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Modular projective representations of direct products of finite groups

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Abstract. Let G be a finite group, S a field of characteristic p or a complete discrete valuation ring of characteristic p. We denote by $S^\lambda G$ a twisted group ring of the group G and the ring S with an S-factor system $\lambda \in Z^2(G, S^*)$ (see [17], pp. 2–4). Let $p\backslash|G|$ and $G = G_p \times B$ be the direct product of a p-subgroup G_p and p'-subgroup B. In this paper we establish necessary and sufficient conditions that every indecomposable $S^\lambda G$-module is the outer tensor product of an indecomposable $S^\lambda G_p$-module and an irreducible $S^\lambda B$-module.

1. Introduction

Let $G = G_1 \times G_2$ be a finite group, S be a Dedekind domain with quotient field T, P a prime ideal in S relatively prime to the order of G_2, and

$$S_P = \left\{ \frac{a}{b} : a, b \in S,\ b \notin P \right\}.$$

A. Jones [14] has shown that if T is a splitting field for G_2, then every indecomposable $S_P G$-module is the outer tensor product $M_1 \# M_2$ of an indecomposable $S_P G_1$-module M_1 and an irreducible $S_P G_2$-module M_2. B. Fein [7] has examined the structure of LG-modules $M_1 \# M_2$, where L is an arbitrary field, and M_i is an irreducible LG_i-module ($i = 1, 2$). In

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particular, he proved that $M_1 \# M_2$ is completely reducible and gave criteria for it to be irreducible. In paper [8] B. Fein generalized his results to the case of arbitrary finite dimensional L-algebras. Outer tensor products of irreducible modules over twisted group algebras were investigated as well in [2].

Let F be a field of characteristic $p > 0$, and $G = G_p \times B$, where G_p is a Sylow p-subgroup. H. I. Blau [3] and P. M. Gudivok [10], [11] proved that every finitely generated indecomposable FG-module is an outer tensor product $V \# W$ of an indecomposable FG_p-module V with an irreducible FB-module W if and only if either G_p is cyclic or F is a splitting field for B. P. M. Gudivok [12] also investigated the similar problem for group rings KG, where K is a complete discrete valuation ring of characteristic $p > 0$. He proved that if F is the quotient field of K, then every indecomposable KG-module is $V \# W$ if and only if either $|G_p| = 2$ or F is a splitting field for B.

In this paper we generalize the results of H. I. Blau and P. M. Gudivok to the case of twisted group rings $S^\lambda G$, where $G = G_p \times B$, $S = F$ or S is a complete discrete valuation ring of characteristic p.

We use the following notations: F is a field of characteristic $p > 0$; $F[[x]]$ is a ring of formal power series in x with coefficients in the field F; G is a finite group and $p \backslash |G|$; G' is the commutant of G; G_p is a Sylow p-subgroup of G; S is an integral domain with an identity element; $S^p = \{a^p : a \in S\}$; S^* is the multiplicative group of the ring S; $Z^2(G, S^*)$ is the group of S-factor systems (2-cocycles) of the group G, where we assume that G acts trivially on S^* (see [15], Chapter 1). Any S-factor system of G is equivalent to some normalized S-factor system of G. From now on we will assume that S-factor systems of G are normalized. An S-basis $\{u_g : g \in G\}$ of $S^\lambda G$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$ will be called natural. Let e be the identity element of G. We will often identify u_e with the identity element of the ring S. That is why, instead of μu_e, we will write μ ($\mu \in S$). If H is a subgroup of the group G, then the restriction of the S-factor system $\lambda \in Z^2(G, S^*)$ to $H \times H$ will also be denoted by λ. In this case $S^\lambda H$ is a subring of the ring $S^\lambda G$. By an $S^\lambda G$-module we understand a unitary left $S^\lambda G$-module which is finitely generated and torsion-free as S-module. If M is any $S^\lambda H$-module, then $M^{S^\lambda G}$ will denote the induced $S^\lambda G$-module of M. Let V be an $S^\lambda G$-module. Then we write $E_{S^\lambda G}(V)$ for the ring of $S^\lambda G$-endomorphisms of V, $\operatorname{rad} E_{S^\lambda G}(V)$ for the Jacobson radical of $E_{S^\lambda G}(V)$, and $\overline{E_{S^\lambda G}(V)}$ for

$$E_{S^\lambda G}(V)/\operatorname{rad} E_{S^\lambda G}(V).$$

Let us briefly present the results obtained. In Section 2, we generalize the result of J. A. Green [9] on induced modules of p-groups (Lemma 2.2). Using this generalization we prove in Propositions 2.1, 2.2 that if G is a p-group and $S = F$ or S is a complete discrete valuation ring of characteristic p, then (under some assumption) for every field K there exists an indecomposable $S^\lambda G$-module V such that $\overline{E_{S^\lambda G}(V)}$ is isomorphic to a field which contains K. Let G be a finite p-group, $\lambda \in Z^2(G, F^*)$, and A be an abelian subgroup of G such that $\operatorname{rad} F^\lambda A$ is cyclic. In Proposition 2.4 we show that $M^{F^\lambda G}$ is an indecomposable $F^\lambda G$-module for every indecomposable $F^\lambda A$-module M and

$$\overline{E_{F^\lambda G}\left(M^{F^\lambda G}\right)}$$

is isomorphic to $F(\rho_1, \dots, \rho_s)$, where ρ_i is a root of the polynomial $x^p - \gamma_i$ with $\gamma_1 \in F$ and $\gamma_i \in F(\rho_1, \dots, \rho_{i-1})$ for $i \geq 2$.

