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Connections for singular foliations on stratified manifolds

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Abstract. In the paper we introduce the notions of a foliated stratified manifold and of a connection adapted to such a stratified manifold. We study the influence of the existence of an adapted connection on the properties of the singular foliation.

In their works from late eighties R. S. Palais and C. Terng studied proper actions of Lie groups which on the principal stratum admitted an integrable transverse subbundle. The leaves of this foliation are totally geodesic sections for the action restricted to the principal stratum. Moreover, the existence of such a section is equivalent to the integrability of the normal bundle. They conjectured that such sections can be extended to global immersed sections of the action, [13]. This conjecture was proved by H. Boualem in a more general case of transversely integrable singular Riemannian foliations (SRF), cf. [4], [3].

The aim of this short note is to present a more general setting in which the Boualem theorem is true. We consider singular foliations on stratified manifolds and we introduce the notion of an adapted connection for such singular foliations, whose properties generalise those of SRF.

In the regular case the Levi–Civita connection of a bundle-like metric is the model of an adapted (foliated or transversely projectable) connection, cf. [7], [6], [17], [19]. The most important property of this connection

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is the fact that a geodesic orthogonal to leaves of the foliation at one point remains orthogonal at any point of its connected domain. Moreover, as the bundle-like metric induces a Riemannian metric on any transverse manifold, any orthogonal geodesic projects to a geodesic on this transverse manifold. Under an additional assumption that the bundle-like metric is complete this ensures that the orthogonal bundle to the tangent bundle (to leaves of the foliation) is an Ehresmann connection, cf. [2]. The existence of an Ehresmann connection is sufficient to prove many important properties of the foliation, cf. [9], [18], [1], [21].

We assume that all objects, manifolds, bundles, foliations, vector fields, etc., are smooth, i.e. of class C^∞. If $(M, \mathcal{F})$ is a foliated manifold, x a point of M, then by L_x we denote the leaf of $\mathcal{F}$ passing through the point x.

1. Basic properties of foliations admitting adapted connections

Let $\mathcal{F}$ be a singular foliation on a stratified manifold (M, Σ), cf. [14], [15], [9], [16]. As the foliation $\mathcal{F}$ is singular the dimension of its leaves can vary from 0 to $n = \dim M$. The codimension of the foliation is the codimension of its leaves of maximal dimension. To develop a meaningful theory we must have some relation between the foliation and the stratification Σ of the manifold M.

We say that the stratification $\{\Sigma_r\}_{r=1}^k$ is *adapted* to the foliation $\mathcal{F}$ if $\mathcal{F}$ induces a regular foliation $\mathcal{F}_r$ on each stratum Σ_r and the dimensions of the foliations $\mathcal{F}_r$ vary from stratum to stratum. The stratum Σ_k on which the leaves of the foliation have the maximal dimension is called the principal stratum.

Definition 1. We call $(M, \Sigma, \mathcal{F})$ a *foliated stratified manifold* if

i) the stratification Σ is adapted to the foliation $\mathcal{F}$;

ii) the principal stratum Σ_k is open and dense in M.

In a foliated stratified manifold the set $\Sigma^s = M - \Sigma_k$ is a closed subset and is called the singular set of the foliation $\mathcal{F}$.

Remark. The definition of a foliated stratified manifold formalises the basic property of a SRF on a compact manifold, cf. [9], [10], see also [20].

Theorem 1 (Molino)**.** *Let $\mathcal{F}$ be a SRF on a compact manifold M. The foliation $\mathcal{F}$ defines a stratification of M and $(M, \Sigma, \mathcal{F})$ is a foliated stratified manifold.*

Let $(M, \Sigma, \mathcal{F})$ be a foliated stratified manifold.

Definition 2. A stratified transverse bundle Q to the foliation $\mathcal{F}$ is a family Q of subbundles Q^r of dimension q_r on the strata Σ_r supplementary to $T\mathcal{F}_r$, i.e. $T\mathcal{F}_r \oplus Q^r = TM|_{\Sigma_r}$, satisfying the compatibility condition:

(C) for any open neighbourhood (a, b) of o in R and any embedding $\gamma : (a, b) \to M$ such that $\gamma(t) \in \Sigma_r$ for $t > 0$ and $\gamma(0) = x_0 \in \Sigma_p$ the limit of $Q^r_{\gamma(t)}$ as t goes to 0 exists in $GR(M; q_r)$ and $\lim_{n\to\infty} Q^r_{\gamma(t)} = Q_0 (\in GR_{x_0}(M; q_r)) \subset Q^p_{x_0}$.

