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Property (β) and orthogonal convexities in some class of Köthe sequence spaces

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Abstract. First we show that property (β) of Rolewicz ([27]) is equivalent to orthogonal uniform convexity $(UC^{\perp})$ ([15]) in symmetric Köthe sequence spaces. To discover precisely the non-symmetric case, we prove criteria for property (β), strict and uniform orthogonal convexity $\left(\left(SC^{\perp}\right), \left(UC^{\perp}\right)\right)$ in Musielak–Orlicz sequence spaces, which essentially extend results from [5], [16] and correspond to respective criteria for strict and uniform convexity ([12], [13]). As a corollary we conclude that in non-symmetric Köthe sequence spaces the property $\left(UC^{\perp}\right)$ is really stronger than three properties $\left(SC^{\perp}\right)$, (β) and uniform monotonicity considered together. The case of Nakano spaces is also discussed.

1. Introduction

Throughout this paper $(X, \|\cdot\|_X)$ is a real Banach space. As usual, $S(X)$ and $B(X)$ stand for the unit sphere and the unit ball of X, respectively. For any subset A of X, we denote by $\operatorname{conv}(A)$ the convex hull of A.

The Banach space $(X, \|\cdot\|_X)$ is *strictly convex* $(X \in (SC))$ if for every $x, y \in X$ with $x \neq y$ and $\|x\|_X = \|y\|_X = 1$ we have $\|x + y\|_X < 2$. X is said to be *uniformly convex* ($X \in (UC)$ for short), if for each $\varepsilon > 0$ there

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is $\delta > 0$ such that for any $x, y \in S(X)$ the inequality $\|x - y\|_X \geq \varepsilon$ implies $\|x + y\|_X \leq 2(1 - \delta)$ (see [3]).

Define for any $x \notin B(X)$ the *drop* $D(x, B(X))$ determined by x by $D(x, B(X)) = \mathrm{conv}(\{x\} \cup B(X))$.

Recall that for any subset C of X, the *Kuratowski measure of noncompactness* of C is the infimum $\alpha(C)$ of those $\varepsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less then ε.

Rolewicz in [26] has proved that $X \in (UC)$ iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that $1 < \|x\|_X < 1 + \delta$ implies $\mathrm{diam}(D(x, B(X)) \setminus B(X)) < \varepsilon$. In connection with this he has introduced in [27] the following property.

A Banach space X has the *property* (β) ($X \in (\beta)$) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(D(x, B(X)) \setminus B(X)) < \varepsilon$ whenever $1 < \|x\|_X < 1 + \delta$.

We say that a sequence $\{x_n\} \subset X$ is ε*-separated* for some $\varepsilon > 0$ if

$$\mathrm{sep}\{x_n\} = \inf\{\|x_n - x_m\|_X : n \neq m\} > \varepsilon.$$

Although the primary definition of property (β) uses the *Kuratowski measure of noncompactness*, more convenient in our considerations is the following equivalent condition proved by Kutzarova in [21].

Theorem 1. *A Banach space X has property (β) if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\mathrm{sep}\{x_n\} \geq \varepsilon$ there is an index k for which*

$$\|x + x_k\|_X \leq 2(1 - \delta).$$

A Banach space is *nearly uniformly convex* ($X \in (NUC)$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $\{x_n\}$ in $B(X)$ with $\mathrm{sep}\{x_n\} > \varepsilon$, we have $\mathrm{conv}(\{x_n\}) \cap (1 - \delta)B(X) \neq \phi$. Rolewicz proved the following implications $(UC) \Rightarrow (\beta) \Rightarrow (NUC)$ (see [27]). Moreover, the class of Banach spaces with an equivalent norm with property (β) coincides neither with that of superreflexive spaces (see [20] and [23]) nor with the class of nearly uniformly convexifiable spaces (see [19]).

Denote by $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{R}_+$ the sets of natural, real and non-negative real numbers, respectively. Let $(\mathcal{N}, 2^{\mathcal{N}}, m)$ be the counting measure space and $l_0 = l_0(m)$ the linear space of all real sequences.

Let $E = (E, \leq, \|\cdot\|_E)$ be a Banach sequence lattice over the measure space $(\mathcal{N}, 2^{\mathcal{N}}, m)$, that is E is a Banach space being a subspace of l_0 endowed with the natural coordinatewise semi-order relation, and E satisfies the conditions:

(i) if $x \in E, y \in l_0, |y| \leq |x|$, i.e. $|y(i)| \leq |x(i)|$ for every $i \in \mathcal{N}$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$,

(ii) there exists a sequence x in E that is positive on the whole $\mathcal{N}$ (see [14] and [22]).

Banach sequence lattices are often called *Köthe sequence spaces*. More generally, if we consider a Banach function lattice E over the measure space (T, Σ, μ) with μ being σ-finite and complete, then we will say that E is a *Köthe space*. If we assume additionally that μ is non-atomic, then E will be called the *Köthe function space*.

A Köthe space E is said to be *strictly monotone* ($E \in (SM)$) if for every $0 \leq y \leq x$ with $y \neq x$ we have $\|y\|_E < \|x\|_E$. We say that a Köthe space E is *uniformly monotone* ($E \in (UM)$) if for every $q \in (0,1)$ there exists $p \in (0,1)$ such that for all $0 \leq y \leq x$ satisfying $\|x\|_E = 1$ and $\|y\|_E \geq q$ we have $\|x - y\|_E \leq 1 - p$ (see [1], [9]).

A Köthe space E is called *order continuous* ($E \in (OC)$) if for every $x \in E$ and each sequence $(x_m) \in E$ such that $0 \leftarrow x_m \leq |x|$ we have $\|x_m\|_E \to 0$ (see [14] and [22]).

The notation $r \wedge s = \min\{r, s\}$, $r \vee s = \max\{r, s\}$ for any $r, s \in \mathcal{R}$ and $A \div B = (A \setminus B) \cup (B \setminus A)$ for every $A, B \in \Sigma$ will be used.

It appears that a geometric property called *orthogonal uniform convexity* is strictly connected with property (β), uniform convexity and uniform monotonicity. It was introduced in [15].

Definition 1. A Köthe space $(E, \|\cdot\|_E)$ is orthogonally uniformly convex ($E \in (UC^{\perp})$), if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in B(E)$ the inequality $\|x\chi_{A_{xy}}\|_E \vee \|y\chi_{A_{xy}}\|_E \geq \varepsilon$ implies $\|x + y\|_E \leq 2(1 - \delta)$, where $A_{xy} = \operatorname{supp} x \div \operatorname{supp} y$.

Obviously, if $E \in (UC)$, then $E \in \left(UC^{\perp}\right)$. It is known that uniformly convex Köthe space is uniformly monotone (see [9]). Moreover,

Lemma 1 (Lemma 3 in [15]). *Let E be a Köthe space. If $E \in \left(UC^{\perp}\right)$, then $E \in (UM)$ and the converse is not true.*

In the sequel we will need the following notion.

Definition 2. A Köthe space $(E, \|\cdot\|_E)$ is orthogonally strictly convex $\left(E \in \left(SC^{\perp}\right)\right)$, if for any $x, y \in B(E)$ the inequality $\|x\chi_{A_{xy}}\|_E \vee \|y\chi_{A_{xy}}\|_E > 0$ implies $\|(x+y)/2\|_E < 1$, where $A_{xy} = \operatorname{supp} x \div \operatorname{supp} y$.

It is clear that if $E \in (SC)$, then $E \in \left(SC^{\perp}\right)$. On the other hand, every strictly convex Köthe space is strictly monotone (see [9]). Furthermore,

Lemma 2. *Let E be a Köthe space. If $E \in \left(SC^{\perp}\right)$, then $E \in (SM)$.*

The proof is analogous as that of Lemma 1, but we apply Theorem 8 from [9] instead of Theorem 6(iii) from [9].

It is known that in the case of Köthe sequence spaces the implications

$$(UC) \Rightarrow \left(UC^{\perp}\right) \Rightarrow (\beta) \tag{1}$$

hold and the converse of any of them is not true in general (see [16]). Furthermore, the implications

$$(UC) \Rightarrow (\beta) \Rightarrow \left(UC^{\perp}\right) \tag{2}$$

are true in every Köthe function space and the second can not be reversed (see [15], [16] and [27]). Notice that property (β) and orthogonal uniform convexity change places in implications (1) and (2).

