Remark 1).

Theorem 1. *Let X and Y be PQD or NQD r.v.'s with densities f_X and f_Y and such that $\mathrm{Cov}_H(X,Y)$ is finite. If*

$$\|f_X\|_\infty < +\infty \quad \text{and} \quad \|f_Y\|_\infty < +\infty \tag{4}$$

then

$$\sup_{x,y\in\mathbb{R}} |H_{X,Y}(x,y)| \le C\left|\mathrm{Cov}_H(X,Y)\right|^{1/3}, \tag{5}$$

where $C = \frac{3}{2}\sqrt[3]{2}\left(\|f_X\|_\infty + \|f_Y\|_\infty\right)^{2/3}$ and

$$\|f\|_\infty = \inf\left\{M : \mu\{x\in\mathbb{R} : |f(x)| > M\} = 0\right\}$$

is the essential supremum of f with respect to the Lebesgue measure μ on $\mathbb{R}$.

PROOF. Assume that X and Y are PQD. It is easy to see that for continuous r.v.'s

$$H_{X,Y}(x,y) = P(X\ge x,\ Y\ge y) - P(X\ge x)P(Y\ge y). \tag{6}$$

Let $a>0$ be fixed and define the set K as follows:

$$K = K_a(x,y) = \left\{(t,s)\in\mathbb{R}^2 : x-a\le t\le x,\ y-a\le s\le y\right\}.$$

Now, consider x, y as fixed. For $(t,s)\in K$ we have

$$\begin{aligned}\mathrm{Cov}_H(X,Y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H_{X,Y}(t,s)dtds \geq \int\int_K H_{X,Y}(t,s)dtds\\ &\geq \int\int_K \left(P(X\geq t,\ Y\geq s) - P(X\geq x-a)P(Y\geq y-a)\right)dtds\\ &\geq a^2\left(P(X\geq x,\ Y\geq y) - P(X\geq x-a)P(Y\geq y-a)\right).\end{aligned} \tag{7}$$

From (7) and by the PQD property we get, for any $x,y\in R$ and $a>0$, the following inequality

$$P(X\geq x,\ Y\geq y) - P(X\geq x-a)P(Y\geq y-a) \leq \frac{\mathrm{Cov}_H(X,Y)}{a^2}. \tag{8}$$

Now we get, by (4) and (8), the following bounds for $H_{X,Y}(x,y)$:

$$\begin{aligned}0 \leq H_{X,Y}(x,y) &= [P(X\geq x,\ Y\geq y) - P(X\geq x-a)P(Y\geq y-a)]\\ &\quad + [P(X\geq x-a)P(Y\geq y-a) - P(X\geq x)P(Y\geq y)]\\ &\leq \frac{\mathrm{Cov}_H(X,Y)}{a^2} + P(x-a\leq X<x) + P(y-a\leq Y<y)\\ &\leq \frac{\mathrm{Cov}_H(X,Y)}{a^2} + a(\|f_X\|_\infty + \|f_Y\|_\infty).\end{aligned} \tag{9}$$

Minimizing the right-hand side of (9) with respect to a we get (5). If X,Y are NQD then $X,-Y$ are PQD and $H_{X,Y}(x,y) = -H_{X,-Y}(x,-y)$ and the conclusion follows from the part already obtained. □

Remark 1. In [1] and [9] (5) was proved for associated r.v.'s with finite second moments. In [9] we find the explicit form of the constant $C = \max(2/\pi^2, 45\|f_X\|_\infty, 45\|f_Y\|_\infty)$. We will give an example which shows that the bound obtained in [9] may be improved upon. Furthermore our Theorem 1 is valid for r.v.'s without any moment assumptions. Let X, Y, ξ be independent r.v.'s such that X, Y have the same d.f.

$$F(x) = \begin{cases} 1-\dfrac{1}{x^p}, & x\geq 1\\ 0, & x<1\end{cases}$$

for some $p>0$ and ξ is such that $P(\xi=0) = P(\xi=1) = 1/2$. For some $\alpha>0$ we put

$$X_1 = X + \alpha\xi,$$

$$X_2 = Y + \alpha\xi.$$

It is obvious that X_1 and X_2 are associated and we check that

$$H_{X_1,X_2}(t,s) = \frac{1}{4}P(t-\alpha < X \le t)P(s-\alpha < Y \le s)$$

thus

$$\begin{aligned}\sup_{t,s\in R} H_{X_1,X_2}(t,s) &= \frac{1}{4}\Big(\sup_{t\in R}\big(F(t)-F(t-\alpha)\big)\Big)^2 \\ &= \frac{1}{4}\left(1-\frac{1}{(1+\alpha)^p}\right)^2.\end{aligned} \tag{10}$$