Let $G = G_p \times B$, $\lambda \in Z^2(G, F^*)$. In Section 3, we establish necessary and sufficient conditions that every indecomposable $F^\lambda G$-module is isomorphic to a module $V\#W$, where V is an indecomposable $F^\lambda G_p$-module, and W is an irreducible $F^\lambda B$-module. In this case we say that the algebra $F^\lambda G$ satisfies the TPIM condition. If G_p is abelian, then $F^\lambda G$ satisfies the TPIM condition if and only if either $\operatorname{rad} F^\lambda G_p$ is cyclic or F is a splitting field for $F^\lambda B$ (Theorem 3.1). If G'_p is not cyclic and F contains a primitive qth root of 1 for every prime $q \backslash |B|$, such that $p \backslash (q-1)$, then $F^\lambda G$ satisfies the TPIM condition if and only if F is a splitting field for $F^\lambda B$ (Theorem 3.2).

In Section 4, we study twisted group rings $S^\lambda G$ satisfying the TPIM condition for the case when S is a complete discrete valuation ring of characteristic p with residue class field F. If G_p is abelian, then we assume that $S = F\big[[x]\big]$ and $\lambda = \mu \times \nu$, where $\mu \in Z^2(G_p, F^*)$ and $\nu \in Z^2(B, S^*)$. If G_p is nonabelian, then we suppose that $|G'_p| \neq 2$ and F contains a primitive qth root of 1 for every prime $q \backslash |B|$ such that $p \backslash (q-1)$. It should be noted that in Sections 3, 4 we also establish conditions that every indecomposable projective S-representation of a group $G = G_p \times B$ is equivalent to a representation $\Gamma \# \Delta$, where Γ is an indecomposable projective S-representation of G_p, Δ is an irreducible projective S-representation of B, and S is the field F or a complete discrete valuation ring of characteristic p.

2. On induced modules

This section begins by reformulation of a well-known result about twisted group rings.

Lemma 2.1 ([6], p. 125)**.** *Let S be a field or a complete discrete valuation ring, and V be an $S^\lambda G$-module. Then V is an indecomposable $S^\lambda G$-module if and only if $\overline{E_{S^\lambda G}(V)}$ is a skewfield.*

Lemma 2.2. *Let $S = F$ or S be a complete discrete valuation ring of characteristic p with residue class field F, G be a finite p-group, H a subgroup of G, and M an indecomposable $S^\lambda H$-module. Suppose that $\overline{E_{S^\lambda H}(M)}$ is isomorphic to a field K, $K \supset F$, and one of the following conditions is satisfied:*

(i) *$G = H \cdot T$, where T is a subgroup of the center of G.*

(ii) *K is a finite Galois extension of F and $[K : F]$ is not divisible by p.*

(iii) *$K = F(\rho_1, \dots, \rho_d)$, where ρ_i is a root of the polynomial*

$$x^{p^{n_i}} - \alpha_i$$

with $\alpha_1 \in F$ and $\alpha_i \in F(\rho_1, \dots, \rho_{i-1})$ for $i \geq 2$.

Then

$$\overline{E_{S^\lambda G}\left(M^{S^\lambda G}\right)}$$

is isomorphic to $K(\theta_1, \dots, \theta_r)$, where θ_j is a root of the polynomial $x^p - \beta_j$ with $\beta_1 \in K$ and $\beta_j \in K(\theta_1, \dots, \theta_{j-1})$ for $j \geq 2$.

Proof. The idea of the proof of the lemma is the same as the one seen in Theorem 8 in [9]. We consider several cases.

I. Suppose that $S = F$ and $|G : H| = p$. Let $G/H = \langle aH \rangle$, and $\{u_g : g \in G\}$ be a natural F-basis of $F^\lambda G$. Then

$$M^{F^\lambda G} = \sum_{i=0}^{p-1} u_a^i \otimes M$$

and

$$u_h(u_a^i \otimes m) = u_a^i \otimes (u_a^{-i} u_h u_a^i)m,$$

for all $h \in H$, $m \in M$. We denote by N the stabilizer of M in G. It is well known ([4], p. 160) that

$$\overline{E_{F^\lambda G}(M^{F^\lambda G})} \cong \overline{E_{F^\lambda N}(M^{F^\lambda N})}.$$

From this it follows that if $N = H$, then

$$\overline{E_{F^\lambda G}(M^{F^\lambda G})} \cong K.$$

Let $N = G$, and φ be an $F^\lambda H$-isomorphism of M onto $u_a \otimes M$. Then $\varphi(m) = u_a \otimes \psi(m)$, $m \in M$, where ψ is an F-automorphism of M, and $\psi u_h = (u_a^{-1} u_h u_a)\psi$ for any h in H. Let ρ be an F-endomorphism of $M^{F^\lambda G}$. Then

$$\rho(u_a^j \otimes m) = \sum_{i=0}^{p-1} u_a^i \otimes \rho_{ij}(m) \quad (j = 0, 1, \ldots, p-1;\ m \in M),$$

where ρ_{ij} is an F-endomorphism of M. Let $\mathcal{L}(M)$ denote the set of F-endomorphisms of M. The correspondence $\rho \to (\rho_{ij})$ is an isomorphism of the ring $\mathrm{Hom}_F\left(M^{F^\lambda G}, M^{F^\lambda G}\right)$ onto the ring of all $p \times p$ matrices with coefficients in $\mathcal{L}(M)$.