It is known that the second of implications (1) can be reversed in Orlicz sequence spaces (see [16]). We extend that result to the case of symmetric Köthe sequence spaces (Theorem 3). On the other hand, property (β) does not even imply strict monotonicity in a non-symmetric Köthe sequence space (Example 1). Note that we have three natural geometric properties weaker than orthogonal uniform convexity in Köthe sequence spaces, namely: property (β), uniform monotonicity and orthogonal strict convexity. Then, it is natural to ask the following question

Question: Does uniformly monotone, orthogonally strictly convex non-symmetric Köthe sequence space with property (β) need to be orthogonally uniformly convex?

We prove that the answer is negative in general by finding criteria for property (β), orthogonal strict convexity and orthogonal uniform convexity in Musielak–Orlicz sequence spaces. However, applying in particular these results for Nakano spaces we will see that the answer may be positive.

We say that properties (P_1) and (P_2) are not comparable, if the sets of spaces having properties (P_1) and (P_2) are not included one into another. Notice that property (β) and uniform monotonicity are not comparable in general (it is enough to take l^1 and the space Y from Example 1). However, it follows from the results given in [11], [27], [28] that property (β) implies uniform monotonicity in symmetric Köthe sequence spaces (see also Corollary 1). Furthermore, from our results we conclude that each two of three properties (β), (UM), $\left(SC^{\perp}\right)$ are not comparable in Musielak–Orlicz sequence spaces.

It is worth mentioning that *orthogonal uniform convexity* plays an important role in studying property (β) in Köthe–Bochner spaces (see [10] and [17]).

2. Results

2.1. Köthe sequence spaces

Theorem 2 (Theorem 2 in [16]). *Every orthogonally uniformly convex Köthe sequence space has property (β).*

Remark 1. Note that if $\dim E < \infty$, then $E \in (\beta)$. Hence the thesis of Theorem 2 is non-trivial for infinite dimensional Köthe sequence spaces. Moreover, the converse of Theorem 2 is not true as the following example shows.

Example 1. The following example is due to Day (see [6]). For $i = 1, 2, \ldots$, let X_i denote R_i with the norm $\|(x_1, x_2, \ldots, x_i)\|_i = \sup_{1\leq j\leq i} |x_j|$. Then define

$$Y = \left\{ y = (y_i) : y_i \in X_i \text{ for every } i = 1, 2, \ldots \text{ and } \sum_{i=1}^{\infty} \|y_i\|_i^2 < \infty \right\}$$

equipped with the norm $\|y\| = \left(\sum_{i=1}^{\infty} \|y_i\|_i^2\right)^{1/2}$. By Proposition 1 from [20] we conclude that $Y \in (\beta)$. However, $Y \notin (SM)$, whence $Y \notin \left(UC^{\perp}\right)$, by Lemma 1.

It follows from Theorem 2 and Example 1 that property $\big(UC^{\perp}\big)$ is essentially stronger than property (β) in Köthe sequence spaces. Notice that the space Y in Example 1 is not symmetric. However, orthogonal uniform convexity and property (β) coincide in Orlicz sequence spaces (see [16]). It appears that this equivalence can be extended to the case of symmetric Köthe sequence spaces.

Recall that E is a *symmetric Köthe sequence space* if for any permutation (n_k) of the set $\mathcal{N}$ we have $\widetilde{x} = \{x(n_k)\}_{k=1}^{\infty} \in E$ and $\|x\|_E = \|\widetilde{x}\|_E$.

Theorem 3. *Let E be a symmetric Köthe sequence space. Then E is orthogonally uniformly convex if and only if E has property (β).*

Proof. The necessity is clear by Theorem 2. Suppose that $E \in (\beta)$. Let $\varepsilon > 0$. Take $x, y \in B(E)$ such that $\|x\chi_A\|_E \vee \|y\chi_A\|_E \geq \varepsilon$, where $A = \operatorname{supp} x \div \operatorname{supp} y$. Applying Theorem 1 take $\delta_1 = \delta(\varepsilon/2)$. Since $(\beta) \Rightarrow (NUC)$ (see [27]), $(NUC) \Rightarrow (KK)$ (see [11], also for the definition of the Kadec–Klee property (KK)) and $(KK) \Rightarrow (OC)$ in any Banach function lattice (see [8]), we get $(\beta) \Rightarrow (OC)$. It is known that a Köthe sequence space is order continuous iff it is absolutely continuous, i.e. for every $x \in E$ we have $\lim_{n\to\infty} \|x - x^{(n)}\|_E = 0$, where $x^{(n)} = (x(1), x(2), \ldots, x(n), 0, 0, \ldots)$ (see [4]). Consequently there exist subsets B_1, B_2 of $\mathcal{N}$ with $\operatorname{card} B_i < \infty$ for $i = 1, 2$ such that

$$\|x\chi_{\mathcal{N}\setminus B_1}\|_E \vee \|y\chi_{\mathcal{N}\setminus B_2}\|_E < \varepsilon/2 \wedge \delta_1/2. \tag{3}$$

Define $x_1 = x\chi_{B_1}$ and $y_1 = y\chi_{B_2}$. Let $A_1 = \operatorname{supp} x_1 \div \operatorname{supp} y_1$. We claim that

$$\|x_1\chi_{A_1}\|_E \vee \|y_1\chi_{A_1}\|_E \geq \varepsilon/2. \tag{4}$$

Indeed, if $\|x\chi_A\|_E \geq \varepsilon$, then, by inequality (3), we get

$$\begin{aligned}\varepsilon &\leq \|x\chi_A\|_E \leq \|x\chi_{A\cap B_1}\|_E + \|x\chi_{A\setminus B_1}\|_E \\ &\leq \|x\chi_{A\cap B_1}\|_E + \varepsilon/2 \leq \|x_1\chi_{A_1}\|_E + \varepsilon/2.\end{aligned}$$

If $\|y\chi_A\|_E \geq \varepsilon$, then, analogously, we get $\|y_1\chi_{A_1}\|_E \geq \varepsilon/2$. Thus the claim is proved. Without loss of generality we assume that $\|x_1\chi_{A_1}\|_E \geq \varepsilon/2$, since otherwise the proof is analogous. Then

$$\|x_1\chi_{A_{11}}\|_E \geq \varepsilon/2, \tag{5}$$

where $A_{11} = \operatorname{supp} x_1 \backslash \operatorname{supp} y_1$. Denote $N_1 = \mathcal{N} \backslash (\operatorname{supp} x_1 \cup \operatorname{supp} y_1)$. Then $\operatorname{card} N_1 = \infty$. Hence there exists a sequence $(E_n)_{n=1}^{\infty}$ in N_1 of pairwise disjoint sets such that $\operatorname{card} E_n = \operatorname{card} A_{11}$ for every $n \in \mathcal{N}$. Denote

$$A_{11} = \{i_1, i_2, \ldots, i_l\} \text{ and } E_n = \{j_1^n, j_2^n, \ldots, j_l^n\} \text{ for every } n \in \mathcal{N},$$

where $l = \operatorname{card} A_{11}$. For every $n \in \mathcal{N}$ define

$$u_n(i) = \begin{cases} x_1(i_k) & \text{if } i \in E_n,\ i = j_k^n,\ k = 1, 2, \ldots, l \\ 0 & \text{if } i \notin E_n \end{cases} \quad \text{and}$$

$$z_n = x_1 \chi_{B_1 \backslash A_{11}} + u_n.$$

Since E is symmetric, so $\|z_n\|_E = \|x_1\|_E$ for each $n \in \mathcal{N}$. Then $z_n \in B(E)$ for every $n \in \mathcal{N}$. Similarly, $\|u_n\|_E \geq \varepsilon/2$ for every $n \in \mathcal{N}$, by (5). Hence $\operatorname{sep}\{z_n\}_E \geq \varepsilon/2$. Thus, by Theorem 1, it follows that $\|y_1 + z_{n_0}\|_E \leq 2(1 - \delta_1)$ for some $n_0 \in \mathcal{N}$. Applying again the symmetry of E we note $\|x_1 + y_1\|_E = \|y_1 + z_n\|_E$ for any $n \in \mathcal{N}$. Thus, by inequality (3), we get

$$\|x + y\|_E \leq \|x_1 + y_1\|_E + \delta_1 = \|y_1 + z_{n_0}\|_E + \delta_1 \leq 2 - \delta_1. \qquad \square$$

Remark 2. Note that Theorem 3 gives a very useful tool to check whether a symmetric Köthe sequence space has property (β). Indeed, it is much easier to check if a Köthe sequence space is orthogonally uniformly convex than if it has property (β). It is enough to see Definition 1 and Theorem 1. We realize also this fact if we compare the proofs of Theorem 1 from [5] (on property (β)) and Theorem 3 from [16] (on property $\left(UC^{\perp}\right)$) concerning Orlicz sequence spaces (the second proof is much simpler and shorter).

