Publ. Math. Debrecen
42 / 3–4 (1993), 397–407

Univoque sequences

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Dedicated to Professor Lajos Tamássy on his 70th birthday

1. Introduction

Let $1 < q \le 2$. For $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) \in \{0,1\}^{\mathbb{N}}$ put

$$\langle \varepsilon, q \rangle := \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q^n}. \tag{1.1}$$

We call the sequence ε *univoque* with respect to q if the equality

$$\langle \varepsilon, q \rangle = \langle \delta, q \rangle \qquad \bigl(\delta \in \{0,1\}^{\mathbb{N}}\bigr) \tag{1.2}$$

holds only for $\delta = \varepsilon$ (i.e., $\delta_n = \varepsilon_n$ for all $n \in \mathbb{N}$).

If $1 < q \le 2$, then

$$L(q) := \sum_{n=1}^{\infty} \frac{1}{q^n} = \frac{1}{q-1}.$$

As it is known, for any $x \in [0, L(q)]$ there exists an $\varepsilon \in \{0,1\}^{\mathbb{N}}$ such that $x = \langle \varepsilon, q \rangle$. For $x \in [0, L(q)]$, we define, by induction on n, the sequence

$$\varepsilon_n(x) = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^{n-1} \frac{\varepsilon_i(x)}{q^i} + \frac{1}{q^n} \le x\,, \\ 0 & \text{if} \quad \sum_{i=1}^{n-1} \frac{\varepsilon_i(x)}{q^i} + \frac{1}{q^n} > x. \end{cases} \tag{1.3}$$

This work was supported by the National Science Foundation Grant OTKA 1652.

Then, for the sequence $\varepsilon(x) := (\varepsilon_1(x), \varepsilon_2(x), \dots)$, we have

$$x = \langle \varepsilon(x), q \rangle, \tag{1.4}$$

and we call the equality (1.4) the *regular* expansion of x with respect to q ([1], [2]).

Let $R(q)$ denote the set of all those sequences $\varepsilon(x)$ which consist of the digits $0, 1$ occurring in the regular expansion of some $x \in [0, L(q)]$, i.e., let

$$R(q) = \{\varepsilon(x) | \quad x \in [0, L(q)]\}. \tag{1.5}$$

Moreover, let us denote by $U(q)$ the set of sequences univoque with respect to q, i.e., let

$$U(q) = \left\{\varepsilon | \quad \varepsilon \in \{0,1\}^{\mathbb{N}}, \ \ \varepsilon \text{ is univoque with respect to } q\right\}. \tag{1.6}$$

In this paper, we investigate the set $U(q)$ of univoque sequences for $q \in]1, 2]$.

2. Characterization of univoque sequences

By definition, the sequences $\underline{0} := (0, 0, \dots)$ and $\underline{1} := (1, 1, \dots)$ are univoque with respect to any $q \in]1, 2]$. If $\varepsilon \in \{0,1\}^{\mathbb{N}}$, then $\underline{1} - \varepsilon = = (1 - \varepsilon_1, 1 - \varepsilon_2, \dots) \in \{0,1\}^{\mathbb{N}}$.

Theorem 2.1. *The sequence $\varepsilon \in \{0,1\}^{\mathbb{N}}$ is univoque with respect to q if and only if $\varepsilon \in R(q)$ and $(\underline{1} - \varepsilon) \in R(q)$.*

Proof. (1) Let the sequence ε be univoque with respect to q, i.e., $\varepsilon \in U(q)$. Then, with the notation $x := \langle \varepsilon, q \rangle$, we necessarily have $\varepsilon = \varepsilon(x)$, i.e. , $\varepsilon \in R(q)$. On the other hand, if $\langle \underline{1} - \varepsilon, q \rangle = \langle \delta, q \rangle$ for some $\delta \in \{0,1\}^{\mathbb{N}}$, and $\underline{1} - \varepsilon \neq \delta$, then

$$\langle \underline{1} - \delta, q \rangle = \langle \varepsilon, q \rangle$$

would hold with $\underline{1} - \delta \neq \varepsilon$, a contradiction, because ε is a univoque sequence. Hence, $(\underline{1} - \varepsilon) \in U(q)$, and this implies $(\underline{1} - \varepsilon) \in R(q)$.