Direct calculation shows that $\rho \in E_{F^\lambda G}(M^{F^\lambda G})$ if and only if the following conditions hold:

$$1) \quad \rho_{ij}\left(u_a^{-j} u_h u_a^j\right) = \left(u_a^{-i} u_h u_a^i\right)\rho_{ij} \quad \text{for every } h \in H;$$

$$2) \quad \rho_{i,j+1} v = w \rho_{i-1,j} \quad (i, j = 0, 1, \ldots, p-1),$$

where the indices $i-1$, $j+1$ being reduced mod p if necessary, and $v, w \in \{u_e, u_a^p\}$, moreover $v = u_a^p$ if and only if $j = p-1$, and $w = u_a^p$ if and only if $i = 0$.

Let $\sigma \in E_{F^\lambda H}(M)$, and

$$\hat{\sigma} = \begin{pmatrix} \sigma & 0 & 0 & \dots & 0 \\ 0 & \sigma & 0 & \dots & 0 \\ 0 & 0 & \sigma & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 & \dots & 0 & u_a^p\psi \\ \psi & 0 & \dots & 0 & 0 \\ 0 & \psi & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \psi & 0 \end{pmatrix}$$

be $p \times p$ matrices. Then

$$(\rho_{ij}) = \hat{\sigma}_0 + \Omega\hat{\sigma}_1 + \cdots + \Omega^{p-1}\hat{\sigma}_{p-1}, \quad \Omega^p = \hat{\omega}, \tag{2.1}$$

where $\omega = u_a^p\psi^p$ and $\omega, \sigma_j \in E_{F^\lambda H}(M)$. It should be noted that $\Omega\widehat{\sigma}_j = \widehat{\mu}_j\Omega$, where $\mu_j = \psi\sigma_j\psi^{-1}$. The mapping $\sigma \to \hat{\sigma}$ is an isomorphism of $E_{F^\lambda H}(M)$ into $E_{F^\lambda G}(M^{F^\lambda G})$.

Let $U = \operatorname{rad} E_{F^\lambda H}(M)$, f be an isomorphism of $\overline{E_{F^\lambda H}(M)}$ onto K,

$$V = \left\{ \sum_{j=0}^{p-1} \Omega^j\hat{\sigma}_j : \sigma_j \in U \right\},$$

and $t = f(u_a^p\psi^p + U)$. The set V is a nilpotent ideal of the ring $E_{F^\lambda G}(M^{F^\lambda G})$. It follows from the hypotheses (i)–(iii) that the algebra

$$E_{F^\lambda G}(M^{F^\lambda G})/V$$

is isomorphic to the twisted group algebra

$$\Lambda = K^\mu(G/H) = K + Kv + \cdots + Kv^{p-1}, \quad v^p = t.$$

If $t \notin K^p$, then Λ is a field. If $t = l^p$, $l \in K$, then $K(v - l)$ is the radical of Λ. It follows that

$$\overline{E_{F^\lambda G}(M^{F^\lambda G})} \cong K.$$

II. Assume $S = F$ and $|G : H| > p$. Let us make an induction on the index $|G : H|$. The subgroup H is contained in some maximal subgroup N of G. By the induction hypothesis,

$$L = \overline{E_{S^\lambda N}\left(M^{S^\lambda N}\right)}$$

is isomorphic to $K(\theta_1,\ldots,\theta_d)$, where θ_j is a root of the polynomial $x^p-\mu_j$ with $\mu_1\in K$ and $\mu_j\in K(\theta_1,\ldots,\theta_{j-1})$ for $j\geq 2$. Suppose that the field K satisfies the condition (ii). If g is an F-automorphism of the field L and g^p is the identity, then g is the identity mapping of K. It follows from this that g is the identity mapping of $K(\theta_1)$, $K(\theta_1,\theta_2),\ldots,K(\theta_1,\ldots,\theta_d)$. Using the result obtained in case I, we complete the proof.

III. Let S be a complete discrete valuation ring of characteristic p, P the maximal ideal of S, $S/P=F$, $|G:H|=p$, and N be the stabilizer of M in G. Arguing as in the proof of Theorem 8 in [9], we obtain that if $N=H$ then $M^{S^\lambda G}$ is an indecomposable $S^\lambda G$-module. By Lemma 2.1 $E_{S^\lambda G}\big(M^{S^\lambda G}\big)$ is a local ring. Let d be the S-rank of M, Γ a matrix S-representation of $S^\lambda H$ realized by M, and $\Gamma^{S^\lambda G}$ a matrix S-representation of $S^\lambda G$ realized by $M^{S^\lambda G}$. We may assume that

$$\Gamma^{S^\lambda G}(u_h)=\begin{pmatrix}\Gamma_1(u_h) & 0 & \ldots & 0\\ 0 & \Gamma_2(u_h) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \Gamma_p(u_h)\end{pmatrix},$$

$$\Gamma^{S^\lambda G}(u_a)=\begin{pmatrix}0 & 0 & \ldots & 0 & A\\ E & 0 & \ldots & 0 & 0\\ 0 & E & \ldots & 0 & 0\\ . & . & \cdots & . & .\\ 0 & 0 & \ldots & E & 0\end{pmatrix},$$

where $\Gamma_j(u_h)=\Gamma\left(u_a^{-j+1}u_hu_a^{j-1}\right)$ for $j=1,2,\ldots,p$, $A=\Gamma(u_a^p)$, and E is the identity matrix of order d. The representations $\Gamma_1,\Gamma_2,\ldots,\Gamma_p$ of $S^\lambda H$ are mutually nonequivalent.