As an immediate consequence of Theorem 3 and Lemma 1 we get the following corollary (see also Example 1 above to compare the symmetric and non-symmetric case).

Corollary 1. *Let E be a symmetric Köthe sequence space. If E has property (β), then E is uniformly monotone.*

Looking at Example 1, Theorems 2 and 3 we may point out a natural question concerning non-symmetric Köthe sequence spaces. Namely, what property must be added to property (β) to assure orthogonal uniform convexity. We consider three natural weaker notions than $\left(UC^{\perp}\right)$ such that property (β), (UM) and $\left(SC^{\perp}\right)$ in Musielak–Orlicz sequence spaces, which are obviously non-symmetric. We prove that each pair of three properties $\left((\beta),(UM) \text{ and } \left(SC^{\perp}\right)\right)$ are not comparable and even these three properties together do not imply orthogonal uniform convexity.

2.2. Musielak–Orlicz sequence spaces

A function Φ is called an *Orlicz function*, if $\Phi : \mathcal{R} \longrightarrow [0,\infty)$ is convex, even, $\Phi(0) = 0$ and Φ is not identically equal to zero. A sequence $\varphi = (\varphi_i)_{i=1}^{\infty}$ of Orlicz functions φ_i is called a *Musielak–Orlicz function.* We will write $\varphi > 0$ if $\varphi_i(u) = 0$ iff $u = 0$ for every $i \in \mathcal{N}$. Define on l_0 a convex modular I_φ by

$$I_\varphi(x) = \sum_{i=1}^{\infty} \varphi_i(x(i))$$

for every $x = (x(i))_{i=1}^{\infty} \in l_0$. By the *Musielak–Orlicz space* l_φ we mean

$$l_\varphi = \{x \in l_0 : I_\varphi(cx) < \infty \text{ for some } c > 0\}.$$

We endow this space with the *Luxemburg norm*

$$\|x\|_\varphi = \inf\{\varepsilon > 0 : I_\varphi(x/\varepsilon) \le 1\}.$$

For every Musielak–Orlicz function φ we will denote by φ^* the sequence $(\varphi_i^*)_{i=1}^{\infty}$ of functions $\varphi_i^* : \mathcal{R} \longrightarrow [0,\infty)$ that are *complementary to* φ_i *in the sense of Young,* i.e. $\varphi_i^*(v) = \sup_{u\ge 0}\{u|v| - \varphi_i(u)\}$ for every $v \in \mathcal{R}$ and $i \in \mathcal{N}$.

We say that a Musielak–Orlicz function φ satisfies δ_2-*condition* ($\varphi \in \delta_2$) if there are constants k_0, $a_0 > 0$ and a sequence $(c_i^0)_{i=1}^{\infty}$ of positive reals with $\sum_{i=1}^{\infty} c_i^0 < \infty$ such that $\varphi_i(2u) \le k_0\varphi_i(u) + c_i^0$ for each $i \in \mathcal{N}$ and $u \in \mathcal{R}$ satisfying $\varphi_i(u) \le a_0$.

Moreover, we can assume without loss of generality that $\varphi_i(1) = 1$ for all $i \in \mathcal{N}$. Otherwise we define a new Musielak–Orlicz function $\psi = (\psi_i)_{i=1}^{\infty}$ by the formula

$$\psi_i(u) = \begin{cases} \varphi_i(b_i u) & \text{for } |u| \le 1 \\ u^2 & \text{for } |u| > 1, \end{cases}$$

where $\varphi_i(b_i) = 1$. The spaces l_φ and l_ψ are isometric (see [13]). Under this assumption we should remember that every function φ_i ($i \in \mathcal{N}$) is non decreasing on $\mathcal{R}_+$ and it is convex on the interval $[0, 1]$, but not necessarily convex on the whole $\mathcal{R}_+$. Then this modified function φ satisfies δ_2-condition iff there are constant $k > 0$ and a nonnegative sequence $(c_i)_{i=1}^\infty$ such that

$$\sum_{i=1}^\infty \varphi_i(c_i) < \infty \quad \text{and} \quad \varphi_i(2u) \le k\varphi_i(u) \tag{6}$$

for each $i \in \mathcal{N}$ and $u \in [c_i, 1]$ (see [7] and [13]).

We say that φ satisfies condition $(*)$ ($\varphi \in (*)$) if for every $\varepsilon \in (0, 1)$ there exists $\delta > 0$ such that $\varphi_i(u) < 1 - \varepsilon$ implies $\varphi_i((1 + \delta)u) \le 1$ for all $u \in \mathcal{R}$ and $i \in \mathcal{N}$ (see [13]).

Lemma 3. *Let φ satisfy δ_2-condition. The following assertions are true:*

([12]) (*a*) $\|x\|_\varphi = 1$ *if and only if* $I_\varphi(x) = 1$.

(*Lemma 9 in* [13]) (*b*) *The function φ satisfies condition $(*)$ if and only if for every $p \in (0, 1)$ there exists $q \in (0, 1)$ such that the inequality $I_\varphi(x) \le 1 - p$ implies $\|x\|_\varphi \le 1 - q$.*

Combining some ideas from Lemma 3 in [7] it is not difficult to show the following two lemmas.

Lemma 4. *Assume that φ^* satisfy δ_2-condition. Then there exists a number $\gamma \in (0, 1)$ and a sequence $\beta = (\beta_i)$ with $I_\varphi(\beta) < \infty$ such that the inequality*

$$\varphi_i(u/2) \le (1 - \gamma)\varphi_i(u)/2 \tag{7}$$

is true for every $i \in \mathcal{N}$ and $u \in [\beta_i, 1]$.

Lemma 5. *Let N be a subset of $\mathcal{N}$. Assume that φ^* satisfy δ_2-condition and φ_i is linear in no neighbourhood of zero for every $i \in N$. Then for every $\varepsilon > 0$ there exists a number $\gamma = \gamma(\varepsilon) \in (0, 1)$ and a sequence $\beta = (\beta_i)$ such that*

$$\sum_{i \in N} \varphi_i(\beta_i) < \varepsilon \quad \text{and} \quad \varphi_i(u/2) \le (1 - \gamma)\varphi_i(u)/2 \tag{8}$$

for each $i \in N$ and $u \in [\beta_i, 1]$.

Lemma 6 (Theorem 0.1 in [13]). *The following conditions are equivalent:*

(i) $\|x_n\|_\varphi \to 0$ *if and only if* $I_\varphi(x_n) \to 0$

(ii) φ *satisfies* δ_2*-condition and* $\varphi > 0$.

Now we find criteria for property (β) in Musielak–Orlicz sequence spaces. But first we need to prove the following modification of Lemma 6.

Lemma 7. *The following statements are equivalent:*

(i) $\|x_n\|_\varphi \to 0$ *if and only if* $I_\varphi(x_n) \to 0$ *for every sequence* (x_n) *in* l_φ *with elements* x_n *having pairwise disjoint supports.*

(ii) $\varphi \in \delta_2$.

Proof. (i) $\Longrightarrow$ (ii): If $\varphi \notin \delta_2$, the sequence (x_n) constructed in the proof of Lemma 6 has elements with pairwise disjoint supports such that $\|x_n\|_\varphi = 1$ and $I_\varphi(x_n) \to 0$.