(2) Let us suppose that $\varepsilon \in R(q)$ and $(\underline{1} - \varepsilon) \in R(q)$. Then, putting

$$x := \langle \varepsilon, q \rangle,$$

we have $\varepsilon = \varepsilon(x)$, and by

$$L(q) - x = \langle \underline{1} - \varepsilon, q \rangle,$$

we get $\underline{1} - \varepsilon = \varepsilon(L(q) - x)$, i.e. ,

$$\varepsilon = \varepsilon(x) = \underline{1} - \varepsilon(L(q) - x). \tag{2.1}$$

Suppose now that, contrarily to our assertion, there exists a $\delta \in \{0,1\}^{\mathbb{N}}$ such that $\delta \neq \varepsilon$ and

$$x = \langle \varepsilon, q\rangle = \langle \delta, q\rangle. \tag{2.2}$$

Then, there exists a smallest $N \in \mathbb{N}$ such that $\delta_i = \varepsilon_i(x) = 1-\varepsilon_i(L(q)-x)$ for $i = 1, 2, \dots, N-1$ and $\delta_N \neq \varepsilon_N(x) = 1-\varepsilon_N(L(q)-x)$. By the 'greedy' property of the regular expansion, $\delta_N = 0$ and $\varepsilon_N(x) = 1-\varepsilon_N(L(q)-x) = 1$ necessarily hold. Hence,

$$x - \sum_{i=1}^{N-1} \frac{\varepsilon_i(x)}{q^i} = \frac{1}{q^N} + \sum_{i=N+1}^{\infty} \frac{\varepsilon_i(x)}{q^i} = \frac{0}{q^N} + \sum_{i=N+1}^{\infty} \frac{\delta_i}{q^i}, \tag{2.3}$$

with $\varepsilon_i(x) = 1 - \varepsilon_i(L(q) - x)$ for $i \geq N + 1$. On the other hand, from $1 - \varepsilon_N(L(q) - x) = 1$, we get $\varepsilon_N(L(q) - x) = 0$, i.e.,

$$\sum_{i=1}^{N-1} \frac{\varepsilon_i(L(q) - x)}{q^i} + \frac{1}{q^N} > L(q) - x.$$

This implies

$$x - \sum_{i=1}^{N-1} \frac{\varepsilon_i(x)}{q^i} > \sum_{i=N+1}^{\infty} \frac{1}{q^i}.$$

Hence, by (2.3),

$$\sum_{i=N+1}^{\infty} \frac{\delta_i}{q^i} > \sum_{i=N+1}^{\infty} \frac{1}{q^i},$$

a contradiction. Thus, (2.2) can be valid only if $\varepsilon = \delta$, i.e. $\varepsilon \in U(q)$. □

Lemma 2.2. *Let $\varepsilon \in \{0,1\}^{\mathbb{N}}$ be such that $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$. Then the sequence ε is an element of $R(q)$ if and only if the inequalities*

$$\frac{1}{q} + \sum_{j=2}^{\infty} \frac{\varepsilon_{p+j}}{q^j} < 1 \qquad \textit{if} \;\; \varepsilon_p = 0 \;\; \textit{and} \;\; \varepsilon_{p+1} = 1 \tag{2.4}$$

are satisfied.

Proof. (1) If $\varepsilon \in R(q)$ such that $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$, and moreover $\varepsilon_p = 0$ and $\varepsilon_{p+1} = 1$, then putting

$$x = \langle \varepsilon, q\rangle, \tag{2.5}$$

we have $\varepsilon = \varepsilon(x)$. Hence, by the algorithm (1.3) ,

$$\sum_{i=1}^{p-1} \frac{\varepsilon_i}{q^i} + \frac{1}{q^p} > x = \sum_{i=1}^{p-1} \frac{\varepsilon_i}{q^i} + \frac{0}{q^p} + \frac{1}{q^{p+1}} + \sum_{i=p+2}^{\infty} \frac{\varepsilon_i}{q^i}, \tag{2.6}$$

and from this, (2.4) immediately follows.

(2) If (2.4) holds for the sequence $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$, then for x as defined by (2.5), we have (2.6), and using the algorithm (1.3), we infer from this, by some computation, that $\varepsilon = \varepsilon(x)$, i.e., $\varepsilon \in R(q)$. □

Theorem 2.3. *Let $\varepsilon \in \{0,1\}^{\mathbb{N}}$ be such that $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$. Then the sequence ε is univoque with respect to q if and only if the inequalities*

$$\frac{1}{q} + \sum_{j=2}^{\infty} \frac{\varepsilon_{p+j}}{q^j} < 1 \qquad \text{if } \ \varepsilon_p = 0 \ \text{ and } \ \varepsilon_{p+1} = 1 \tag{2.7}$$

and

$$\frac{1}{q} + \sum_{j=2}^{\infty} \frac{1-\varepsilon_{r+j}}{q^j} < 1 \qquad \text{if } \ \varepsilon_r = 1 \ \text{ and } \ \varepsilon_{r+1} = 0 \tag{2.8}$$

hold.