Let

$$C=\begin{pmatrix}C_{11} & \ldots & C_{1p}\\ . & \ldots & .\\ C_{p1} & \ldots & C_{pp}\end{pmatrix},$$

where C_{ij} is $d\times d$ matrix over S $(i,j=1,\ldots,p)$. If $C\cdot\Gamma^{S^\lambda G}(u_g)=\Gamma^{S^\lambda G}(u_g)\cdot C$ for arbitrary $g\in G$, then $C_{11}=\cdots=C_{pp}$, $C_{11}\Gamma(u_h)=\Gamma(u_h)C_{11}$ for every $h\in H$, and C_{ij} is not invertible for $i\neq j$. It follows from this

$$C = \begin{pmatrix} C_{11} & & 0 \\ & \ddots & \\ 0 & & C_{pp} \end{pmatrix} + B,$$

where B is not invertible and $B \cdot \Gamma^{S^\lambda G}(u_g) = \Gamma^{S^\lambda G}(u_g) \cdot B$ for arbitrary $g \in G$. Hence, $\overline{E_{S^\lambda G}\left(M^{S^\lambda G}\right)} \cong K$.

Let $N = G$. By the same arguments as in the case I we can establish existence of the isomorphism $\rho \to (\rho_{ij})$ of the ring $E_{S^\lambda G}\big(M^{S^\lambda G}\big)$ onto the ring of all $p \times p$ matrices (2.1), where $\sigma_j \in E_{S^\lambda H}(M)$. Let W be an $S^\lambda G$-module, and

$$\tilde{E}_{S^\lambda G}(W) = E_{S^\lambda G}(W)/PE_{S^\lambda G}(W).$$

By Proposition 5.22 ([6], p. 112)

$$\overline{E_{S^\lambda G}(W)} \cong \tilde{E}_{S^\lambda G}(W)/\operatorname{rad}\tilde{E}_{S^\lambda G}(W).$$

Therefore after we obtain formula (2.1) we may replace S by residue class field F and then we may argue as in the case I. This completes the proof of the lemma. □

Proposition 2.1. *Let K be a finite separable extension of the field F, G a finite p-group, $\lambda \in Z^2(G, F^*)$, and there exist a noncyclic subgroup H in G, such that $F^\lambda H$ is equivalent to the group algebra FH. Suppose that either*

(i) *$G = HT$, where T is a subgroup of the center of G, or*

(ii) *K is a finite Galois extension of F and $[K : F]$ is not divisible by p.*

Then there exists an indecomposable $F^\lambda G$-module V such that $\overline{E_{F^\lambda G}(V)}$ is isomorphic to a field that contains K.

Proof. We will assume $F^\lambda H = FH$. By the hypothesis, $K = F(\theta)$, where θ is algebraic over F. We denote by $f(x)$ the monic irreducible polynomial for θ over F. Let d be the degree of $f(x)$. In H there exists a normal subgroup N such that $H/N = (aN) \times (bN)$ is a noncyclic group of order p^2. Let $\bar{H} = H/N$, U be a $F\bar{H}$-module by which the following F-representation of the group $\bar{H}$ is realized:

$$aN \to \begin{pmatrix} E & E \\ 0 & E \end{pmatrix}, \quad bN \to \begin{pmatrix} E & \Gamma \\ 0 & E \end{pmatrix},$$

where E is the identity matrix of order d, and Γ is the companion matrix of $f(x)$. The module U is also an FH-module. It is known (see [3], [10], [11]) that $\overline{E_{FH}(U)} \cong K$. Applying Lemma 2.2 to U we conclude that $V = U^{F^\lambda G}$ is a required $F^\lambda G$-module, and the proof is complete. $\square$

Proposition 2.2. *Let K be a finite separable extension of the field F, S be a complete discrete valuation ring of characteristic p with residue class field F; G a finite p-group; $\lambda \in Z^2(G, S^*)$; H a subgroup of G such that $|H| > 2$ and $S^\lambda H$ be equivalent to the group ring SH. Assume that either*

(i) *$G = HT$, where T is a subgroup of the center of G, or*

(ii) *K is a finite Galois extension of F and $[K : F]$ is not divisible by p.*

Then there exists an indecomposable $S^\lambda G$-module V such that $\overline{E_{S^\lambda G}(V)}$ is isomorphic to a field that contains K.