(ii) $\Longrightarrow$ (i): Take a sequence (x_n) in l_φ with elements x_n having pairwise disjoint supports such that $I_\varphi(x_n) \to 0$. We need to prove that $I_\varphi(2x_n) \to 0$ (see [24]). Denoting $\varphi \circ x_n = (\varphi_i(x_n(i)))_{i=1}^\infty$ we have $\|\varphi \circ x_n\|_{l^1} = I_\varphi(x_n) \to 0$. Then there exist $y \in l^1_+$ and a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $|\varphi \circ x_{n_k}| \le y$ (see Lemma 2 in [14], p. 138). Since $\varphi \in \delta_2$, there are constants $k_0, a_0 > 0$ and a sequence $c^0 = (c_i^0)_{i=1}^\infty \in l^1$ of positive reals such that $\varphi_i(2u) \le k_0\varphi_i(u) + c_i^0$ for each $i \in \mathcal{N}$ and $u \in \mathcal{R}$ satisfying $\varphi_i(u) \le a_0$. Moreover, we can find a number $k_1 \in \mathcal{N}$ such that $I_\varphi(x_{n_k}) < a_0$ for every $k \ge k_1$, by $I_\varphi(x_n) \to 0$. Then $|\varphi \circ (2x_{n_k})| \le k_0 y + c^0 \in l^1_+$ for every $k \ge k_1$. Furthermore, functions $\varphi \circ (2x_{n_k})$ for $k = 1, 2, \dots,$ have pairwise disjoint supports, so $\varphi \circ (2x_{n_k}) \to 0$ pointwisely as $k \to \infty$. Since $l^1 \in (OC)$, so $\|\varphi \circ 2x_{n_k}\|_{l^1} = I_\varphi(2x_{n_k}) \to 0$ as $k \to \infty$. In virtue of the double extract subsequence theorem, we get $I_\varphi(2x_n) \to 0$ as $n \to \infty$. $\square$

From Lemma 7 we conclude immediately

Corollary 2. *If* $\varphi \in \delta_2$, *then for every* $\varepsilon > 0$ *there exists* $\sigma = \sigma(\varepsilon) > 0$ *such that for every sequence* (x_n) *in* l_φ *with elements* x_n *having pairwise disjoint supports and satisfying* $\|x_n\|_\varphi \ge \varepsilon$ *for every* $n \in \mathcal{N}$ *the inequality* $I_\varphi(x_n) \ge \sigma$ *holds for almost every* $n \in \mathcal{N}$.

Theorem 4. *Suppose that φ is a Musielak–Orlicz function satisfying the condition $(*)$. Then the following statement are equivalent:*

(i) *l_φ has property (β).*

(ii) *l_φ is nearly uniformly convex.*

(iii) *functions φ and φ^* satisfy δ_2-condition, i.e. l_φ is reflexive.*

Proof. The implication (i) $\implies$ (ii) was proved by Rolewicz in [27]. Moreover, every nearly uniformly convex Banach space is reflexive (see [11]), so (ii) $\implies$ (iii).

(iii) $\implies$ (i). Let $\varepsilon > 0$. Take $x, x_n \in B(l_\varphi)$, $n = 1, 2, \dots$, such that $\operatorname{sep}\{x_n\} \geq \varepsilon$. Take numbers $\sigma = \sigma(\varepsilon/8)$ and $\gamma \in (0, 1)$ from Corollary 2 and Lemma 4, respectively. Let $q = q(\gamma\sigma/4)$ be from Lemma 3b. Notice that under our assumptions $l_\varphi \in (OC)$. Then, analogously as in the proof of Theorem 3, we conclude that there exists set $A \subset \mathcal{N}$ with $\operatorname{card} A < \infty$ such that

$$\|x\chi_{\mathcal{N}\setminus A}\|_\varphi < q. \tag{9}$$

Since $x_n \in B(l_\varphi)$, $n = 1, 2, \dots$, so for each i the sequence $(x_n(i))_{n=1}^\infty$ in $\mathcal{R}$ has a convergent subsequence. Then $(x_n\chi_A)_{n=1}^\infty$ has a norm convergent subsequence in l_φ, by $\operatorname{card} A < \infty$. Thus, passing to a subsequence, if necessary, we may assume that $\operatorname{sep}\{x_n\chi_{\mathcal{N}\setminus A}\}_{n=N_1}^\infty \geq \varepsilon/2$ for some N_1, by $\operatorname{sep}\{x_n\} \geq \varepsilon$. Hence, by the triangle inequality, $\|x_{n_1}\chi_{\mathcal{N}\setminus A}\|_\varphi \geq \varepsilon/4$ for some $n_1 \geq N_1$. Moreover, by $l_\varphi \in (OC)$, there exists set $A_1 \subset \mathcal{N}\setminus A$ with $\operatorname{card} A_1 < \infty$ such that $\|x_{n_1}\chi_{A_1}\|_\varphi \geq \varepsilon/8$. Similarly as above, there exists $n_2 \in \mathcal{N}$ such that $\|x_{n_2}\chi_{\mathcal{N}\setminus(A\cup A_1)}\|_\varphi \geq \varepsilon/4$. Furthermore, there exists set $A_2 \subset \mathcal{N}\setminus(A \cup A_1)$ with $\operatorname{card} A_2 < \infty$ such that $\|x_{n_2}\chi_{A_2}\|_\varphi \geq \varepsilon/8$. Proceeding in such a way infinitely many times and denoting $y_k = x_{n_k}\chi_{A_k}$, $k = 1, 2, \dots$, we get that the sequence $(y_k)_{k=1}^\infty$ has elements with pairwise disjoint supports and $\|y_k\|_\varphi \geq \varepsilon/8$ for every $k \in \mathcal{N}$. Applying Corollary 2 we conclude that

$$I_\varphi(x_{n_k}\chi_{A_k}) = I_\varphi(y_k) \geq \sigma \tag{10}$$

for almost every $k \in \mathcal{N}$. Without loss of generality we may assume that the inequality (10) is true for each $k \in \mathcal{N}$. Put x_k in place of x_{n_k}, for simplicity. Denote

$$A_k^1 = \{i \in A_k : |x_k(i)| \geq \beta_i\} \quad \text{and} \quad A_k^2 = \{i \in A_k : |x_k(i)| < \beta_i\},$$

where (β_i) is from Lemma 4. We claim that $I_\varphi(x_{k_0}\chi_{A^1_{k_0}}) \geq \sigma/2$ for some $k_0 \in \mathcal{N}$. Suppose conversely that

$$I_\varphi(x_k\chi_{A^1_k}) < \sigma/2 \quad \text{for every } k \in \mathcal{N}. \tag{11}$$

We have $I_\varphi(x_k\chi_{A^2_k}) \leq \sum_{i\in A^2_k} \varphi_i(\beta_i) \to 0$ as $k\to\infty$, because $\sum_{i=1}^\infty \varphi_i(\beta_i)<\infty$ and $A_k \cap A_l = \emptyset$ for any $k \neq l$. Then $I_\varphi(x_k\chi_{A^2_k}) < \sigma/2$ for sufficiently large k. Then, in view of (10) and (11), we get a contradiction, which proves the claim. Note that $A_k \cap A = \emptyset$ for any k. Consequently, by Lemma 4,

$$I_\varphi((x\chi_A + x_{k_0})/2) \leq \frac{1}{2}(I_\varphi(x\chi_A) + I_\varphi(x_{k_0})) - \frac{\gamma}{2} I_\varphi\big(x_{k_0}\chi_{A^1_{k_0}}\big) \leq 1 - \frac{\gamma\sigma}{4}.$$

Thus Lemma 3b yields $\|(x\chi_A + x_{k_0})/2\|_\varphi \leq 1 - q$. Finally, by (9), we get $\|(x + x_{k_0})/2\|_\varphi \leq 1 - q/2$. □