Let L be any leaf of $\mathcal{F}$, L is a submanifold of a stratum Σ_r and a leaf of the regular foliation $\mathcal{F}_r$. The subbundle Q^r restricted to L can be identified with the normal bundle $N(L)$ of L in M. The infinitesimal action of vector fields tangent to the leaves of the foliation $\mathcal{F}$ define a foliation $\mathcal{F}_L$ in the normal bundle $Q^r|_L$. The leaves of the foliation $\mathcal{F}_L$ have the dimension at least equal to the dimension of L. Any curve in L can be lifted to a leaf curve in $Q^r|_L$ starting at the chosen vector. The relation between the foliation $\mathcal{F}_L$ and the foliation $\mathcal{F}$ in a neighbourhood of L is an important one. If the exponential mapping with respect to some connection is a foliation preserving diffeomorphism of some open neighbourhood of the zero section (i.e. L) of $Q^r|_L$ onto the image (a neighbourhood of L in M) we say that the foliation $\mathcal{F}$ is linearisable in a neighbourhood of L. For the discussion of this condition for SRFs see [10].

A curve (geodesic) $\gamma : (a, b) \to M$ is called a Q-curve (geodesic) if for any $t \in (a, b)$ the fact that $\gamma(t) \in \Sigma_r$ implies that $\dot\gamma(t) \in Q_r$.

Definition 3. A connection ∇ on a stratified foliated manifold $(M, \Sigma, \mathcal{F})$ is said to be *adapted* if there exists a stratified supplementary subbundle Q such that:

i) Let $\gamma : (a, b) \to M$ be a geodesic of ∇. If for some $t_0 \in (a, b)$ $\gamma(t_0) \in \Sigma_r$ and $\dot\gamma(t_0) \in Q^r$ then γ is a Q-geodesic;

ii) Let $\alpha : [a, b] \to \Sigma_r \subset M$ be any leaf curve and let $\gamma : [c, d] \to M$ be a Q-geodesic such that $\gamma(c) = \alpha(a)$. Let $v = \dot{\gamma}(c) \in Q^r$ and let α_v be a lift of α to v. Then the mapping $\sigma : [a, b] \times [c, d] \to M$, $\sigma(t, s) = \exp^{\nabla}(s\alpha_v(t))$ is defined on the entire rectangle $[a, b] \times [c, d]$, where $\exp^{\nabla}$ is the exponential mapping defined by the connection ∇, has the following properties:

a) $\sigma_s : [a, b] \to M$, $\sigma|_{[a,b]\times\{s\}}$ is a leaf curve for any $s \in [c, d]$;

b) $\sigma^t : [c, d] \to M$, $\sigma|_{\{t\}\times[c,d]}$ is a Q-geodesic for any $t \in [a, b]$.

On the principal stratum Σ_k the connection ∇ is adapted to the regular foliation $\mathcal{F}_k$, the restriction of ∇ to the transverse bundle Q^k is a transversely projectable connection.

Lemma 1. *Let ∇ be an adapted connection on the stratified foliated manifold $(M, \Sigma, \mathcal{F})$. Then the subbundle Q^k is an (reduced) Ehresmann connection for the regular foliation $\mathcal{F}_k$ on the principal stratum Σ_k.*

Proof. It is a straightforward consequence of the definition of an adapted connection and the fact that the singular set Σ^s is a closed. □

Corollary 1. *The leaves of the regular foliation $\mathcal{F}_k$ have the same universal covering space.*

Before passing to more advanced problems let us look once again at a Q-geodesic say γ; it passes through points of various strata, through points of leaves of different dimensions, let us choose a point $\gamma(t)$ which belongs to a leaf of the greatest dimension along $\gamma - p$. So the point $\gamma(t)$ belongs to the stratum Σ_q. As the dimension of leaves is lower semi-continuous the set Γ_p of the parameters of points of the trace of the geodesic γ belonging to the stratum Σ_q is open in R. We call these points the regular points of the geodesic, the others the singular ones. The trace of the geodesic γ is in the closure of the stratum Σ_q.

The following lemmas are simple but useful technical results.

Lemma 2. *Let x be any point of a stratum Σ_r. The exponential mapping $\exp|_{Q^r_x} : Q^r_x \to M$ is a diffeomorphism onto the image V_x when restricted to some ball $B_x(O, \rho)$ in Q^r_x. Taking a sufficiently small contractible neighbourhood P_x of x in the leaf L_x the restriction of $\exp$ to $Q^r|_{P_x} =$*

$P_x \times Q^r_x$ defines a tubular neighbourhood of P_x. Then $\exp(P_x \times \{v\})$ is contained in a leaf of $\mathcal{F}$ (for any $v \in Q^r_x$).