Proof. By Theorem 2.1, the relation $\varepsilon \in U(q)$ holds if and only if $\varepsilon \in R(q)$ and $(\underline{1} - \varepsilon) \in R(q)$. Applying Lemma 2.2, we obtain the inequalities (2.7) and (2.8). □

Theorem 2.4. *Let $1 < q < q' \leq 2$. Then*

$$U(q) \subset U(q'). \tag{2.9}$$

Proof. If $\varepsilon = \underline{0}$ or $\varepsilon = \underline{1}$, then ε belongs to both $U(q)$ and $U(q')$. If $\varepsilon \neq \underline{0}$, $\varepsilon \neq \underline{1}$ and $\varepsilon \in U(q)$, then the inequalities (2.7) and (2.8) are satisfied. In view of $1/q' < 1/q$, the inequalities (2.7) and (2.8) remain valid if we replace q by q'. Hence by Theorem 2.3, $\varepsilon \in U(q')$. □

3. Stable numbers

By Theorem 2.4, the set $U(q)$ of univoque sequences can only become larger as q increases. We shall need the following

Definition 3.1. We call the number $q \in]1,2[$ *stable* if there exists $d > 0$ such that $q + d < 2$ and

$$U(q) = U(q+d). \tag{3.1}$$

Let $k \geq 0$ be a natural number and

$$P_k(t) = \frac{1}{t^1} + \frac{1}{t^2} + \cdots + \frac{1}{t^k} \tag{3.2}$$

if $t \in [1,2]$. Moreover, let

$$Q_k(t) = P_k(t) + \frac{1}{t^{2k}} \tag{3.3}$$

if $t \in [1,2]$.

The functions $P_k,\ Q_k : [1,2] \to \mathbb{R}$ are strictly monotone decreasing and continuous in $[1,2]$, and satisfy

$$P_k(1) = k, \quad Q_k(1) = k+1 \quad \text{and} \quad 0 < P_k(2) < 1, \quad 0 < Q_k(2) < 1.$$

Moreover, $P_k(t) < Q_k(t)$ holds for all $t \in [1,2]$.

Hence, there exist the uniquely determined values $q^*(k)$ and $q(k)$ in $[1,2]$ for which

$$P_k(q^*(k)) = Q_k(q(k)) = 1,$$

and for these

$$q(k-1) < q^*(k) < q(k) \qquad (k \geq 3)$$

holds.

On the other hand,

$$1 = q^*(1) < q^*(2) < q^*(3) < \dots$$

and

$$\frac{1+\sqrt{5}}{2} = q^*(2) = q(1) < q(2) < q(3) < \dots,$$

moreover $\lim\limits_{k\to\infty} q^*(k) = \lim\limits_{k\to\infty} q(k) = 2$.

Theorem 3.2. *If $1 < q < q(1) = \frac{1+\sqrt{5}}{2}$, then q is a stable number and*

$$U(q) = \{\underline{0}, \underline{1}\} = U(q(1)).$$

Moreover, the number $q(1)$ is not stable.

Proof. By Theorem 2.4, it suffices to prove that $U(q(1)) = \{\underline{0}, \underline{1}\}$, and $q(1)$ is not stable. Then

$$1 = \frac{1}{q(1)} + \frac{1}{q(1)^2}.$$

Hence, if $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$, moreover ε is a univoque sequence with respect to $q(1)$, then by Theorem 2.3 the inequalities (2.7) and (2.8) are valid for $q := q(1)$. Now, there are two possibilities:

(1) $\varepsilon_1 = 0$ and there exists a smallest natural number p such that $\varepsilon_p = 0$ and $\varepsilon_{p+1} = 1$; or else

(2) $\varepsilon_1 = 1$ and there exists a smallest natural number r such that $\varepsilon_r = 1$ and $\varepsilon_{r+1} = 0$.