Proof. Let $P = (\pi)$ be a maximal ideal of S, $S/P = F$, $K = F\left(\overline{\theta}\right)$, and let $f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n \in S[x]$ such that $\overline{f}(x) = \overline{a}_0 + \overline{a}_1 x + \cdots + \overline{a}_{n-1}x^{n-1} + x^n$, $\overline{a}_j = a_j + P$, be an irreducible polynomial for $\overline{\theta}$ over F. We denote by Γ the companion matrix of $f(x)$. Suppose $S^\lambda H = SH$, and either H is cyclic of order $p^d > 2$ or H is an group of type $(2,2)$. If $H = \langle a \rangle$, then on the basis of [12] (see also [13], p. 87) the mapping

$$a \to \begin{pmatrix} E & \pi E & \Gamma \\ 0 & E & \pi E \\ 0 & 0 & E \end{pmatrix} \tag{2.2}$$

is an indecomposable S-representation of this group. Assume that the representation (2.2) is realized by SH-module U. Then by [12] we have $\overline{E_{SH}(U)} \cong K$. If $H = \langle a \rangle \times \langle b \rangle$ is a group of type $(2,2)$, then as U we take an SH-module by which the following representation is realized [12]:

$$a \to \begin{pmatrix} E & E \\ 0 & E \end{pmatrix}, \quad b \to \begin{pmatrix} E & \Gamma \\ 0 & E \end{pmatrix}.$$

Now if we apply Lemma 2.2 to U, then we obtain $V = U^{S^\lambda G}$ is a required $S^\lambda G$-module. This completes the proof. $\square$

Proposition 2.3. *Let G be a finite p-group, H a subgroup of G, and $\lambda \in Z^2(G, F^*)$. If M is an irreducible $F^\lambda H$-module, then $M^{F^\lambda G}$ is an indecomposable $F^\lambda G$-module.*

Proof. The ring $E_{F^\lambda H}(M)$ is isomorphic to a field $F(\rho_1, \dots, \rho_d)$, where ρ_i is a root of the polynomial

$$x^{p^{n_i}} - \alpha_i$$

with $\alpha_1 \in F$ and $\alpha_i \in F(\rho_1, \dots, \rho_{i-1})$ for $i \geq 2$ [2]. Application of Lemma 2.2 completes the proof. $\square$

Proposition 2.4. *Let G be a finite p-group; $\lambda \in Z^2(G, F^*)$; A an abelian subgroup of G such that $F^\lambda A$ has the cyclic radical. If M is an indecomposable $F^\lambda A$-module, then $M^{F^\lambda G}$ is an indecomposable $F^\lambda G$-module. Furthermore,*

$$\overline{E_{F^\lambda G}\left(M^{F^\lambda G}\right)}$$

is isomorphic to a field $F(\rho_1, \dots, \rho_d)$, where ρ_i is a root of the polynomial $x^p - \alpha_i$ with $\alpha_1 \in F$ and $\alpha_i \in F(\rho_1, \dots, \rho_{i-1})$ for $i \geq 2$.

Proof. Assume $A = N \times H$, $H = \langle a \rangle$, $|a| = p^n$; $K = F^\lambda N$ is a field;

$$F^\lambda A = \bigoplus_{i=0}^{p^n-1} K u_a^i, \quad u_a^{p^n} = \alpha^{p^m},$$

where $\alpha \in K$, $\alpha \notin K^p$ for $0 \leq m < n$, and $\alpha = 1$ for $m = n$. Let

$$v = u_a^{p^{n-m}} - \alpha, \quad v_t = v^{p^m - t}, \quad V_t = F^\lambda A \cdot v_t.$$

Then $\operatorname{rad} F^\lambda A = F^\lambda A \cdot v$. The ideal V_t is $F^\lambda A$-isomorphic to $F^\lambda A / \left(\operatorname{rad} F^\lambda A\right)^t$.

Any indecomposable $F^\lambda A$-module is isomorphic to some V_t, where $1 \leq t \leq p^m$. The ideals $V_1, \dots, V_{p^m}$ are pairwise nonisomorphic. For every t we have $\overline{E_{F^\lambda A}(V_t)} \cong K(\theta)$, where θ is a root of the irreducible polynomial

$$x^{p^{n-m}} - \alpha \in K[x].$$

Using Lemma 2.2, the proposition is proved. $\square$

3. Indecomposable projective representations of direct products of finite groups over a field

Let S be an integral domain of characteristic p, G a finite group, $\lambda \in Z^2(G, S^*)$, $p \backslash |G|$, and $G = G_p \times B$. Then $S^\lambda G \cong S^\lambda G_p \otimes_S S^\lambda B$. If every indecomposable $S^\lambda G$-module is isomorphic to a module of the form $V\#W$, where V is an indecomposable $S^\lambda G_p$-module and W is an irreducible $S^\lambda B$-module, then we will say that the ring $S^\lambda G$ satisfies the TPIM (Tensor Product of Indecomposable Modules) condition. In this section we study such rings for the case when S is a field of characteristic p.

Lemma 3.1. *Let S be a field of characteristic $p > 0$ or a complete discrete valuation ring with residue class field of characteristic $p > 0$, $G = G_1 \times G_2$ a finite group, $\lambda_i \in Z^2(G_i, S^*)$, $\lambda = \lambda_1 \times \lambda_2$, and $p \not\mid |G_2|$. The following statements are equivalent:*

(i) *Every indecomposable $S^\lambda G$-module is the outer tensor product of an indecomposable $S^\lambda G_1$-module and an irreducible $S^\lambda G_2$-module.*

(ii) *The outer tensor product of any indecomposable $S^\lambda G_1$-module and any irreducible $S^\lambda G_2$-module is an indecomposable $S^\lambda G$-module.*

The proof is similar to that of the corresponding fact for a group ring (see [3], [11], [13]).