Proof. It is a straightforward consequence of the definition of an adapted connection. □

Lemma 3. *Let $\sigma : [a,b] \times [c,d] \to M$ be a smooth mapping as in the definition of an adapted connection. Then for any $s_0 \in [c,d]$ the curve $\hat{\alpha}_{s_0} : t \mapsto \partial/\partial s\sigma(t,s)$ $t \in [a,b]$ is a leaf curve in $N(L_s)$.*

Proof. We have to prove that the vectors of the curve $\hat{\alpha}_s$ are in the orbit of the vector $\hat{\alpha}_s(0)$. Let k be the maximal dimension of leaves for parameters near s and k_0 the dimension of the leaf passing through the point $\gamma(s)$. If $k = k_0$ the geodesics are in a stratum Σ_q (locally) and our supposition is a well-known fact. It $k > k_0$ we define a mapping $\Phi^k : R^k \times R \to M$ such that $\Phi(t_1, \ldots, t_k, s) = \varphi^k_{t_k} \ldots \varphi^1_{t_1}(\gamma(s))$ where φ^i is the flow of the vector field X_i – the vector fields locally spanning the bundle tangent to the leaves of the foliation. For any s $\Phi^k(R^k \times \{s\})$ is a neighbourhood of $\gamma(s)$ in the leaf. Using this mapping it is not difficult to show that if $\sigma(t,s) = \Phi(t_1(s), \ldots, t_k(s), s)$ for some $(t_1(s), \ldots, t_k(s))$, then $\hat{\alpha}_s(t) =^Q \Phi(t_1(s), \ldots t_k(s), s)(\dot{\gamma}(s))$; ^{Q} denotes the induced action on the bundle Q. □

Finally, we can formulate and demonstrate the homothety lemma.

Lemma 4 (The Homothety Lemma)**.** *Let x be any point of a stratum Σ_r. The exponential mapping $\exp|_{Q^r_x} : Q^r_x \to M$ is a diffeomorphism onto the image V_x when restricted to some ball $B_x(O,\rho)$ in Q^r_x. Taking a sufficiently small contractible neighbourhood P_x of x in the leaf L_x the restriction of $\exp$ to $Q^r|_{P_x} = P_x \times Q^r_x$ defines a tubular neighbourhood of P_x. Then for any $t \in (0,1]$ the leaves passing through the points $\exp(x,tv)$ belong to the same stratum.*

Proof. Let $y = \exp(x,v)$ and $\gamma_y : [0,1] \to M$ be the geodesic with the initial condition v; let P_y be a small contractible neighbourhood of y in L_y. The geodesic γ_y joins x to y. Any point z of P_y can be reached from y by a short leaf curve $\alpha_z : [0,\epsilon] \to M$. Consider the rectangle σ_z defined by the pair $(\alpha_z, \gamma_y^{-1})$. The geodesics σ^s end in the plaque P_x and the vectors $\dot{\sigma}^s_z(1) \in Q^r$. For P_y sufficiently small all $\dot{\sigma}^s_z(1) \in P_x \times B_x(0,\rho)$. Then the geodesics starting at these vectors do not intersect, so the plaque P_y is mapped by the mapping (homothety) H_t, $H_t(\exp(x,v)) = \exp(x,tv)$, into an open neighbourhood of the point $H_t(y)$ in the corresponding plaque. Therefore the dimension of the leaf passing through the point $H_t(y)$ is greater or equal to the dimension of the leaf passing through the point y. However, owning to Lemma 3, the inverse mapping H_t^{-1} maps plaques into plaques in a neighbourhood of the point $H_t(y)$, thus the leaves L_y and $L_{H_t(y)}$ are of the same dimension. □

We have just demonstrated that any point $x \in \Sigma_r \subset M$ admits an open neighbourhood U diffeomorphic to $P_x \times B^r_x(O,\rho)$ where P_x is an open contractible neighbourhood of x in the leaf L_x and $B^r_x(O,\rho)$ is a ball in Q^r_x. Therefore any curve in U is homotopic relative to its ends to a curve of the form $\beta{*}\alpha$ where α is a leaf curve and β a geodesic tangent to Q. Let γ be any curve in M, then γ can be covered by a finite number of open sets of the form of Lemma 2. It means that γ is homotopic relative to its ends to a piecewise regular curve of the form $\beta_k{*}\alpha_k{*}\dots{*}\beta_1{*}\alpha_1$ where α_i are leaf curves and β_i are segments of geodesics tangent to Q.