In case (1), the inequality (2.7) yields

$$\frac{1}{q(1)} + \sum_{j=2}^{\infty} \frac{\varepsilon_{p+j}}{q(1)^j} < 1 = \frac{1}{q(1)} + \frac{1}{q(1)^2},$$

i.e., necessarily $\varepsilon_{p+2} = 0$. Hence, putting $r := p+1$, we get $\varepsilon_r = 1$ and $\varepsilon_{r+1} = 0$, and by (2.8) this implies

$$\frac{1}{q(1)} + \sum_{j=2}^{\infty} \frac{1-\varepsilon_{r+j}}{q(1)^j} < 1 = \frac{1}{q(1)} + \frac{1}{q(1)^2},$$

whence $1 - \varepsilon_{r+2} = 0$, i.e., $\varepsilon_{p+3} = \varepsilon_{r+2} = 1$ follows. Continuing this procedure, we see that in case (1) the sequence univoque with respect to $q(1)$ is of the form

$$(\varepsilon_1, \varepsilon_2, \dots) = (0, 0, \dots, \overset{p}{\overbrace{0}}, \overset{p+1}{\overbrace{0}}, 0, 1, 0, 1, \dots). \tag{3.4}$$

Similarly, we obtain that, in case (2), the sequence univoque with respect to $q(1)$ is of the form

$$(\varepsilon_1, \varepsilon_2, \dots) = (1, 1, \dots, \overset{r}{\overbrace{1}}, \overset{r+1}{\overbrace{0}}, 1, 0, 1, 0, \dots). \tag{3.5}$$

The considerations, effected so far, show that if, contrarily to our assertion, there exists a sequence $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$ univoque with respect to $q(1)$, then this sequence is necessarily of the form (3.4) or (3.5). Let us now show that this is impossible, at the same time the sequences of the form (3.4) or (3.5) are, for any $q \in]q(1), 2]$, univoque sequences with respect to q, i.e., $q(1)$ is not stable. As a matter of fact, by Theorem 2.3, (3.4) or (3.5) is univoque with respect to q if and only if

$$\frac{1}{q} + \frac{1}{q^3} + \frac{1}{q^5} + \cdots < 1,$$

i.e., $q/(q^2-1) < 1$, whence $q^2 - q - 1 > 0$. However, this latter condition is satisfied if and only if $1/q + 1/q^2 < 1$, i.e., $q > q(1) = (1+\sqrt{5})/2$. □

Remark. In another form, Theorem 3.2 has been stated and proved in our paper [3] too. The method of proof given here will, however, be applicable also in a more general situation.

4. Properties of the set $U(q)$

In the course of further investigations concerning the set $U(q)$, a useful role will be played by the following

Lemma 4.1. *If $1 < q \leq 2$, then $U(q)$ has the following properties :*

(1) $\underline{0}, \underline{1} \in U(q)$;

(2) If $\varepsilon \in U(q)$, then $(\underline{1} - \varepsilon) \in U(q)$;

(3) If $\varepsilon \in U(q)$, then $T\varepsilon \in U(q)$, where T is the shift operator defined by the equations $(T\varepsilon)_i := \varepsilon_{i+1}$ $(i \in \mathbb{N})$;

(4) If $\varepsilon \in U(q)$ such that $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$, then the sets $1_\varepsilon := \{n| \quad \varepsilon_n = 1\}$ and $0_\varepsilon := \{n| \quad \varepsilon_n = 0\}$ are infinite.

Proof. Properties (1) and (2) are clear. If $\varepsilon \in U(q)$ and $x := \langle \varepsilon, q\rangle = \sum_{n=1}^\infty \frac{\varepsilon_n}{q^n}$, then let $y := \langle T\varepsilon, q\rangle = \sum_{n=1}^\infty \frac{\varepsilon_{n+1}}{q^n}$. Suppose that, contrarily to our assertion, there exists $\delta \in \{0,1\}^{\mathbb{N}}$ such that $T\varepsilon \neq \delta$ and $y = \langle \delta, q\rangle = = \sum_{n=1}^\infty \frac{\delta_n}{q^n}$. Then the sequence $\delta' := (\varepsilon_1, \delta_1, \delta_2, \dots) \in \{0,1\}^{\mathbb{N}}$ satisfies $\delta' \neq \varepsilon$, and we have

$$
\begin{aligned}
x = \langle \varepsilon,\ q\rangle = \sum_{n=1}^\infty \frac{\varepsilon_n}{q^n} = \frac{\varepsilon_1}{q} + \frac{\varepsilon_2}{q^2} + \frac{\varepsilon_3}{q^3} + \cdots = \\
= \frac{\varepsilon_1}{q} + \frac{1}{q}\left(\frac{\varepsilon_2}{q} + \frac{\varepsilon_3}{q^2} + \cdots\right) = \frac{\varepsilon_1}{q} + \frac{1}{q}\left(\frac{\delta_1}{q} + \frac{\delta_2}{q^2} + \cdots\right) = \\
= \frac{\varepsilon_1}{q} + \frac{\delta_1}{q^2} + \frac{\delta_2}{q^3} + \cdots = \langle \delta', q\rangle,
\end{aligned}
$$

a contradiction. This proves property (3).