Furthermore

$$\begin{aligned}\mathrm{Cov}_H(X_1,X_2) &= \lim_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a} H_{X_1,X_2}(t,s)\,dtds \\ &= \frac{1}{4}\lim_{a\to\infty}\left(\int_{-a}^{a} P(t-\alpha < X \le t)dt\right)^2 \\ &= \frac{1}{4}\lim_{a\to\infty}\left(\alpha-\int_{a-\alpha}^{a}\frac{1}{t^p}dt\right)^2 = \frac{1}{4}\alpha^2.\end{aligned} \tag{11}$$

From (11) we see that $\mathrm{Cov}_H(X_1,X_2)$ is finite for every $p > 0$, while $\mathrm{Cov}(X_1,X_2)$ is defined only for $p > 1$. Denoting by $G(x)$ the d.f. of X_1 and X_2, we have

$$G(x) = \frac{1}{2}F(x) + \frac{1}{2}F(x-\alpha)$$

thus the p.d.f. of X_1 and X_2 for $t \ne 1, 1+\alpha$ is equal to

$$g(t) = G'(t) = \frac{1}{2}f(t) + \frac{1}{2}f(t-\alpha).$$

We easily find that

$$\|g\|_\infty = \frac{p}{2}\left(1+\frac{1}{(1+\alpha)^{p+1}}\right). \tag{12}$$

Now by choosing $\alpha = \alpha(n) = n^{-6}$ and $p = p(n) = n^5$, we see that by (10) the left-hand side of (5) tends to 0 as $n \to \infty$, the bound obtained in [9] is, by (12), of order $\|g\|_\infty \alpha^{2/3} = O(n)$, while our bound is of order $\|g\|_\infty^{2/3} \alpha^{2/3} = O(n^{-2/3})$.

Theorem 2. *Let X and Y be integer valued PQD or NQD r.v.'s such that* $\mathrm{Cov}_H(X,Y)$ *is finite, then*

$$\sup_{x,y\in\mathbb{R}} |H_{X,Y}(x,y)| \le |\mathrm{Cov}_H(X,Y)|\,. \tag{13}$$

Proof. Assume that X and Y are PQD. For $x \in \mathbb{R}$ define a function

$$h_x(t) = \begin{cases} 1, & t \le x \\ 0, & t \ge x+1 \\ -t+1+x, & t \in (x, x+1). \end{cases}$$

Then, noting that X, Y take only integer values, we get by Remark 4 of [11]:

$$\begin{aligned} |H_{X,Y}(x,y)| &= \left|\mathrm{Cov}\left(I_{[X\le x]}, I_{[Y\le y]}\right)\right| \\ &= \left|\mathrm{Cov}\left(h_{[x]}(X), h_{[y]}(Y)\right)\right| \\ &\le \mathrm{Cov}_H(X,Y), \end{aligned}$$

where $[\,.\,]$ stands for the integer part of a number. The NQD case follows similarly as in the proof of Theorem 1. □

Remark 2. The bound obtained in Theorem 2 is sharp. For the r.v.'s X, $Y = X$ such that $P(X=1) = p$, $P(X=0) = 1-p$ we get equality in (13).

Remark 3. As noted previously, the bound obtained in Theorem 2 is optimal but it is still an open question whether the upper bound in Theorem 1 may be improved. Let us only note that for any bounded (by 1 say) r.v.'s (not necessarily PQD or NQD) one has

$$\begin{aligned} |\mathrm{Cov}_H(X,Y)| = |\mathrm{Cov}(X,Y)| &= \left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H_{X,Y}(t,s)\,dtds\right| \\ &\le \int_{-1}^{1}\int_{-1}^{1} |H_{X,Y}(t,s)|\,dtds \le 4\sup_{t,s\in R}|H_{X,Y}(t,s)|\,. \end{aligned}$$

Now we will turn our attention to inequalities for the difference between joint density function and the product of its marginal densities. In the main theorem we do not make any assumptions on the dependence of the r.v.'s.

Theorem 3. *Let X and Y be any random variables with joint and marginal p.d.f. $f_{X,Y}(x,y)$, $f_X(x)$ and $f_Y(y)$ respectively. If the function $\widetilde{f}_{X,Y}(x,y) = f_{X,Y}(x,y) - f_X(x)f_Y(y)$ satisfies the Lipschitz condition*

$$\left|\widetilde{f}_{X,Y}(x+\Delta x, y+\Delta y) - \widetilde{f}_{X,Y}(x,y)\right| \le L\left(|\Delta x| + |\Delta y|\right) \tag{14}$$

for some constant $L > 0$ and every $\Delta x, \Delta y \in \mathbb{R}$, then

$$\sup_{x,y\in\mathbb{R}} \left|\widetilde{f}_{X,Y}(x,y)\right| \le 3\sqrt[3]{4}L^{2/3}\Big(\sup_{x,y\in\mathbb{R}} |H_{X,Y}(x,y)|\Big)^{1/3}. \tag{15}$$

