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On an oblique derivative problem of finite index for nonlinear elliptic discontinuous equations in the plane

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Abstract. An uniqueness and existence theorem in Sobolev spaces is proved for a tangential oblique derivative problem of finite index in the plane for second order nonlinear elliptic equations with discontinuous coefficients. The nonlinear operator $\mathcal{A}\left(x, D^2u\right)$ is assumed to be of Carathéodory type and to satisfy an ellipticity condition. Only measurability with respect to the independent variable x is required.

1. Introduction

The paper deals with the strong solvability in Sobolev space $W^{2,2}(B)$ of the following tangential oblique derivative problem in a ball B of the plane

$$\begin{cases} \mathcal{A}\left(x, D^2u\right) = f(x) & \text{a.e. in } B \\ \dfrac{\partial u}{\partial x_2} = 0 & \text{on } \partial B \\ u(-1,0) = 0, \ u(1,0) = 0, \end{cases} \tag{1}$$

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where $x = (x_1, x_2) \in \mathbb{R}^2$, $D^2u = \{D_{ij}u\}_{i,j=1,2}$ is the Hessian matrix of u, $l = (0, 1)$ is an axis tangential to the boundary ∂B in the points $(-1, 0)$, $(1, 0)$.

The operator $\mathcal{A}\left(x, D^2u\right)$ is assumed to be of Carathéodory type and to satisfy the following ellipticity condition, introduced by S. CAMPANATO [2], that does not imply the continuity of the mapping $\mathcal{A}\left(x, \xi\right)$ with respect to x:

There exist α, γ, $\delta > 0$, $\gamma + \delta < 1$, such that, for almost all $x \in B$, for all $\xi, \tau \in \mathbb{R}^{2\times 2}$, one has

$$\left|\sum_{i=1}^{2} \xi_{ii} - \alpha\left[\mathcal{A}(x, \xi + \tau) - \mathcal{A}(x, \tau)\right]\right| \leq \gamma\|\xi\| + \delta\left|\sum_{i=1}^{2} \xi_{ii}\right|. \tag{A}$$

For a discussion about condition (A) see Section 2.

In multidimensional case the regular oblique derivative problem with uniformly continuous data has been very well studied. The case of discontinuous operators is less studied and exhaustive results are not available (see [17] for up-to-date survey).

New effects occur, in case $n \geq 3$, when the vector l, which prescribes boundary condition, becomes tangential to the boundary. In this case Shapiro–Lopatinskii condition is not fulfilled, then the problem may have infinite dimensional kernel or cokernel, and its solvability and regularity properties depend mainly on the structure of the set of tangency and the local behaviour of l near this set (see [23], [24]). The first results in the study of linear tangential problem are due to BICADZE [1] and HÖRMANDER [15], who indicated how the solvability and uniqueness properties depend on the sign of the scalar product $l\cdot\chi$, with χ unit outward normal to the boundary $\partial\Omega$. Hörmander's results were refined by EGOROV and KONDRAT'EV [7], who proved that the linear problem is of Fredholm type if $l \cdot \chi$ does not change its sign on $\partial\Omega$, i.e., l is of neutral type. Moreover in the emergent case, e.g. $l\cdot\chi$ changes its sign from $-$ to $+$ on the trajectories of l through the set of tangency, sc Egorov and KONDRA'TEV ([7]) proved that the linear problem becomes of Fredholm type if an extra condition is prescribed on the set of tangency. Later regularity and classical or strong solvability have been studied in [6], [12], [19], [20], [25], [28], [29] and most recently in [13], [14], [18].

A different situation arises in case of two dimensions. In fact if the field l is nowhere zero, Shapiro–Lopatinskii condition is fulfilled, that is the oblique derivative problem is a regular one. Planar linear problem with boundary condition $\frac{\partial u}{\partial l} = \text{const.}$ has been studied by G. Talenti [26], who proved a priori estimate for the second derivatives.