Example 1. Let G be a group isometries of a semi-Riemannian manifold M. Consider the foliation defined by the action of the group G on M. If the orbits are space-like then the Levi–Civita connection is an adapted connection for the bundle Q defined as the orthogonal complement of the tangent spaces to the orbits.

Example 2. Let (M,D) be an affine manifold and G a group of affine transformations of (M,D). First, assume that the orbits are of the same dimension and that it admits a supplementary invariant and integrable totally geodesic subbundle. The so-called Atiyah–Molino class of this action supplies an obstruction to the existence of such a transverse subbundle, cf. [8], [18]. Then, of course, our subbundle is adapted to the foliation by orbits. The study of such group actions and associated foliations (often transversely affine) is of practical interest as such structures are natural generalisations of bilagrangian structures – when one retains only the affine structures, cf. [5], [11], [22], [23], [12]. One can call such foliations singular transversely affine foliations.

2. Transversely integrable foliations

Following Boualem we call an adapted connection ∇ on the foliated stratified manifold $(M,\Sigma,\mathcal{F})$ *transversely integrable* if the transverse subbundle Q^k on the principal stratum Σ_k is integrable, i.e. on Σ_k we have a pair of transverse regular foliations and Q^k is a (reduced) Ehresmann connection for $\mathcal{F}_k$. Therefore we have the following, cf. [1], [2]:

– the universal coverings of leaves of $\mathcal{F}_k$ (resp. Q^k) are diffeomorphic;

– any leaf of $\mathcal{F}_k$ intersects any leaf of Q^k.

Let x be any point of M and let L_x be a leaf of the stratum Σ_r. When restricted to some ball $B(O,\rho)$ in Q^r_x, the mapping $\exp_x|_{Q^r_x} : Q^r_x \to M$ is a diffeomorphism onto the image. The image is a q_r-dimensional submanifold V_x of M containing the point x such that $T_xV_x = Q^r_x$. The following lemma formulates a very important property of these "sections"; compare "Lemme fondamental" 1.2.1 of [3].

Lemma 5. *Let y be a regular point of V_x (i.e. $y \in V_x \cap \Sigma_k$). Then $Q^k_y \subset T_yV_x$.*

Proof. Let $\gamma : [0,1] \to M$ be a geodesic joining x to y, $\dot\gamma(0) = v \in B(O,\rho)$ $y = \gamma(1) = \exp_x(v)$. γ is traced in V_x. The homothety lemma ensures that the points $\gamma(t)$ $t \in (0,1]$, are regular, so they belong to the same leaf K_y of Q^k passing through the point y. It is worth recalling that the foliation Q^k is totally geodesic with respect to ∇.

In a neighbourhood of the origin there exists exactly one vector $Y_t \in Q^k_{\gamma(t)}$ such that $\exp_{\gamma(t)}(Y_t) = y$.

Obviously

$$[T_{Y_t}\exp_{\gamma(t)}](T_{Y_t}Q^k_{\gamma(t)}) = T_yK_y = Q^k_y.$$

Thus

$$T_{Y_t}Q^k_{\gamma(t)} = [T_{Y_t}\exp_{\gamma(t)}]^{-1}(T_yK_y).$$

Let us go to 0 with t. Then $\gamma(t) \to x$, $Y_t \to v$. Thus $T_{Y_t}Q^k_{\gamma(t)}$ goes to a q_k-dimensional subspace $W \subset T_vT_xM$, but we know that $\lim_{t\to 0} Q^k_{\gamma t)} = W_0 \subset Q^r_x$. Hence $W \subset T_vQ^r_x$, but $[T_v\exp_x]^{-1}(Q^k_y) = W$. Therefore $Q^k_y \subset T_v\exp_x(T_v(Q^r_x)) = T_yV_x$. □

As in the case of SRFs the homothety lemma and the above lemma ensure that the trace of the singular stratum Σ^s on V_x is the image of a closed vector subspace X of $W \subset Q^r_x$, i.e.

$$\Sigma^s \cap V_x = \exp_x(X).$$

Thus the traces of leaves of Q^k on V_x are of the form $\exp_x(W-X)$ (the sets $W - X$ are connected unless the subspace X is of codimension 1 in W). Thus any leaf of Q^k can be extended to a generalised section, i.e. an immersed submanifold of M which meets any leaf of $\mathcal{F}$, compare [3], [13]. The above considerations permit us to formulate the following theorem:

Theorem 2. *Let $(M, \Sigma, \mathcal{F})$ be a foliated stratified manifold. If ∇ is a transversely integrable adapted connection on $(M, \Sigma, \mathcal{F})$, then the foliation admits generalised sections.*

Proof. We adopt the notation of Lemma 5. Let us take a leaf of Q^k. Then its trace on V_x is of the form $\exp_x(W - X)$. We extend it over the singular stratum by taking $\exp_x(W)$. So the extension remains a submanifold. Problems appear when the leaf returns to the section V_x. The extension at this stage can intersect our previous extension, so the result is an immersed submanifold K only.

