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Some global properties of quasi Einstein manifolds

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Abstract. In this paper some global properties of a compact, orientable quasi Einstein manifold without bourdary are obtained.

Introduction

The notion of a quasi Einstein manifold was introduced in a recent paper [1] by the first author M. C. Chaki and R. K. Maity. According to them a non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \quad X, Y, \dots \in \mathfrak{X}(M) \tag{1}$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X), \quad \forall X, \tag{2}$$

U being a unit vector field. In such a case, a, b are called the associated scalars, A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol $(QE)_n$.

In this paper some global properties of a compact orientable $(QE)_n$ without boundary are obtained.

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1. Preliminaries

From (1) we get

$$S(X,X) = ag(X,X) + bA(X)A(X) = a|X|^2 + b[g(X,U)]^2 \quad \forall X. \quad (1.1)$$

Let θ be the angle between U and any vector X. Then

$$\cos\theta = \frac{g(X,U)}{\sqrt{g(U,U)}\,\sqrt{g(X,X)}} = \frac{g(X,U)}{\sqrt{g(X,X)}} \; [\because \text{ by hypothesis } g(U,U) = 1].$$

Hence $[g(X,U)]^2 \leq g(X,X) = |X|^2$. Therefore

$$|X|^2 \geq [g(X,U)]^2 \quad (1.2)$$

Hence if $a > 0$, then $a|X|^2 \geq a[g(X,U)]^2$. Adding $b[g(X,U)]^2$ on both sides we get

$$a|X|^2 + b[g(X,U)]^2 \geq (a+b)[g(X,U)]^2.$$

Hence $S(X,X) \geq (a+b)[g(X,U)]^2$. Again if $b > 0$, then from (1.2) we get $b|X|^2 \geq b[g(X,U)]^2$. Adding $a|X|^2$ on both sides we have

$$(a+b)|X|^2 \geq a|X|^2 + b[g(X,U)^2 \geq S(X,X).$$

Thus

$$S(X,X) \geq (a+b)[g(X,U)]^2 \quad \text{when} \quad a > 0$$

and

$$S(X,X) \leq (a+b)|X|^2 \quad \text{when} \quad b > 0. \quad (1.3)$$

Next, contracting (1) over X and Y we get

$$r = na + b, \quad (1.4)$$

where r is the scalar curvature.

These results will be used in the sequel.

2. Sufficient condition for a compact orientable $(QE)_n$ to be conformal to a sphere in E_{n+1}

We begin with the definition of conformality of one Riemannian manifold to another.

Let (M,g) and (M',g') be two n-dimensional Riemannian manifolds. If there exists a one-one differentiable mapping $(M,g) \to (M',g')$ such that the angle between any two vectors at a point p of M is always equal to that of the corresponding two vectors at the corresponding point p' of M', then (M,g) is said to be conformal to (M',g'). Y. Watanabe [2] has given a sufficient condition of conformality of an n-dimensional Riemannian manifold to an n-dimensional sphere immersed in E_{n+1}. Its statement is as follows:

If in an n-dimensional Riemannian manifold M, there exists a non parallel vector field X such that the condition

$$\int_M S(X,X)dv = 1/2\int_M |dX|^2 dv + n - 1/n \int_M (\partial X)^2 dv \tag{2.1}$$

holds, then M is conformal to a sphere in E_{n+1}, where dv is the volume element of M and dX and ∂X are the curl and divergence of X respectively.

In this section we consider a compact and orientable quasi Einstein manifold $(QE)_n = M$ without boundary having associated scalars a, b and generator U. It satisfies (1) and (2). Hence

$$S(U,U) = a + b.$$

In virtue of this and by taking U for X the condition (2.1) takes the following form

$$\int_M (a+b)dv = 1/2\int_M |dU|^2 dv + (n-1)/n \int_M (\partial U)^2 dv. \tag{2.2}$$

We now suppose that $U = \operatorname{grad} f$. Then U cannot be parallel, for othetwise $\nabla U = 0$ or $\nabla \operatorname{grad} f = 0$ or $\Delta f = 0$, where Δ denotes Laplacian of f, and ∇ denotes the covariant differentiation with respect to the metric of M. Hence by Bochner's lemma [3, p. 39], f is constant, which implies that $U = 0$ which is not admissible.

Since by assumption $U = \operatorname{grad} f$, $|dU|^2 = 0$. Hence (2.2) takes the form

$$\int_M (a+b)dU = (n-1)/n \int_M (\partial U)^2 dv. \tag{2.3}$$

Thus if in an n-dimensional compact orientable quasi Einstein manifold M without boundary the generator U is grad f, then U is a non-parallel vector field. If in such a case the condition (2.3) is satisfied, then by Watanabe's condition (2.1) M is conformal to a sphere in E_{n+1}. We can therefore state the following.

Theorem 1. *If in a compact, orientable quasi Einstein manifold $M = (QE)_n$ $(n \geq 3)$ without boundary having a, b as associated scalars, the generator U is the gradient of a function and satisfies the condition* (2.3), *then the manifold $(QE)_n$ is conformal to a sphere immersed in E_{n+1}.*

3. Killing vector field in a compact orientable $(EQ)_n$ $(n \geq 3)$ without boundary

In this section we consider a compact, orientable $(QE)_n = M$ $(n \geq 3)$ without boundary with a, b as associated scalars and U as the generator.