Remark 3. Let Φ be an Orlicz function and l_Φ the Orlicz sequence space. Obviously, l_Φ is symmetric. Recall that $\Phi \in \delta_2$ if there are $u_0 > 0$ with $\Phi(u_0) > 0$ and $k > 2$ such that $\Phi(2u) \leq k\Phi(u)$ for every $|u| \leq u_0$. Applying Theorem 4 for Orlicz sequence space we get that $l_\Phi \in (\beta)$ iff $\Phi \in \delta_2$ and $\Phi^* \in \delta_2$ (notice that every Orlicz function satisfies the condition $(*)$). That fact was proved directly in [5] and it was extended in [16]). Note that if $\Phi \in \delta_2$, then $\Phi > 0$. However, for a Musielak–Orlicz function $\varphi = (\varphi_i)_{i=1}^\infty$, the condition $\varphi \in \delta_2$ not guarantee that $\varphi > 0$. Let

$$\varphi_i(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 2^{-i} \\ u^2 - 2^{-2i} & \text{if } u > 2^{-i}. \end{cases}$$

Then a simple computation gets $\varphi_i(2u) \leq 3 \cdot 2^{-2i}$ for every $0 \leq u \leq 2^{-i}$ and $i \in \mathcal{N}$. Moreover, $\varphi_i(2u) \leq 4\varphi_i(u) + 3 \cdot 2^{-2i}$ for every $u > 2^{-i}$ and $i \in \mathcal{N}$. Then $\varphi_i(2u) \leq 4\varphi_i(u) + 3 \cdot 2^{-2i}$ for each $u \geq 0$ and $i \in \mathcal{N}$. Since $3\sum_{i=1}^\infty 2^{-2i} = 3/2$, so $\varphi \in \delta_2$. Furthermore,

$$\varphi_i^*(v) = \begin{cases} 2^{-i}v & \text{if } 0 \leq v \leq 2^{-i+1} \\ \frac{1}{4}v^2 + 2^{-2i} & \text{if } v > 2^{-i+1}. \end{cases}$$

It is easy to calculate that $\varphi_i^*(2v) < 17\varphi_i^*(v)$ for every $v \geq 0$ and $i \in \mathcal{N}$. Hence $\varphi^* \in \delta_2$. One can check also that $\varphi \in (*)$. Then l_φ has property

(β), by Theorem 4. On the other hand, $l_\varphi \notin (SM)$, because φ_i vanishes outside zero for every $i \in \mathcal{N}$ (Lemma 2.1 in [18]). Hence $l_\varphi \notin \big(UC^\perp\big)$.

We thank Professor Henryk Hudzik for the above example, which was an inspiration for these investigations of Musielak–Orlicz spaces.

Now we find criteria for strict and uniform orthogonal convexity in Musielak–Orlicz sequence spaces. But first we must recall some necessary terminology.

We say that an Orlicz function Φ is strictly convex on an interval $[a,b]$ if $\Phi(\frac{u+v}{2}) < (\Phi(u)+\Phi(v))/2$ for all $u,v \in [a,b]$, $u \neq v$.

Given a Musielak–Orlicz function $\varphi = (\varphi_i)_{i=1}^\infty$ we define the function $h_i : \mathcal{R} \times \mathcal{R} \to [0,\infty)$ by

$$h_i(u,v) = \begin{cases} \dfrac{2\varphi_i(\frac{u+v}{2})}{\varphi_i(u)+\varphi_i(v)} & \text{if } \varphi_i(u) \vee \varphi_i(v) > 0, \\ 0 & \text{if } \varphi_i(u) \vee \varphi_i(v) = 0 \end{cases}$$

for every $i \in \mathcal{N}$.

Let $c > 0$. It is said that a Musielak–Orlicz function $\varphi = (\varphi_i)_{i=1}^\infty$ is uniformly convex in the c-neighbourhood of zero if for every $a \in [0,1)$ there exist $\delta \in (0,1)$ and a nonnegative sequence $d = (d_i)$ with $I_\varphi(d) < \infty$ and $\varphi_i(d_i) \le c$ for every $i \in \mathcal{N}$ such that

$$h_i(u, au) \le 1 - \delta$$

for all $u \in (d_i,\, \varphi_i^{-1}(c)]$, $i \in \mathcal{N}$. Let N be a subset of $\mathcal{N}$. We say that a family $(\varphi_i)_{i \in N}$ is uniformly convex in the c-neighbourhood of zero if the function $\psi = (\psi_i)_{i=1}^\infty$ has this property, where $\psi_i = \varphi_i$ for $i \in N$ and $\psi_i = 0$ for $i \notin N$.

Lemma 8 (Lemma 5 in [13]). *If the function φ is uniformly convex in the c-neighbourhood of zero and each φ_i is strictly convex on the interval $[0, \varphi_i^{-1}(c)]$, respectively, then for each $\varepsilon \in (0,1)$ there exists $p \in (0,1)$ and a nonnegative sequence (d_i) with $\sum_{i=1}^\infty \varphi_i(d_i) < \varepsilon$ such that*

$$h_i(u,v) \le 1 - p$$

if $|u-v| \ge \varepsilon(u \vee v)$ and $u \vee v \in (d_i, \varphi_i^{-1}(c)]$, $i \in \mathcal{N}$.

In the sequel we use the symbol $e_n = (\underbrace{0,\dots,0}_{n-1}, 1, 0, \dots)$.

Theorem 5. *Let φ be a Musielak–Orlicz function. Then l_φ is orthogonally strictly convex if and only if:*

(i) *each function φ_i vanishes only at zero and the function φ fulfills the δ_2-condition.*

(ii) *φ_i can be linear in a neighborhood of zero for at most one $i \in \mathcal{N}$.*

(iii) *if φ_{i_0} is linear in a neighborhood of zero for some $i_0 \in \mathcal{N}$, then φ_i is strictly convex on the interval $[0,1]$ for every $i \neq i_0$.*

Proof. *Necessity.* If $l_\varphi \in \big(SC^\perp\big)$, then $l_\varphi \in (SM)$, by Lemma 2. Consequently $\varphi \in \delta_2$ and $\varphi > 0$ (Lemma 2.1 in [18]). We will show the necessity of condition (ii). Suppose conversely that there exist $i_1, i_2 \in \mathcal{N}$ and $\alpha_1, \alpha_2 > 0$ such that φ_i is linear on the interval $[0, \alpha_i]$ for $i \in \{i_1, i_2\}$. Take $0 < a_i < \alpha_i$ for $i \in \{i_1, i_2\}$ with $\varphi_{i_1}(a_{i_1}) = \varphi_{i_2}(a_{i_2}) < 1$. Let $b > 0$ and $i_3 \notin \{i_1, i_2\}$ be such that $\varphi_{i_1}(a_{i_1}) + \varphi_{i_3}(b) = 1$. Define

$$x = a_{i_1} e_{i_1} + b e_{i_3} \text{ and } y = a_{i_2} e_{i_2} + b e_{i_3}.$$

Then $I_\varphi(x) = I_\varphi(y) = 1$. Hence $x, y \in S(l_\varphi)$. Denote $A = \operatorname{supp} x \div \operatorname{supp} y$. Then $\|x\chi_A\|_\varphi = \|a_{i_1} e_{i_1}\|_\varphi > 0$. Moreover

$$I_\varphi((x+y)/2) = \varphi_{i_1}(a_{i_1})/2 + \varphi_{i_2}(a_{i_2})/2 + \varphi_{i_3}(b) = 1.$$

Hence $\|(x+y)/2\|_\varphi = 1$. Thus $l_\varphi \notin \big(SC^\perp\big)$.

Assume now that the condition (iii) is not satisfied, i.e. there exist $i_0, i_1 \in \mathcal{N}$, $i_0 \neq i_1$, such that φ_{i_0} is linear on an interval $[0, \alpha_{i_0}]$ for some $\alpha_{i_0} > 0$ and φ_{i_1} is affine on an interval $[a_{i_1}, b_{i_1}]$ for some $0 \le a_{i_1} < b_{i_1} \le 1$. Take $\alpha \in (0, \alpha_{i_0})$ and $c \in (a_{i_1}, b_{i_1})$ such that $\varphi_{i_1}(a_{i_1}) + \varphi_{i_0}(\alpha) = \varphi_{i_1}(c)$. Let $i_2 \notin \{i_0, i_1\}$ and $d > 0$ be such that $\varphi_{i_1}(c) + \varphi_{i_2}(d) = 1$. Define

$$x = \alpha e_{i_0} + a_{i_1} e_{i_1} + d e_{i_2} \quad \text{and} \quad y = c e_{i_1} + d e_{i_2}.$$

Then it is easy to conclude that $l_\varphi \notin \big(SC^\perp\big)$ similarly as in the above proof of the necessity of condition (ii).