Our extended immersed submanifold K meets any leaf of $\mathcal{F}_k$. We have only to check that it meets any leaf of $\mathcal{F}$ in the singular stratum.

Let L be any leaf in the singular set Σ^s and let x be a point of L. Let $\gamma : [a, b] \to M$ be a Q-geodesic starting at x and ending at a point y of the regular stratum Σ_k. Then there exists a leaf curve $\alpha : [0, 1] \to L_y \subset M$ starting at the point y and ending at a point z of $K \cap \Sigma_k$ (it is true as any leaf L of $\mathcal{F}$ in the regular stratum meets any leaf of Q^k). Let us extend the pair (α, γ^{-1}) to the rectangle σ. The curve σ_b is a leaf curve in L_x ending at the point $\sigma^1(b)$ which is a point of K, as σ^1 is a Q-geodesic and as such it must be a curve in the leaf of Q^k passing through the point z, and thus contained in K. □

To complete our study of foliated stratified manifolds with adapted connections we would like to investigate the structure of the universal covering of the manifold, compare [1], [3].

We have already demonstrated that leaves of $\mathcal{F}$ have the same universal covering space. Let us have a look at the generalised section of

Theorem 2, the extensions of leaves of Q^k. Let K be such a generalised section and let x be a point of $\Sigma_k \cap K$. Then any curve γ in K starting at x is homotopic to piecewise regular curve whose pieces are Q-geodesics with ends in the regular stratum with the possible exception of the last piece if the end point of the curve is in the singular stratum. Moreover if two such curves are homotopic relative to its ends, one can construct a homotopy between these curves consisting of curves of the same type.

Therefore the standard procedure of the theory of Ehresmann connections, cf. [2], [3], permits us to prove that the generalised sections have the same universal covering space $\tilde{K}$.

Let us take the Cartesian product $\hat{M} = \tilde{L} \times \tilde{K}$ of $\tilde{L}$ the universal covering space of leaves of $\mathcal{F}_k$ and $\tilde{K}$ the universal covering space of generalised section of Theorem 2. Let us choose a point $x_0 \in \Sigma_k \subset M$. Let L_0 be the leaf of $\mathcal{F}$ passing through x_0 and K the generalised section passing through the point x_0. We identify $\tilde{L}$ with the universal covering of the leaf L_0 and $\tilde{K}$ with the universal covering of K. Then any point of $\hat{M}$ is represented by a pair of curves (α, β) such that

i) $\alpha : [a, b] \to L_0$;

ii) $\beta : [0, 1] \to K_0$;

iii) $\alpha(a) = \beta(0) = x_0 \in L_0 \cap K_0$;

iv) β is a piece-wise Q-geodesic.

We can define a "natural" smooth mapping h from $\hat{M}$ into $\tilde{M}$: Let $\sigma_{(\alpha,\beta)}$ be the rectangle defined by the pair (α, β), $\sigma : [a, b] \times [0, 1] \to M$. Put $h((\alpha, \beta)) = \sigma^b{*}\alpha$. It is not difficult to see that the mapping h is well defined and smooth. Moreover h is surjective due to the considerations following Lemma 4.

Directly from the definition of an adapted connection one gets that the mapping h is foliated for the natural product foliation of $\hat{M}$ with leaves of the form $\tilde{L} \times \{k\}$, $k \in \tilde{K}$.

Therefore we have the following theorem.

Theorem 3. *Let* $(M, \Sigma, \mathcal{F})$ *be a stratified foliated manifold with a transversely integrable adapted connection* ∇*. Then there exists a blow-up* $\hat{M}$ *of* M *diffeomorphic to the foliated manifold* $\tilde{L} \times \tilde{K}$*, where* $\tilde{L}$ *is the universal covering space of leaves of the foliation on the regular stratum* *and $\tilde{K}$ is the universal covering space of generalised sections, and there is a smooth foliated mapping $h : \hat{M} \to M$ sending $\tilde{L} \times \{k\}$ into leaves of $\mathcal{F}$.*

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