Recently planar $W^{2,q}(\Omega)$ solvability has been proved by S. Giuffrè [8], [9], [10], for oblique derivative problem for discontinuous nonlinear operator $\mathcal{A}(x, u, Du, D^2u)$, under the above mentioned ellipticity condition, in case of finite number of points of tangency, under condition $l \cdot \chi \geq 0$.

Concerning Problem (1) where $l = (0, 1)$, the scalar product $l \cdot \chi$ changes its sign from $-$ to $+$ on the trajectories of l through the points of tangency. In this case the problem for Laplacian

$$\begin{cases} \Delta u = f(x) & \text{a.e. in } B \\ \dfrac{\partial u}{\partial l} = 0 & \text{on } \partial B \end{cases} \tag{2}$$

as we already observed, is a regular one, that is it has a finite dimensional kernel and a finite dimensional cokernel, i.e. a finite index. On the other hand since the field l is tangential to the boundary, the kernel of (2) is nontrivial (see [23], [24]), but, prescribing the values of u at the points of tangency, a new problem arises having a trivial kernel and then a unique solution in the class $W^{2,2}(B)$. A pleminінary note of this paper has been published in C.R. Acad. Sci. Bulg. (see [11]).

2. Basic assumptions and main results

Let B be a ball of $\mathbb{R}^2$ of radius 1, and let $x_1 = \cos\varphi$, $x_2 = \sin\varphi$, $\varphi \in [0, 2\pi]$, be parametric equations of ∂B. Let $\mathcal{A} : B \times \mathbb{R}^{2\times 2} \to \mathbb{R}$ be a mapping measurable in $x \in B$ for all $\xi \in \mathbb{R}^{2\times 2}$, $\xi = \{\xi_{ij}\}_{i,j=1,2}$, and continuous in ξ for almost all $x \in B$, such that $\mathcal{A}(x, 0) = 0$ and verifying the following assumption:

There exist α, γ, $\delta > 0$, $\gamma + \delta < 1$, such that, for almost all $x \in B$, for all $\xi, \tau \in \mathbb{R}^{2\times 2}$, one has

$$\left| \sum_{i=1}^{2} \xi_{ii} - \alpha \left[\mathcal{A}(x, \xi + \tau) - \mathcal{A}(x, \tau) \right] \right| \leq \gamma \|\xi\| + \delta \left| \sum_{i=1}^{2} \xi_{ii} \right| \tag{A}$$

where $\|\xi\| = (\sum_{i,j=1}^{2} \xi_{ij}^2)^{\frac{1}{2}}$.

Our aim is to study strong solvability of problem (1), for which we have

$$\cos\theta(\varphi) = l \cdot \chi = \sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right), \tag{3}$$

where $\chi = (\cos\varphi, \sin\varphi)$ is the unit outward normal to ∂B and $\theta(\varphi)$ is the angle between χ and $l = (0,1)$.

Then in $[0, 2\pi]$ l is tangential to ∂B in the points corresponding to $\varphi = 0$, $\varphi = \pi$, that are $(1,0)$, $(-1,0)$. The scalar product $l \cdot \chi$ changes its sign in such a way that the direction of l is inward and then outward to ∂B.

Moreover it is easily seen that

$$\theta(\varphi) = \frac{\pi}{2} - \varphi. \tag{4}$$

Then, if we denote by κ the curvature of ∂B, it results $\kappa = -1$ and

$$\frac{d\theta}{d\varphi} - \kappa = 0. \tag{5}$$

This condition allows us to apply Lemma 3.1 (see Section 3). Let us note that we can not apply Theorem 3 of [26] in order to get the existence of the solution, at least in the linear case, because this theorem requires $\frac{d\theta}{d\varphi} - \kappa > 0$.

Moreover from (4) it derives that

$$h = \frac{\theta(2\pi) - \theta(0)}{2\pi} = -1,$$

that is the vector l makes a turn around the normal χ, while the point $(x_1(\varphi), x_2(\varphi))$ describes an anticlockwise turn of the boundary ∂B.