Lemma 3.2. *Assume that the hypotheses of Lemma 3.1 are satisfied. If V_i is an indecomposable $S^\lambda G_i$-module $(i = 1, 2)$, then*

$$\overline{E_{S^\lambda G}(V_1 \# V_2)} \cong \overline{E_{S^\lambda G_1}(V_1)} \otimes_{\bar{S}} \overline{E_{S^\lambda G_2}(V_2)},$$

where $\bar{S}$ is the residue class field of S.

The lemma follows immediately from Lemma 2.1 and Proposition 2 [3].

Lemma 3.3. *Assume that the hypotheses of Lemma 3.1 are satisfied, and T is the quotient field of S. If T is a splitting field for the algebra $T^\lambda G_2$, then every indecomposable $S^\lambda G$-module can be represented uniquely, up to isomorphism, in the form $V\#W$, where V is an indecomposable $S^\lambda G_1$-module and W is an irreducible $S^\lambda G_2$-module.*

The proof of the lemma is analogous to the one of Theorem 1 [14] (see too [13], p. 84).

Lemma 3.4. *Let F be a field of characteristic p, B a finite p'-group, and suppose that F contains a primitive qth root of 1 for every prime $q\backslash|B|$ such that $p\backslash(q-1)$. Then for any algebra $F^{\lambda}B$ there exists a splitting field K such that K is a finite Galois extension of F and $[K:F]$ is not divisible by p.*

Proof. It is well known ([5], §53) that

$$\lambda_{a,b}^{|B|} = \frac{\alpha_a \alpha_b}{\alpha_{ab}}$$

for all $a, b \in B$. Let K be a splitting field over F of the polynomial

$$\left(x^{|B|^2} - 1\right) \prod_{a \in G} \left(x^{|B|} - \alpha_a\right).$$

Then $K^{\lambda}B = K^{\mu}B$, where

$$\mu_{a,b}^{|B|} = 1$$

for all $a, b \in B$. The algebra $K^{\mu}B$ is a homomorphic image of some group algebra KH, where H is a central extension of a cyclic group of order $|B|$ by the group B. Since K is a splitting field for H ([5], §70), K is a splitting field for $F^{\lambda}B$.

Let π be an F-automorphism of K and $\pi^p = 1_K$. Denote by ε a primitive qth root of 1. If $\varepsilon \in K$, then $\pi(\varepsilon) = \varepsilon$. Therefore we may assume that F contains primitive qth root of 1 for every prime $q\backslash|B|$. In K there exists a sequence of fields

$$L_0 = F \subset L_1 \subset \cdots \subset L_s = K$$

such that L_{i+1} is obtained by adjoining a root of the polynomial $x^q - \alpha_i$ to L_i, where $\alpha_i \in L_i$ and q is a prime divisor of $|B|$. Moving along this sequence from left to right, we find that π is the identity automorphism of K. Then this implies that $[K:F]$ is not divisible by p, and the proof is complete. $\square$

Lemma 3.5. *Let F be a field of characteristic p, $G = G_p \times B$, H a noncyclic subgroup of G_p, $\lambda \in Z^2(G, F^*)$ and the restriction of λ to $H \times H$ be a coboundary. Suppose that either*

(i) *$G_p = H \cdot T$, where T is a subgroup of the center of G, or*

(ii) *F contains a primitive qth root of 1 for every prime $q \backslash |B|$ such that $p \backslash (q-1)$.*

Then the algebra $F^\lambda G$ satisfies the TPIM condition if and only if F is a splitting field for the subalgebra $F^\lambda B$.

Proof. The sufficiency of the lemma follows from Lemma 3.3. Let us prove the necessity. Assume F is not a splitting field for $F^\lambda B$. There is an irreducible $F^\lambda B$-module W with $D = E_{F^\lambda B}(W)$ being a division F-algebra of dimension greater than one. In view of Lemma 3.4 we may find a splitting field K for $F^\lambda B$, which is a finite Galois extension of F and satisfies $[K : F] \not\equiv 0 \pmod{p}$. On the basis of Proposition 2.1 there exists an indecomposable $F^\lambda G_p$-module V, for which $\overline{E_{F^\lambda G_p}(V)}$ is isomorphic to a field $L \supset K$. By Lemma 3.2

$$\overline{E_{F^\lambda G}(V\#W)} \cong L \otimes_F D.$$

Since $L \otimes_F D$ is not a skewfield, by Lemma 2.1 $V\#W$ is a decomposable $F^\lambda G$-module, and the lemma is proved. □

Theorem 3.1. *Let F be a field of characteristic p, $G = G_p \times B$, and $\lambda \in Z^2(G, F^*)$. If G_p is abelian, then $F^\lambda G$ satisfies the TPIM condition if and only if either* $\operatorname{rad} F^\lambda G_p$ *is cyclic or F is a splitting field for $F^\lambda B$.*

Proof. Suppose that the radical of $F^\lambda G_p$ is cyclic. If V is an indecomposable $F^\lambda G_p$-module, then by Proposition 2.4 $\overline{E_{F^\lambda G_p}(V)}$ is isomorphic to a field K which is a purely inseparable extension of the field F. Let W be an irreducible $F^\lambda B$-module, and $D = E_{F^\lambda B}(W)$. Since the algebra $F^\lambda B$ is separable, the center of the division ring D is a separable extension of F ([5], §71). The index of D is relatively prime to $[K : F]$ (see [5], §68 and [16]). From this we obtain that $K \otimes_F D$ is a division ring. Applying Lemmas 2.1 and 3.2, we conclude that $V\#W$ is an indecomposable $F^\lambda G$-module. On the basis of Lemma 3.1 the algebra $F^\lambda G$ satisfies the TPIM condition.