Sufficiency. Take any $x, y \in S(l_\varphi)$ such that $\|x\chi_{A_{xy}}\|_\varphi \vee \|y\chi_{A_{xy}}\|_\varphi > 0$, where $A_{xy} = \operatorname{supp} x \div \operatorname{supp} y$. We divide the proof into three parts.

I. Suppose that $\|x\chi_{A_{xy}}\|_\varphi \wedge \|y\chi_{A_{xy}}\|_\varphi > 0$. Then $\operatorname{card} A_{xy} > 1$. Consequently, by the assumption (ii), there is $j_0 \in A_{xy}$ such that φ_{j_0} is linear in no neighborhood of zero. Hence $\varphi_{j_0}(u/2) < \varphi_{j_0}(u)/2$ for every $u > 0$. Thus

$\varphi_{j_0}((x(j_0)+y(j_0))/2) < (\varphi_{j_0}(x(j_0))+\varphi_{j_0}(y(j_0)))/2$. So $I_\varphi((x+y)/2) < 1$. This implies $\|(x+y)/2\|_\varphi < 1$, since $\varphi \in \delta_2$.

II. Assume that $\|x\chi_{A_{xy}}\|_\varphi > 0$ and $\|y\chi_{A_{xy}}\|_\varphi = 0$. If there is $i \in A_{xy}$ such that φ_i is linear in no neighborhood of zero, then we proceed as in Case I. Otherwise, in view of assumption (ii), we get $\operatorname{card} A_{xy} = 1$. Put $A_{xy} = \{i_0\}$. Denote $B_{xy} = \operatorname{supp} x \cap \operatorname{supp} y$. By (i), we get $l_\varphi \in (SM)$ (see [18]). Then , since $x \in S(l_\varphi)$ and $\|x\chi_{A_{xy}}\|_\varphi > 0$, so $\|x\chi_{B_{xy}}\|_\varphi < 1$ (Theorem 8 from [9]). However $\|y\chi_{B_{xy}}\|_\varphi = 1$, by $\|y\chi_{A_{xy}}\|_\varphi = 0$. Thus we conclude that there exists $i_1 \in B_{xy}$ such that $y(i_1) > x(i_1)$. Moreover, since φ_{i_0} is linear in a neighborhood of zero, so φ_{i_1} is strictly convex on the interval $[0,1]$, by (iii). Then we get that $\|(x+y)/2\|_\varphi < 1$ similarly as in Case I.

III. If $\|x\chi_{A_{xy}}\|_\varphi = 0$ and $\|y\chi_{A_{xy}}\|_\varphi > 0$, the proof is analogous as in Case II. $\square$

Theorem 6. *Let φ be a Musielak–Orlicz function. Then l_φ is orthogonally uniformly convex if and only if*

(i) *each function φ_i vanishes only at zero, functions φ and φ^* fulfill the δ_2-condition and φ satisfies the condition $(*)$.*

(ii) *φ_i can be linear in a neighborhood of zero for at most one $i \in \mathcal{N}$.*

(iii) *if φ_{i_0} is linear in a neighborhood of zero for some $i_0 \in \mathcal{N}$, then*
   (1) *φ_i is strictly convex on the interval $[0,1]$ for every $i \neq i_0$ and*
   (2) *$(\varphi_i)_{i\neq i_0}$ is uniformly convex in 1-neighbourhood of zero.*

Proof. *Necessity.* It is known that $l_\varphi \in (UM)$ iff $\varphi \in \delta_2$, $\varphi > 0$ and $\varphi \in (*)$ (see [18]). Then the necessity of conditions $\varphi \in \delta_2$, $\varphi > 0$ and $\varphi \in (*)$ follows immediately from Lemma 1. Moreover, if $l_\varphi \in \left(UC^\perp\right)$, then $l_\varphi \in (\beta)$, by Theorem 2, and consequently l_φ is reflexive, so $\varphi \in \delta_2$ and $\varphi^* \in \delta_2$. The necessity of condition (ii) follows from Theorem 5.

We prove the necessity of condition (iii). We will apply some technics from the proof of necessity of Theorem 1 in [13]. First note that uniform convexity of φ in the c-neighbourhood of zero is equivalent to the following condition.

For every $a \in (0,1)$ there exists $\delta \in (0,1)$ such that $\sum_{i=1}^{\infty} \varphi_i(u_i(\delta,a)) < \infty$, where $u_i(\delta,a) = \sup\{u \in [0,\varphi_i^{-1}(c)] : h_i(u,au) \geq 1-\delta\}$.

It follows from Theorem 5 that if φ_{i_0} is linear in some neighborhood of zero, then φ_i is strictly convex on $[0,1]$ for every $i \neq i_0$. Consequently, assuming for the contrary that the condition (iii) is not satisfied, we may suppose that φ_{i_0} is linear on the interval $[0, \alpha_{i_0}]$ for some $i_0 \in \mathcal{N}$, $\alpha_{i_0} > 0$ and $(\varphi_i)_{i \neq i_0}$ is not uniformly convex in 1-neighbourhood of zero. Denote $N_0 = \mathcal{N} \backslash \{i_0\}$. Hence there exist $a \in (0,1)$ and a sequence (δ_k) included in $(0,1)$ and $\delta_k \downarrow 0$ such that

$$\sum_{i \in N_0} \varphi_i(u_{ik}) = \infty, \tag{12}$$

for every $k \in \mathcal{N}$, where $u_{ik} = u_i(\delta_k, a) \in [0, \varphi_i^{-1}(1)]$. By the definition of the sequence $u_i(\delta, a)$ we get

$$h_i(u_{ik}, au_{ik}) \geq 1 - \delta_k \tag{13}$$

for each $k \in \mathcal{N}$ and $i \in N_0$. We divide the proof into two parts.

I. There exists $b \in (0,1)$ such that $\overline{\lim}_{k \to \infty} \sup_{i \geq m} \varphi_i(u_{ik}) > b$ for every $m \in N_0$. Then there exist increasing subsequences (k_j) of $\mathcal{N}$ and (i_j) of N_0 such that $\varphi_{i_j}(u_{i_j k_j}) \in (b, 1]$ for every $j \in \mathcal{N}$. We denote j and v_j in place of i_j and $u_{i_j k_j}$, for simplicity. We consider two subcases.

a. Suppose that

$$\varphi_j(v_j) - \varphi_j(av_j) \in (0, \varphi_{i_0}(\alpha_{i_0})] \tag{14}$$

for infinitely many $j \in N_0$. Without loss of generality we may assume that the condition (14) holds for every $j > i_0$. Then there exists a sequence (α_j) in $(0, \alpha_{i_0}]$ such that

$$\varphi_{i_0}(\alpha_j) = \varphi_j(v_j) - \varphi_j(av_j) \tag{15}$$

for every $j > i_0$. Let $b_j \geq 0$ be such that

$$\varphi_j(v_j) + \varphi_{j+1}(b_j) = 1 \tag{16}$$

for every $j > i_0$. Define

$$x_j = \alpha_j e_{i_0} + av_j e_j + b_j e_{j+1} \text{ and } y_j = v_j e_j + b_j e_{j+1}.$$

Then, by (15) and (16), $I_\varphi(x_j) = I_\varphi(y_j) = 1$ for every $j > i_0$. Hence $\|x_j\|_\varphi = \|y_j\|_\varphi = 1$ for every $j > i_0$. Moreover, by (15),

$$\varphi_{i_0}(\alpha_j) = \varphi_j(v_j) - \varphi_j(av_j) \geq (1-a)\varphi_j(v_j) \geq (1-a)b$$

for every $j > i_0$. Hence, denoting $A_j = \operatorname{supp} x_j \div \operatorname{supp} y_j$, we get $I_\varphi(\frac{x_j}{(1-a)b}\chi_{A_j}) \geq 1$, so $\|x_j\chi_{A_j}\|_\varphi \geq (1-a)b$ for every $j > i_0$. On the other hand, by (13) and by linearity of φ_{i_0} on the interval $[0, \alpha_{i_0}]$, we get

$$I_\varphi\left(\frac{x_j + y_j}{2}\right) \geq \frac{1}{2}\varphi_{i_0}(\alpha_j) + (1-\delta_{k_j})\frac{\varphi_j(v_j) + \varphi_j(av_j)}{2} + \varphi_{j+1}(b_j).$$