Proof. Let $x_0, y_0 \in \mathbb{R}$ and $a > 0$ be fixed. By (14) for $x_0 \le t \le x_0+a$ and $y_0 \le s \le y_0 + a$ we have

$$\left|\widetilde{f}_{X,Y}(t,s) - \widetilde{f}_{X,Y}(x_0,y_0)\right| \le 2La \tag{16}$$

thus

$$\begin{aligned} &\Big|\int_{x_0}^{x_0+a}\int_{y_0}^{y_0+a} \widetilde{f}_{X,Y}(t,s)dtds - a^2\widetilde{f}_{X,Y}(x_0,y_0)\Big| \\ &= \Big|\int_{x_0}^{x_0+a}\int_{y_0}^{y_0+a} \widetilde{f}_{X,Y}(t,s)dtds - \int_{x_0}^{x_0+a}\int_{y_0}^{y_0+a} \widetilde{f}_{X,Y}(x_0,y_0)dtds\Big| \\ &\le 2La^3. \end{aligned} \tag{17}$$

Let us observe that

$$\begin{aligned} &\int_{x_0}^{x_0+a}\int_{y_0}^{y_0+a} \widetilde{f}_{X,Y}(t,s)dtds \\ &\quad = H_{X,Y}(x_0+a, y_0+a) - H_{X,Y}(x_0, y_0+a) \\ &\qquad - H_{X,Y}(x_0+a, y_0) + H_{X,Y}(x_0,y_0). \end{aligned} \tag{18}$$

From (17) and (18) we get

$$\left|\widetilde{f}_{X,Y}(x_0,y_0)\right| \le \frac{4\sup_{x,y\in\mathbb{R}}|H_{X,Y}(x,y)|}{a^2} + 2La \tag{19}$$

for any $x_0, y_0 \in \mathbb{R}$ and $a > 0$. Minimalization of the right-hand side of (19) with respect to a completes the proof of (15). □

Combining Theorems 1 and 3 yields the following corollary.

Corollary 1. *Let X and Y be PQD or NQD r.v.'s satisfying the assumptions of Theorem 1 and Theorem 3. Then*

$$\sup_{x,y\in\mathbb{R}} \left|\widetilde{f}_{X,Y}(x,y)\right| \leq 3^{4/3}2^{4/9}L^{2/3}\left(\|f_X\|_\infty + \|f_Y\|_\infty\right)^{2/9}\left|\mathrm{Cov}_H(X,Y)\right|^{1/9}. \tag{20}$$

3. Applications

In this section we present some applications of Theorem 3 in investigating the weak consistency of kernel-type estimators.

Let the kernel $K(u)$ be a p.d.f. such that

$$\int_{-\infty}^{+\infty} K^2(u)du < +\infty \quad \text{and} \quad uK(u) \to 0 \text{ as } |u| \to \infty. \tag{21}$$

In order to estimate the common p.d.f. of the random variables $X_1, X_2, \ldots$ we often use the kernel estimator

$$f_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n} K\left(\frac{x - X_j}{h_n}\right), \tag{22}$$

where the bandwidhs h_n are such that

$$0 < h_n \to 0 \text{ and } nh_n \to \infty, \quad \text{as } n \to \infty. \tag{23}$$

Theorem 4. *Let $(X_n)_{n\in N}$ be a sequence of r.v.'s with the same bounded density $f(x)$ such that the functions $\widetilde{f}_{X_i,X_j}(x,y)$ satisfy the Lipschitz condition with the same constant $L > 0$:*

$$\left|\widetilde{f}_{X_i,X_j}(x+\Delta x, y+\Delta y) - \widetilde{f}_{X_i,X_j}(x,y)\right| \leq L\left(|\Delta x| + |\Delta y|\right), \tag{24}$$

for any $x, y, \Delta x, \Delta y \in \mathbb{R}$ and $i \neq j$. Assume that the functions $H_{X_i,X_j}(x,y)$ satisfy

$$\sup_{x,y\in\mathbb{R}} \left|H_{X_i,X_j}(x,y)\right| \leq \alpha(|j-i|), \quad i \neq j \tag{25}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} \alpha^{1/3}(k) \to 0, \quad \text{as } n \to \infty. \tag{26}$$

If the conditions (21) *and* (23) *are satisfied then*

$$f_n(x) \xrightarrow{P} f(x), \quad \text{as } n \to \infty \tag{27}$$

for all continuity points of f.