As it is well known from Riemann–Hilbert theory, a classical solution of the problem

$$\begin{cases} \Delta u = 0 & \text{a.e. in } B \\ \dfrac{\partial u}{\partial l} = g(x) & \text{on } \partial B \end{cases}$$

contains $-2h$ arbitrary constants. In our case $-2h = 2$, then this fact perfectly agrees with our result about the solvability of Problem (1).

Let us specify the Sobolev class V in which we are searching the solution to Problem (1). We define V the closure of $\big\{u \in C^2(\overline{B}) \cap C^3(B) : \frac{\partial u}{\partial x_2} = 0 \text{ on } \partial B,\ u(-1,0) = 0,\ u(1,0) = 0\big\}$ with respect to the norm

$$\|u\|_V = \|\Delta u\|_{L^2(B)}. \tag{6}$$

We will prove later that (6) is a norm.

We are in position to formulate the existence and uniqueness result.

Theorem 2.1. *Let us assume* (A) *condition. Then, for all* $f \in L^2(B)$, *Problem* (1) *admits a unique solution* $u \in V$ *and it results*

$$\|D^2 u\|_{L^2(B)} \leq \frac{\alpha}{1 - (\gamma + \delta)} \|f\|_{L^2(B)}.$$

For what concern (A) assumption let us observe that it is equivalent to the following "monotonicity" conditions (see [17])

$$|\mathcal{A}(x, \xi + \tau) - \mathcal{A}(x, \tau)| \leq \frac{\gamma + \sqrt{2}(1 + \delta)}{\alpha} \|\xi\| \tag{a}$$

$$(\mathcal{A}(x, \xi + \tau) - \mathcal{A}(x, \tau)) \sum_{i=1}^{2} \xi_{ii} \geq \frac{1 - \delta(\gamma + \delta)}{2\alpha} \left| \sum_{i=1}^{2} \xi_{ii} \right|^2 - \frac{\gamma(\gamma + \delta)}{2\alpha} \|\xi\|^2 \tag{b}$$

for almost all $x \in B$, for all $\xi, \tau \in \mathbb{R}^{2\times 2}$. From Condition (a) it follows that the mapping $\xi \to \mathcal{A}(x, \xi)$ is a.e. differentiable for a.a. $x \in B$ and that the derivatives $\frac{\partial \mathcal{A}(x,\xi)}{\partial \xi_{ij}}$ belong to $L^\infty(B \times \mathbb{R}^{2\times 2})$ and they are bounded by

$\Lambda = \frac{\gamma+\sqrt{2}(1+\delta)}{\alpha}$. From this and from (b) it follows the strong ellipticity condition:

$$\lambda \|\zeta\|^2 \leq \sum_{i,j=1}^{2} \frac{\partial \mathcal{A}(x,\xi)}{\partial \xi_{ij}} \zeta_i \zeta_j \leq 2\, \Lambda \|\zeta\|^2 \tag{7}$$

for all $\zeta \in \mathbb{R}^2$, for almost all $x \in B$, for all $\xi, \tau \in \mathbb{R}^{2\times 2}$, where $\lambda = \frac{1-(\gamma+\delta)^2}{2\alpha}$. Let us show an example of nonlinear operator which satisfies condition (A):

$$\mathcal{A}(x, D^2 u) = \rho \Delta u + \arctan(a(x)\Delta u) \tag{8}$$

where $a(x) \in L^\infty(\Omega)$ is a discontinuous nonnegative function (for example $\frac{|x_1 x_2|}{x_1^2+x_2^2}$) and ρ is a positive number. It is easily seen that, choosing $0 < \alpha < \frac{1}{\rho+\sup_\Omega a(x)}$, condition (A) is fulfilled by operator (8).