Since $\delta_{k_j} \to 0$, so $I_\varphi(\frac{x_j+y_j}{2}) \to 1$, by (15) and (16). Hence $\|\frac{x_j+y_j}{2}\|_\varphi \to 1$, because we have $I_\varphi(z) \leq \|z\|_\varphi$ for every $z \in B(l_\varphi)$. Thus $l_\varphi \notin \left(UC^\perp\right)$.

b. Suppose that

$$\varphi_j(v_j) - \varphi_j(av_j) > \varphi_{i_0}(\alpha_{i_0}) \tag{17}$$

for almost every $j \in N_0$. We may assume without loss of generality that the last inequality holds for every $j > i_0$. Then there exists a sequence (a_j) in $(a, 1)$ such that $\varphi_{i_0}(\alpha_{i_0}) = \varphi_j(v_j) - \varphi_j(a_j v_j)$ for every $j > i_0$. Let $b_j \geq 0$ be such that $\varphi_j(v_j) + \varphi_{j+1}(b_j) = 1$. Define

$$x_j = \alpha_{i_0}e_{i_0} + a_j v_j e_j + b_j e_{j+1} \quad \text{and} \quad y_j = v_j e_j + b_j e_{j+1}.$$

Similarly, as in case a, we get $x_j, y_j \in S(l_\varphi)$ and $\|x_j\chi_{A_j}\|_\varphi = \|\alpha_{i_0}e_{i_0}\|_\varphi$ for every $j > i_0$. The function $h_j(u, au)$ is nondecreasing function of a. Hence, by (13), $h_j(v_j, a_j v_j) > 1 - \delta_{k_j}$ for every $j > i_0$. Then the proof can finished the same way as in case a.

II. Contrary to I, suppose that for every $b \in (0,1)$ there exists $m \in N_0$ such that $\overline{\lim}_{k\to\infty} \sup_{i\geq m} \varphi_i(u_{ik}) \leq b$. Then we find subsequences (m_j) of N_0 and (k_j) of $\mathcal{N}$, which is increasing such that $\varphi_i(u_{ik_j}) \leq \frac{1}{2^{j+1}}$ for every $j \in \mathcal{N}$ and $i \geq m_j, i \neq i_0$. Then, by (12), we conclude that for each j there exists a set N_j of N_0 such that

$$1 - \frac{1}{2^{j-1}} \leq \sum_{i\in N_j} \varphi_i(u_{ik_j}) \leq 1 - \frac{1}{2^j}. \tag{18}$$

In the remaining part of the proof we will consider three cases.

a. Suppose that

$$\varphi_{i_0}(\alpha_{i_0}) + \sum_{i\in N_j} \varphi_i(au_{ik_j}) < 1 - \frac{1}{2^{j-1}}$$

for infinitely many j. Without loss of generality we may assume that the last inequality holds for every j. Hence, by (18), there exists a sequence (a_j) such that $a_j \in [a,1]$ for each j and $\varphi_{i_0}(\alpha_{i_0}) = \sum_{i\in N_j}(\varphi_j(u_{ik_j}) - \varphi_j(a_j u_{ik_j}))$ for every j. Define

$$x_j = \alpha_{i_0} e_{i_0} + \sum_{i\in N_j} a_j u_{ik_j} e_i \quad \text{and} \quad y_j = \sum_{i\in N_j} u_{ik_j} e_i.$$

Then $x_j, y_j \in B(l_\varphi)$, by (18). The function $h_j(u, au)$ is nondecreasing function of a. Hence, the same way as in case Ib, applying inequality (18) we conclude that $l_\varphi \notin \left(UC^\perp\right)$.

b. Suppose that

$$\varphi_{i_0}(\alpha_{i_0}) + \sum_{i\in N_j} \varphi_i(au_{ik_j}) > 1 - \frac{1}{2^j}$$

for infinitely many j. Without loss of generality we may suppose that the above inequality is true for each j. Then, by (18), there exists a sequence (α_j) in $(0, \alpha_{i_0}]$ such that

$$\varphi_{i_0}(\alpha_j) + \sum_{i\in N_j} \varphi_i(au_{ik_j}) = \sum_{i\in N_j} \varphi_i(u_{ik_j}) \tag{19}$$

for every j. Define

$$x_j = \alpha_j e_{i_0} + \sum_{i\in N_j} au_{ik_j} e_i \text{ and } y_j = \sum_{i\in N_j} u_{ik_j} e_i.$$

Then $x_j, y_j \in B(l_\varphi)$, by (18) and (19). Moreover, again by (18) and (19),

$$\varphi_{i_0}(\alpha_j) = \sum_{i\in N_j} \varphi_i(u_{ik_j}) - \varphi_i(au_{ik_j}) \geq (1-a) \sum_{i\in N_j} \varphi_i(u_{ik_j}) \geq (1-a)/2.$$

for every j. Hence, denoting $A_j = \operatorname{supp} x_j \div \operatorname{supp} y_j$, we get $I_\varphi(\frac{x_j}{(1-a)/2}\chi_{A_j}) \geq 1$, so $\|x_j \chi_{A_j}\|_\varphi \geq (1-a)/2$ for every j. Then, similarly as in previous cases, one can conclude that $l_\varphi \notin \left(UC^\perp\right)$.

c. If

$$1 - \frac{1}{2^{j-1}} \leq \varphi_{i_0}(\alpha_{i_0}) + \sum_{i\in N_j} \varphi_i(au_{ik_j}) \leq 1 - \frac{1}{2^j}$$

for infinitely many j, then, in order to show that $l_\varphi \notin \left(UC^\perp\right)$, it is enough to define

$$x_j = \alpha_{i_0} e_{i_0} + \sum_{i \in N_j} a u_{ik_j} e_i \quad \text{and} \quad y_j = \sum_{i \in N_j} u_{ik_j} e_i.$$

Sufficiency. Let $\varepsilon > 0$. Take $x, y \in S(l_\varphi)$ such that $\|x\chi_A\|_\varphi \vee \|y\chi_A\|_\varphi \geq \varepsilon$, where $A = \operatorname{supp} x \div \operatorname{supp} y$. Applying Lemma 6 take number $\sigma_1 > 0$ such that

$$\text{if } \|z\|_\varphi \geq \varepsilon/2, \quad \text{then } I_\varphi(z) \geq \sigma_1. \tag{20}$$

Applying Lemma 5 with the number $\sigma_1/2$ we conclude that there exists $\gamma \in (0,1)$ and a sequence $\beta = (\beta_i)$ such that

$$\sum_{i \neq i_0} \varphi_i(\beta_i) < \sigma_1/2 \quad \text{and} \quad \varphi_i(u/2) \leq (1-\gamma)\varphi_i(u)/2 \tag{21}$$

holds true for each $i \in \mathcal{N}\backslash\{i_0\}$ and $u \in [\beta_i, 1]$, where i_0 is from condition (iii). Let $q_1 = q(\gamma\sigma_1/4)$ be from Lemma 3b. Hereafter we assume that

$$\|x\chi_A\|_\varphi \geq \varepsilon,$$

because otherwise the proof is analogous. We consider two cases.

I. Suppose that $\|x\chi_{A\backslash\{i_0\}}\|_\varphi \geq \varepsilon/2$. Then

$$I_\varphi(x\chi_{A\backslash\{i_0\}}) \geq \sigma_1, \tag{22}$$

by (20). Denote

$$A_1 = \{i \in A\backslash\{i_0\} : |x(i)| \geq \beta_i\} \quad \text{and} \quad A_2 = \{i \in A\backslash\{i_0\} : |x(i)| < \beta_i\}.$$

Then $I_\varphi(x\chi_{A_2}) < \sigma_1/2$, by (21). Hence $I_\varphi(x\chi_{A_1}) \geq \sigma_1/2$, by (22). Consequently, applying (21), we get $I_\varphi((x+y)/2) \leq 1 - \gamma I_\varphi(x\chi_{A_1})/2 \leq 1 - \gamma\sigma_1/4$. Thus $\|(x+y)/2\|_\varphi \leq 1 - q_1$, where $q_1 = q(\gamma\sigma_1/4)$ is from Lemma 3b.