Now we come to the case when the radical of $F^\lambda G_p$ is not cyclic. Let $\bar{G}_p$ be the socle of G_p. The radical of $F^\lambda \bar{G}_p$ is also not cyclic. Arguing as in the proof of Theorem 2.4 [1], we obtain $F^\lambda \bar{G}_p = F^\mu N$, where N is an elementary abelian p-group of order $|\bar{G}_p|$ and N contains a noncyclic subgroup H such that $F^\mu H$ is equal to the group algebra FH. We may assume that H is a subgroup of G_p. By Lemma 3.5 $F^\lambda G$ satisfies the TPIM condition if and only if F is a splitting field for $F^\lambda B$. Thus the proof is finished. □

Lemma 3.6. *Let S be an integral domain of characteristic $p > 0$, G a finite group, and H a p-subgroup of G'. Then the restriction of every cocycle $\lambda \in Z^2(G, S^*)$ to $H \times H$ is a coboundary.*

The lemma follows immediately from Corollary 4.10 ([15], p. 42).

Let $t_p = \sup\{0, m\}$, where m is a natural number such that for some $\gamma_1, \dots, \gamma_m \in F^*$ the algebra

$$F[x]/(x^p - \gamma_1) \otimes_F \dots \otimes_F F[x]/(x^p - \gamma_m)$$

is a field. If F is a perfect field, then $t_p = 0$. We also remark that for any natural number d there exists a field F for which $t_p = d$.

Proposition 3.1. *Let F be a field of characteristic p, $G = G_p \times B$, and $\lambda \in Z^2(G, F^*)$. If G_p/G'_p decomposes into a direct product of no less than $t_p + 2$ cyclic subgroups, then the algebra $F^\lambda G$ satisfies the TPIM condition if and only if F is a splitting field for $F^\lambda B$.*

Proof. By Lemma 3.6 $F^\lambda G'_p$ is equivalent to the group algebra FG'_p. Let $F^\lambda G'_p = FG'_p$, and $U = F^\lambda G(\operatorname{rad} FG'_p)$. Then $F^\lambda G/U \cong F^\mu H$, where $H = H_p \times B$, $H_p \cong G_p/G'_p$, and $F^\mu B \cong F^\lambda B$. It follows from the hypothesis that the radical of $F^\mu H_p$ is not cyclic. From this and Theorem 3.1 we conclude that if $F^\lambda G$ satisfies the TPIM condition, then F is a splitting field for $F^\lambda B$. This proves the necessity. The sufficiency of the proposition follows from Lemma 3.1. □

Theorem 3.2. *Let F be a field of characteristic p, $G = G_p \times B$, and $\lambda \in Z^2(G, F^*)$. Suppose that G'_p is not cyclic, and that F contains a primitive qth root of 1 for every prime $q \backslash |B|$ such that $p \backslash (q-1)$. The algebra $F^\lambda G$ satisfies the TPIM condition if and only if F is a splitting field for $F^\lambda B$.*

The theorem follows from Lemmas 3.5 and 3.6.

Proposition 3.2. *Let F be a field of characteristic p, K a perfect subfield of F, $G = G_p \times B$. If $\lambda \in Z^2(G, K^*)$, then the algebra $F^\lambda G$*

satisfies the TPIM condition if and only if either G_p is cyclic or F is a splitting field for $F^\lambda B$.

Since $F^\lambda G_p$ is equivalent to FG_p ([15], p. 68), the proposition follows from Lemma 3.5 and Theorem 3.1.

Proposition 3.3. *Let F be a field of characteristic p, and $G = G_p \times B$. Every indecomposable projective F-representation of G is equivalent to an outer tensor product of an indecomposable projective F-representation of G_p with an irreducible projective F-representation of B if and only if either G_p is cyclic or any irreducible projective F-representation of B is absolutely irreducible.*

The proposition follows immediately from Theorem 3.1 and Proposition 3.2.

4. Indecomposable projective representations of direct products of finite groups over a complete discrete valuation ring

Let S be a complete discrete valuation ring of characteristic p, T the quotient field of S, and $G = G_p \times B$.

Theorem 4.1. *Let $S = F[[x]]$, $p \neq 2$, G_p be abelian, and $\lambda = \mu \times \nu$, where $\mu \in Z^2(G_p, F^*)$ and $\nu \in Z^2(B, S^*)$. The ring $S^\lambda G$ satisfies the TPIM condition if and only if either $F^\lambda G_p$ is a field or T is a splitting field for $T^\lambda B$.*

Proof. Let $K = F^\lambda G_p$. If K is a field, then $S^\lambda G_p = K\big[[x]\big]$ is a principal ideal ring. Because of this, every indecomposable $S^\lambda G_p$-module is isomorphic to the left regular module. We may further observe that $\overline{E_{S^\lambda G_p}(S^\lambda G_p)} \cong K$ and K is a purely inseparable extension of F. Arguing as in the proof of Theorem 3.1, we obtain that $S^\lambda G$ satisfies the TPIM condition.