It is known ([2] or [3] p. 43) that in such a manifold M the following relation holds.

$$\int_M [S(X,X) - |\nabla X|^2 - (\operatorname{div} X)^2] dv = 0, \quad \forall X \tag{3.1}$$

If X is a Killing vector field, then $\operatorname{div} X = 0$ ([3], p. 43). Hence (3.1) takes the form

$$\int_M [S(X,X) - |\nabla X|^2] dv = 0. \tag{3.2}$$

Let $b > 0$. Then by (1.3) $(a+b)|X|^2 \geq S(X,X)$. Therefore $(a+b)|X|^2 - |\nabla X|^2 \geq S(X,X) - |\nabla X|^2$. Consequently,

$$\int_M [(a+b)|X|^2 - |\nabla X|^2] dv \geq \int_M [S(X,X) - |\nabla X|^2] dv$$

and by (3.2)

$$\int_M (a+b)|X|^2 - |\nabla X|^2 dv \geq 0.$$

If $a + b < 0$, then

$$\int_M [(a+b)|X|^2 - |\nabla X|^2 dv = 0.$$

Therefore $X = 0$. This leads to the following result:

Theorem 2. *If in a compact, orientable $(QE)_n$ $(n \geq 3)$ without boundary the associated scalars are such that $b > 0$ and $a + b < 0$, then there exists no nonzero Killing vector field in this manifold.*

We now state the following two corollaries of the above theorem which follow easily from it.

Corollary 1. *If in a compact, orientable $M = (QE)_n$ $(n \geq 3)$ without boundary, the generator U is a Killing vector field, then the following relation holds.*

$$\int_M [(a+b) - |\nabla U|^2]\, dv \geq 0.$$

Corollary 2. *If in a compact, orientable $M = (QE)_n$ $(n \geq 3)$ without boundary $b > 0$ and $a+b < 0$, then the generator U of the manifold cannot be a Killing vector field.*

4. Killing p-form in a compact, orientable conformally flat $M = (QE)_n$ $(n > 3)$ without boundary

Let ω be a p-form in a compact orientable conformally flat $M = (QE)_n$ $(n > 3)$ without boundary and $F_p(\omega, \omega)$ be the well known [3] quadratic form given by

$$F_p(\omega,\omega) = S_{ij}\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} + \frac{p-1}{2} R_{ijk\ell}\omega^{iji_3\ldots i_p}\,\omega^{k\ell}i_3\ldots i_p, \tag{4.1}$$

where $R_{ijk\ell}$ and S_{ij} are the components of the curvature tensor R of type $(0,4)$ and the Ricci tensor S of type $(0,2)$ of the $(QE)_n$.

Since $(QE)_n$ is conformally flat the tensor R can be expressed as follows ([4], p. 234; [5], p. 40]:

$$\begin{aligned} R(X,Y,Z,W) = {} & \frac{1}{n-2}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & + g(X,W)S(Y,Z) - g(Y,W)S(X,Z)] \\ & - \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{4.2}$$

By (1)

$$R(X,Y,Z,W) = a'[g(Y,Z)g(X,W) - g(Y,W)g(X,Z)] + b'[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) + g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)],$$

where

$$a' = \frac{a}{n-1} - \frac{b}{(n-1)(n-2)}, \qquad b' = \frac{b}{n-2}. \tag{4.3}$$

In virtue of (1) and (4.2) we can express (4.1) as follows:

$$F_p(\omega,\omega) = [(n-p)a' + b']|\omega|^2 + (n-2p)b'[U.\omega]^2, \tag{4.4}$$

where the components of ω are $\omega_{i_1 i_2 \dots i_p}$, those of U are U^i and $U.\omega$ is a tensor of type $(0, p-1)$ with components $U^j \omega_{j i_1 \dots i_{p-1}}$ and $|\omega|^2 = \omega_{i_1 i_2 \dots i_p} \omega^{i_1 i_2 \dots i_p}$.

Using (4.3) the relation (4.4) can be expressed as follows:

$$F_p(\omega,\omega) = \frac{[(n-p)(n-2)a + b(p-1)]}{(n-1)(n-2)}|\omega|^2 + (n-2p)\frac{b}{n-2}[U.\omega]^2 \tag{4.5}$$

We now suppose that ω is a Killing p-form. Then it is known [3] that

$$\int_{(QE)_n} [F_p(\omega,\omega) - |\nabla\omega|^2] dv = 0. \tag{4.6}$$

In virtue of (4.5) we can express (4.6) as follows:

$$\int_{(QE)_n} \left[\frac{(n-p)(n-2)a + b(p-1)}{(n-1)(n-2)}|\omega|^2 + \frac{n-2p}{n-2} b[U.\omega]^2 - |\nabla\omega|^2 \right] dv = 0. \tag{4.7}$$

If $(n-2p)b < 0$ and $(n-p)(n-2)a + b(p-1) < 0$ then from (4.7) it follows that $\omega = 0$. This leads to the following result.

Theorem 3. *If in a compact, orientable conformally flat* $(QE)_n$ $(n>3)$ *without boundary,* $(n-2p)b < 0$ *and* $(n-p)(n-2)a + b(p-1) < 0$*, where* $p > 1$ *but* $< n$*, then there exists no nonzero Killing* p*-form in such a manifold.*

References

[1] M. C. Chaki and R. K. Maity, On quasi Einstein manifolds, *Publ. Math. Debrecen* **57** (2000), 297–306.

[2] Y. Watanabe, Integral inequalities in compact orientable manifold, *Riemannian or Kählerian. Kodai Math. Sem. Report* **20** (1968), 264–271.

[3] K. Yano, Integral formulas in Riemannian Geometry, *Marcel Dekker*, *New York*, 1970.

[4] M. C. Chaki, A Text book of tensor Calculus. 3rd, Ed., *Calcutta Publishers*, 2000.

[5] K. Yano and M. Kon, Structures on manifolds, World Scientific, 1994.

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