II. Assume that $\|x\chi_{A\cap\{i_0\}}\|_\varphi \geq \varepsilon/2$. Hence $\|x\chi_{\{i_0\}}\|_\varphi \geq \varepsilon/2$. Note that, by the assumptions, l_φ is a uniformly monotone Köthe space (see [18]). Moreover, a Köthe space $E \in (UM)$ iff for any $\nu \in (0,1)$ there is $\eta(\nu) > 0$ such that for any $u \in E_+$ with $\|u\|_E = 1$ and for any $A \in \Sigma$

$$\text{if } \|u\chi_A\|_E \geq \nu \text{ then } \|u\chi_{T\backslash A}\|_E \leq 1 - \eta(\nu) \quad \text{(Theorem 6 in [9]).} \tag{23}$$

Consequently, applying (23) with $p_1 = \eta(\varepsilon/2)$, we get $\|x\chi_{\mathcal{N}\setminus\{i_0\}}\|_\varphi \le 1-p_1$. Take $\xi > 1$ with $\xi(1-p_1) < 1$ and $a \in (0,1)$ such that $\xi(1-p_1) < a$. Applying again Lemma 6 take number $\sigma_2 > 0$ such that

$$\text{if } \|z\|_\varphi \ge 1-a, \quad \text{then } I_\varphi(z) \ge \sigma_2. \tag{24}$$

Let $N_0 = (\operatorname{supp} x \cup \operatorname{supp} y)\setminus\{i_0\}$ and

$$B_1 = \left\{ i \in N_0 : \frac{|x(i)| \wedge |y(i)|}{|x(i)| \vee |y(i)|} < \frac{1}{\xi} \right\} \qquad \text{and}$$

$$B_2 = \left\{ i \in N_0 : \frac{|x(i)| \wedge |y(i)|}{|x(i)| \vee |y(i)|} \ge \frac{1}{\xi} \right\}.$$

First note that $\|y\chi_{B_2}\|_\varphi < a$, because otherwise we get a contradiction $a \le \|y\chi_{B_2}\|_\varphi \le \xi\|x\chi_{B_2}\|_\varphi \le \xi(1-p_1)$. Consequently $\|y\chi_{B_1}\|_\varphi \ge 1-a$, because $i_0 \in \operatorname{supp} x\setminus \operatorname{supp} y$, whence $\operatorname{supp} y \subset B_1 \cup B_2$ and $\|y\chi_{N_0}\|_\varphi = 1$. Thus, by (24)

$$I_\varphi(y\chi_{B_1}) \ge \sigma_2. \tag{25}$$

We apply Lemma 8 with $\widetilde{\varepsilon} = (1-1/\xi) \wedge \sigma_2/2$ for the Musielak–Orlicz function $(\varphi_i)_{i\ne i_0}$. Then there exists $p \in (0,1)$ and a nonnegative sequence (d_i) with $\sum_{i\ne i_0} \varphi_i(d_i) < \widetilde{\varepsilon}$ such that

$$h_i(u,v) \le 1-p \tag{26}$$

if $|u-v| \ge \widetilde{\varepsilon}(u \vee v)$ and $u \vee v \in (d_i, 1]$, $i \in \mathcal{N}\setminus\{i_0\}$. First note that $|x(i)-y(i)| \ge \widetilde{\varepsilon}(|x(i)| \vee |y(i)|)$ for every $i \in B_1$, because $|x(i)| \wedge |y(i)| < (|x(i)| \vee |y(i)|)\frac{1}{\xi}$. Moreover, denoting $B_{11} = \{i \in B_1 : |y(i)| \ge d_i\}$, we get $I_\varphi(y\chi_{B_1\setminus B_{11}}) < \sigma_2/2$. Then $I_\varphi(y\chi_{B_{11}}) \ge \sigma_2/2$, by (25). Then, applying (26), we get

$$I_\varphi((x+y)/2) \le 1 - pI_\varphi(y\chi_{B_{11}})/2 \le 1 - p\sigma_2/4.$$

Thus $\|(x+y)/2\|_\varphi \le 1-q_2$, where $q_2 = q(p\sigma_2/4)$ is from Lemma 3b. $\square$

Taking into account Theorems 5, 6 and Theorems 0.2, 1 in [13] we can exactly compare the difference between (UC), (SC) and $\left(UC^\perp\right)$, $\left(SC^\perp\right)$, respectively, in Musielak–Orlicz sequence spaces.

Now we are able to answer the question which has been pointed out in the introduction. Namely, applying Theorems 4, 5, 6 and Theorem 2.5 from [18] we get the following

Corollary 3. *There exists uniformly monotone, orthogonally strictly convex Musielak–Orlicz sequence space with property* (β) *which is not orthogonally uniformly convex.*

Recall that a Nakano space $l_{(p_i)}$ is the Musielak–Orlicz space l_φ where $\varphi_i(u) = |u|^{p_i}$, $1 \le p_i < \infty$, $i = 1, 2, \ldots$ (see [25]). This space can be isometrically transformed in such a way that $\varphi_i(u) = u^{p_i}$ if $u \in [0,1]$ and $\varphi_i(u) = u$ if $u > 1$ (see [13]). Notice that $\varphi \in \delta_2$ iff $\overline{\lim}_{i\to\infty} p_i < \infty$ and $\varphi^* \in \delta_2$ iff $\underline{\lim}_{i\to\infty} p_i > 1$. Moreover $\varphi \in (*)$ whenever $\overline{\lim}_{i\to\infty} p_i < \infty$ and $\underline{\lim}_{i\to\infty} p_i > 1$ (see the proof of Theorem 2 in [13]). Then, applying Theorems 4, 5, 6, Theorems 0.2 and 2 from [13], we get

Corollary 4. (a) $l_{(p_i)}$ *has property* (β) *iff* $\overline{\lim}_{i\to\infty} p_i < \infty$ *and* $\underline{\lim}_{i\to\infty} p_i > 1$.

(b) *Thefollowing statements are equivalent:*

- (i) $l_{(p_i)}$ *is strictly convex.*
- (ii) $l_{(p_i)}$ *is orthogonally strictly convex.*
- (iii) $\overline{\lim}_{i\to\infty} p_i < \infty$ *and* $p_i = 1$ *for at most one* i.

(c) *The following statements are equivalent:*

- (i) $l_{(p_i)}$ *is uniformly convex.*
- (ii) $l_{(p_i)}$ *is orthogonally uniformly convex.*
- (iii) $\overline{\lim}_{i\to\infty} p_i < \infty$, $\underline{\lim}_{i\to\infty} p_i > 1$ *and* $p_i = 1$ *for at most one* i.

Note that orthogonally uniformly convex Orlicz sequence space need not be strictly convex (see [16]). We also conclude that, for Nakano spaces, the answer for the question pointed out in the introduction is positive in contrast to the general case of Musielak–Orlicz sequence spaces (see Corollary 3).

Corollary 5. $l_{(p_i)}$ *is orthogonally uniformly convex if and only if* $l_{(p_i)}$ *is orthogonally strictly convex and it has property* (β).

Applying Theorem 5 to the case of Orlicz sequence spaces we get

Corollary 6. *The Orlicz sequence space* l_Φ *is orthogonally strictly convex if and only if* Φ *fulfills the* δ_2*-condition and* Φ *is linear in no neighborhood of zero.*

Similarly, applying Theorem 6 or Theorems 3 and 4 we get the following criteria for orthogonal uniform convexity in Orlicz sequence spaces proved in [16].

Corollary 7. *The Orlicz sequence space l_Φ is orthogonally uniformly convex if and only if $\Phi \in \delta_2$ and $\Phi^* \in \delta_2$.*

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