PROOF. Let us introduce the following notation

$$\varphi_n(x, X_j) = \frac{1}{h_n}\left(K\left(\frac{x - X_j}{h_n}\right) - EK\left(\frac{x - X_j}{h_n}\right)\right). \tag{28}$$

For $i \neq j$ we have

$$\begin{aligned}
&|\mathrm{Cov}\,(\varphi_n(x, X_i), \varphi_n(x, X_j))| \\
&\quad = \frac{1}{h_n^2}\left|\mathrm{Cov}\left(K\left(\frac{x - X_i}{h_n}\right), K\left(\frac{x - X_j}{h_n}\right)\right)\right| \\
&\quad = \frac{1}{h_n^2}\left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K\left(\frac{x-u}{h_n}\right) K\left(\frac{x-v}{h_n}\right)\widetilde{f}_{X_i,X_j}(u,v)dudv\right| \\
&\quad = \left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(u')K(v')\widetilde{f}_{X_i,X_j}(x - u'h_n, x - v'h_n)du'dv'\right| \\
&\quad \leq \sup_{x,y\in\mathbb{R}}\left|\widetilde{f}_{X_i,X_j}(x,y)\right|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(u')K(v')du'dv',
\end{aligned}$$

so that by Theorem 3 and (25) we get

$$|\mathrm{Cov}\,(\varphi_n(x, X_i), \varphi_n(x, X_j))| \leq C_1\alpha^{1/3}(|j - i|), \tag{29}$$

where C_1 depends only on the constant L.

Furthermore $\left\|f_{X_j}\right\|_\infty = \|f\|_\infty < \infty$ so that by (21) we obtain

$$\begin{aligned}
\mathrm{Var}\,(\varphi_n(x, X_j)) &\leq \frac{1}{h_n^2}E\left(K\left(\frac{x - X_j}{h_n}\right)\right)^2 \\
&= \frac{1}{h_n^2}\int_{-\infty}^{+\infty} K^2\left(\frac{x-u}{h_n}\right) f_{X_j}(u)du \\
&= \frac{1}{h_n}\int_{-\infty}^{+\infty} K^2(v)f(x - vh_n)dv \\
&\leq \frac{1}{h_n}\|f\|_\infty \int_{-\infty}^{+\infty} K^2(v)dv = \frac{C_2}{h_n}.
\end{aligned} \tag{30}$$

By (29), (30) and (26) we get

$$
\begin{aligned}
\operatorname{Var}(f_n(x)) &= \frac{1}{n^2} \operatorname{Var}\Big(\sum_{j=1}^n \varphi_n(x, X_j)\Big) \\
&= \frac{1}{n^2} \sum_{j=1}^n \operatorname{Var}(\varphi_n(x, X_j)) + \frac{2}{n^2} \sum\sum_{1 \le i < j \le n} \operatorname{Cov}\left(\varphi_n(x, X_i), \varphi_n(x, X_j)\right) \\
&\le \frac{C_2}{n h_n} + \frac{2C_1}{n} \sum_{j=1}^n \alpha^{1/3}(j) \longrightarrow 0, \text{ as } n \to \infty.
\end{aligned}
\tag{31}
$$

Let us observe that if x is a continuity point of f then by the Lebesgue dominated convergence theorem we have

$$
\begin{aligned}
Ef_n(x) &= \frac{1}{h_n} EK\left(\frac{x - X_1}{h_n}\right) = \frac{1}{h_n} \int_{-\infty}^{+\infty} K\left(\frac{x-u}{h_n}\right) f(u) du \\
&= \int_{-\infty}^{+\infty} K(v) f(x - v h_n) dv \longrightarrow f(x), \text{ as } n \to \infty.
\end{aligned}
\tag{32}
$$

From (31) and (32) it follows that for a continuity point x of the density f,

$$f_n(x) \xrightarrow{L^2} f(x)$$

and the proof is completed. □

Corollary 2. *Let $(X_n)_{n \in N}$ be a sequence of pairwise PQD or NQD r.v.'s with the same p.d.f. $f(x)$ such that* (24) *is satisfied and*

$$|\operatorname{Cov}_H(X_i, X_j)| \le \alpha(|j - i|), \quad i \ne j, \tag{33}$$

where

$$\frac{1}{n} \sum_{k=1}^n \alpha^{1/9}(k) \longrightarrow 0, \quad \text{as } n \to \infty. \tag{34}$$

If (21) *and* (23) *are satisfied then* (27) *holds.*

Proof. It is easy to see that, by Theorem 1, (33) and (34) imply (25) and (26). □

Remark 4. If $(X_n)_{n\in\mathbb{N}}$ is a stationary (wide sense) sequence of associated r.v.'s then from the assumption

$$\sum_{k=2}^{\infty} \operatorname{Cov}(X_1, X_k) < +\infty$$

it follows that $\alpha(n) = \operatorname{Cov}(X_1, X_{n+1}) \to 0$ so that (34) is satisfied. For stationary sequences of NA r.v.'s the series $\sum_{k=2}^{\infty} |\operatorname{Cov}(X_1, X_k)|$ is automatically convergent (cf. [8]) and therefore (34) holds.

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