Suppose now that K is not a field and T is not a splitting field for $T^\lambda B$. There exists an irreducible $S^\lambda B$-module W such that a division ring $D = \overline{E_{S^\lambda B}(W)}$ is not isomorphic to F ([5], §76). By Lemma 3.4 we can find a splitting field L' for D, which is a finite Galois extension of F and satisfies $[L' : F] \not\equiv 0 \pmod p$. In view of Proposition 2.2 there exists an indecomposable $S^\lambda G_p$-module V, for which $\overline{E_{S^\lambda G_p}(V)}$ is isomorphic to a field L with $L' \subset L$. By Lemmas 3.2 and 2.1 $V\#W$ is a decomposable $S^\lambda G$-module. We now use Lemma 3.1 to complete the proof. □

Theorem 4.2. *Let $S = F[[x]]$, $p = 2$, G_2 be abelian, and $\lambda = \mu \times \nu$, where $\mu \in Z^2(G_p, F^*)$ and $\nu \in Z^2(B, S^*)$. The ring $S^\lambda G$ satisfies the TPIM condition if and only if either $\dim(F^\lambda G_2/\operatorname{rad} F^\lambda G_2) \geq \frac{|G_2|}{2}$ or T is a splitting field for $T^\lambda B$.*

Proof. The necessary part of the theorem may be proved by arguments similar to those in the proof of Theorem 4.1. Let us prove the sufficiency. On the basis of Lemma 3.3 and the proof of Theorem 4.1 we may assume $\dim(F^\lambda G_2/\operatorname{rad} F^\lambda G_2) = \frac{|G_2|}{2}$. Then there exists a factorization $G_2 = H \times A$, with $A = \langle a \rangle$ a cyclic group of order 2^n, such that $K = F^\lambda H$ is a field, and

$$F^\lambda G_2 = \bigoplus_{i=0}^{2^n-1} K u_a^i, \quad u_a^{2^n} = \gamma^2,$$

where $\gamma \in K$ and $\gamma \notin K^2$. One can consider the ring $S^\lambda G_2$ as a twisted group ring $R^\sigma A$ with $R = K\big[[x]\big]$. Let M be a finitely generated S-torsion free $S^\lambda G_2$-module. Then M is finitely generated R-torsion free $R^\sigma A$-module. The isomorphism and the indecomposability of $S^\lambda G_2$-modules are equivalent to those of $R^\sigma A$-modules.

Let ρ be a root of the irreducible polynomial

$$y^{2^{n-1}} - \gamma \in K[y],$$

and $\tilde\rho$ the matrix corresponding to the operator of the multiplication by ρ in the R-basis

$$1, \rho, \ldots, \rho^{2^{n-1}-1}$$

of the ring $R[\rho]$. Then up to equivalence the indecomposable R-representations of the ring $R^\sigma A$ are the following [1]:

$$\Delta : u_a \to \tilde\rho; \quad \Gamma_j : u_a \to \begin{pmatrix} \tilde\rho & \langle x^j \rangle \\ 0 & \tilde\rho \end{pmatrix} \quad (j = 0, 1, 2, \ldots),$$

where $\langle x^j \rangle$ is the matrix in which all columns but last one being zero, and the last column consisting of $x^j, 0, \dots, 0$.

Let L be the quotient field of R. Continuing every representation Γ_j to an L-representation of the algebra $L^\sigma A$, we obtain a representation which is equivalent to the left regular representation of $L^\sigma A$. It follows from this, Lemma 3.1, and Theorem 3.1 that the ring $S^\lambda G$ satisfies the TPIM condition. $\square$

Theorem 4.3. *Let S be a complete discrete valuation ring of characteristic p with residue class field F, $G = G_p \times B$, $|G'_p| > 2$, $\lambda \in Z^2(G, S^*)$, and F contain a primitive qth root of 1 for every prime $q \backslash |B|$ such that $p \backslash (q-1)$. The ring $S^\lambda G$ satisfies the TPIM condition if and only if T is a splitting field for $T^\lambda B$.*

The proof of the theorem is analogous to one of Theorem 4.1.

Proposition 4.1. *Let S be a complete discrete valuation ring of characteristic p, K a perfect subfield of S, and $G = G_p \times B$. If $\mu \in Z^2(G_p, K^*)$, $\nu \in Z^2(B, S^*)$ and $\lambda = \mu \times \nu$, then the ring $S^\lambda G$ satisfies the TPIM condition if and only if either $|G_p| = 2$ or T is a splitting field for $T^\lambda B$.*

The proposition follows from [12], and Lemmas 3.1–3.3.

Proposition 4.2. *Let S be a complete discrete valuation ring of characteristic p, $G = G_p \times B$, and $|G_p| > 2$. Every indecomposable projective S-representation of G is equivalent to an outer tensor product of an indecomposable projective S-representation of G_p and an irreducible projective S-representation of B if and only if any irreducible projective T-representation of B with S-factor system is absolutely irreducible.*

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