Moreover in the plane the linear operator

$$\mathcal{A}(x,\xi) = \sum_{i,j=1}^{2} a_{ij}(x)\xi_{ij}$$

with $a_{ij} \in L^\infty(\Omega)$, satisfies condition (A). In fact condition (A) is equivalent to the so called Cordes condition:

$$\frac{\sum_{i,j=1}^{2} a_{ij}^2(x)}{\left(\sum_{i=1}^{2} a_{ii}(x)\right)^2} < \frac{1}{1+\varepsilon} \quad \text{with} \quad 0 < \varepsilon < \frac{4\lambda\Lambda}{4\Lambda^2+\lambda^2}$$

which in our case is equivalent to ellipticity condition (7).

Thus, in the linear case, the mere assumption $a_{ij} \in L^\infty(B)$ together (7) guarantees the solvability of Problem (1).

Remark 2.1. It is assumed that B is a ball only for the sake of simplicity. The result holds true under the general assumption that B is a convex subset of $\mathbb{R}^2$ with boundary of class C^2.

Remark 2.2. The result still holds in case of an unit vector field $l = (Y_1, Y_2)$, Y_1, $Y_2 \in \mathbb{R}$, tangential to the boundary in the points $(-Y_2, Y_1)$, $(Y_2, -Y_1)$.

3. Preliminary result

Since condition (5) is fulfilled, we are in condition to use the following TALENTI estimate (see [26] Theorem 1, for $g = 0$, $\mathcal{L}u = \Delta u$, $\lambda_1 = \lambda_2 = 1$).

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and open set with C^2-smooth boundary $\partial\Omega$. Let $l = (Y_1(\varphi), Y_2(\varphi))$ be a unit vector field of class $C^1(\partial\Omega)$ and assume $\frac{d\theta(\varphi)}{d\varphi} - \kappa(\varphi) \geq 0$. Then $\forall u \in C^2(\overline{\Omega}) \bigcap C^3(\Omega)$ such that $\frac{\partial u}{\partial l} =$ const. on $\partial\Omega$, it results*

$$\left\| D^2 u \right\|_{L^2(\Omega)} \leq \|\Delta u\|_{L^2(\Omega)} . \tag{9}$$

By means of this lemma we may prove that (6) is a norm. In fact

$$\|u\|_V = 0 \to \|\Delta u\|_{L^2(B)} = 0$$

and by estimate (9) it follows

$$\|D^2 u\|_{L^2(B)} = 0$$

that is

$$u = \alpha x_1 + \beta x_2 + \gamma.$$

By conditions $\frac{\partial u}{\partial x_2} = 0$, $u(-1, 0) = 0$, $u(1, 0) = 0$, we obtain $\alpha = \beta = \gamma = 0$ and then $u = 0$. It also follows that the kernel of Problem (10) is trivial.

Remark 3.1. G. TALENTI [26] also provides some examples from which the necessity of the condition $\frac{d\theta}{d\varphi} \geq \kappa$ can be derived.

4. Proof of the theorem

In order to prove Theorem 2.1, we need the following uniqueness and existence result for the Laplacian in the class $W^{2,2}(B)$, the proof of which, for utility of the reader, we report (see also [1]).

Theorem 4.1. *For all $f \in L^2(B)$, there exists a unique solution of the problem*

$$\begin{cases} \Delta u = f(x) \quad \text{a.e. in} \quad B \\ u \in V. \end{cases} \tag{10}$$

Proof. In order to prove the existence of a solution to Problem (10), let $\{f_n\} \subset C_0^\infty(B)$ such that $\|f_n - f\|_{L^2(B)} \to 0$. Consider the following Dirichlet problem

$$\begin{cases} \Delta v_n = \dfrac{\partial f_n}{\partial x_2} & \text{a.e. in } B \\ v_n(x_1, x_2) = 0 & \text{on } \partial B, \end{cases}$$

it admits a unique solution $v_n \in C^\infty(\overline{B})$.

Moreover the problem

$$\begin{cases} \dfrac{d^2 w_n(x_1)}{dx_1^2} = f_n(x_1, 0) - \dfrac{\partial v_n(x_1, 0)}{\partial x_2} & \text{a.e. in } B \\ w_n(-1) = 0, w_n(1) = 0 \end{cases}$$

admits a unique solution $w_n(x_1) \in C^\infty([-1, 1])$.

The function $u_n(x_1, x_2) = \int_0^{x_2} v_n(x_1, t)dt + w_n(x_1)$ is a solution of problem

$$\begin{cases} \Delta u_n = f_n(x) & \text{a.e. in } B \\ \dfrac{\partial u_n}{\partial x_2} = 0 & \text{on } \partial B \\ u_n(-1, 0) = 0, u_n(1, 0) = 0. \end{cases}$$

Moreover it results

$$\|u_n - u_m\|_V = \|f_n - f_m\|_{L^2(B)},$$

and hence there exists $u \in V$, such that

$$\|u_n - u\|_V \to 0.$$

Finally we get

$$\|\Delta u - f\|_{L^2(B)} \leq \|f - f_n\|_{L^2(B)} + \|u - u_n\|_V \to 0$$

and then it follows the solvability of Problem (10). The uniqueness of the solution follows from the fact that the kernel of (10) is trivial. $\square$

Now we are in position to prove Theorem 2.1.

To get our result it is enough to prove that, $\forall f \in L^2(B)$, there exists a unique $u \in V$ such that

$$\Delta u = \left[\Delta u - \alpha \mathcal{A}(x, D^2 u)\right] + \alpha f(x).$$

From Theorem 4.1 it follows that the problem

$$\begin{cases} \Delta u = \left[\Delta w - \alpha \mathcal{A}(x, D^2 w)\right] + \alpha f(x) \quad \text{a.e. in } B \\ u \in V \end{cases} \tag{11}$$

admits a unique solution, then we may define the operator $T : V \to V$ in such a way that $u = T(w)$, $\forall w \in V$, is the unique solution to Problem (11). We may show that T is a contraction mapping. In fact, for every $w_1, w_2 \in V$, such that $T(w_1) = u_1$, $T(w_2) = u_2$, we have, in virtue of condition (A) and estimate (9)

$$\begin{aligned} \|T(w_1) - T(w_2)\|_V &= \|u_1 - u_2\|_V = \|\Delta(u_1 - u_2)\|_{L^2(B)} \\ &= \|\Delta(w_1 - w_2) - \alpha[\mathcal{A}(x, D^2 w_1) - \mathcal{A}(x, D^2 w_2)]\|_{L^2(B)} \\ &\leq \gamma \|D^2(w_1 - w_2)\|_{L^2(B)} + \delta \|\Delta(w_1 - w_2)\|_{L^2(B)} \\ &\leq (\gamma + \delta)\|\Delta(w_1 - w_2)\|_{L^2(B)} = (\gamma + \delta)\|w_1 - w_2\|_V. \end{aligned}$$

Since $\gamma + \delta < 1$, T is a contraction mapping. From this it follows existence of a unique fixed point of T, i.e., a unique solution of (1).

Moreover by virtue of estimate (9) and condition (A), taking into account that $\mathcal{A}(x, 0) = 0$, we get

$$\begin{aligned} \|\Delta u\|_{L^2(B)} &= \|\Delta u - \alpha \mathcal{A}(x, D^2 u) + \alpha f(x)\|_{L^2(B)} \\ &\leq \gamma \|D^2 u\|_{L^2(B)} + \delta \|\Delta u\|_{L^2(B)} + \alpha \|f(x)\|_{L^2(B)} \\ &\leq (\gamma + \delta)\|\Delta u\|_{L^2(B)} + \alpha \|f(x)\|_{L^2(B)} \end{aligned}$$

and hence

$$\|D^2 u\|_{L^2(B)} \leq \|\Delta u\|_{L^2(B)} \leq \frac{\alpha}{1 - (\gamma + \delta)} \|f(x)\|_{L^2(B)}.$$

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