Finally, (4) follows from the fact that if ε is a univoque sequence such that $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$, then for the digits $\varepsilon_n = \varepsilon_n(x)$ occurring in the regular expansion of the number $x := \langle \varepsilon, q\rangle$ the stated property is satisfied. □

Remark. Let $H \subset \{0,1\}^{\mathbb{N}}$ denote the set of all those sequences ε for which the sets $1_\varepsilon = \{n| \quad \varepsilon_n = 1\}$ and $0_\varepsilon = \{n| \quad \varepsilon_n = 0\}$ are infinite, moreover let $\underline{0}, \underline{1} \in H$. Then H satisfies the properties (1) – (4) , and one can easily see that that $H = U(2)$.

5. Definition and properties of the set H_k

Let $k \geq 2$ be a natural number. Let H_k denote the set of those sequences $\varepsilon \in \{0,1\}^{\mathbb{N}}$ for which the following property is satisfied:

If $\varepsilon_p = 0$, then $\varepsilon_{p+1} + \varepsilon_{p+2} + \cdots + \varepsilon_{p+k} \leq k-1$, and if $\varepsilon_p = 1$, then $\varepsilon_{p+1} + \varepsilon_{p+2} + \cdots + \varepsilon_{p+k} \geq 1$.

This means that the sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) \in \{0,1\}^{\mathbb{N}}$ will be an element of H_k if and only if it has the following property:

If some coordinate in ε is 0, then it cannot be followed by k consecutive coordinates 1, and conversely if some coordinate in ε is 1, then it cannot be followed by k consecutive coordinates 0.

Lemma 5.1. *The set H_k satisfies the following properties :*
(1) $\underline{0}, \underline{1} \in H_k$;
(2) If $\varepsilon \in H_k$, then $(\underline{1} - \varepsilon) \in H_k$;
(3) If $\varepsilon \in H_k$, then $T\varepsilon \in H_k$;

(4) If $\varepsilon \in H_k$ such that $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$, then the sets 1_ε and 0_ε are infinite.

Proof. Properties (1) – (3) follow immediately from the definition of H_k. If $\varepsilon \in H_k$ such that $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$, then there exists $p \in \mathbb{N}$ such that $\varepsilon_p = 0$ and $\varepsilon_{p+1} = 1$. Now, by the definition of H_k, any section of length k of the sequence ε contains both digits 0 and 1, and from this the validity of property (4) immediately follows. □

Lemma 5.2. *If $q^*(k) < q \leq 2$ for some $k \geq 2$, then*

$$H_k \subset U(q). \tag{5.1}$$

Proof. Let $\varepsilon \in H_k$. It will be sufficient to show that $q^*(k) < q \leq 2$ implies $\varepsilon \in R(q)$. Indeed, by $(\underline{1} - \varepsilon) \in H_k$, in this case, we also have $(\underline{1} - \varepsilon) \in R(q)$, and from this, in view of Theorem 2.1, $\varepsilon \in U(q)$ follows.

By Lemma 2.2, an $\varepsilon \in H_k$ with $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$ is an element of $R(q)$ if and only if

$$\frac{1}{q} + \sum_{j=2}^{\infty} \frac{\varepsilon_{p+j}}{q^j} < 1 \qquad \text{if } \varepsilon_p = 0 \text{ and } \varepsilon_{p+1} = 1.$$

Now, by $\varepsilon \in H_k$, for $\varepsilon_p = 0$ and $\varepsilon_{p+1} = 1$ we have

$$\begin{aligned} \frac{1}{q} + \sum_{j=2}^{\infty} \frac{\varepsilon_{p+j}}{q^j} &\leq \left(\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{k-1}}\right)\left(1 + \frac{1}{q^k} + \frac{1}{q^{2k}} + \cdots\right) = \\ &= \left(\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{k-1}}\right)\frac{q^k}{q^k - 1}, \end{aligned} \tag{5.2}$$

and for

$$T^{p-1}\varepsilon = (\varepsilon_p, \varepsilon_{p+1}, \dots) = (0, \underbrace{1, 1, \dots, 1}_{k-1}, 0, \underbrace{1, 1, \dots, 1}_{k-1}, 0, \dots) \in H_k$$

the equality sign holds in (5.2). On the other hand, one easily verifies that

$$\left(\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{k-1}}\right)\frac{q^k}{q^k - 1} < 1$$

is valid if and only if $q^*(k) < q \leq 2$. □

Lemma 5.3. *The number $q^*(k)$, for $k \geq 2$, is not stable.*

Proof. We have

$$\begin{aligned} 1 &= \frac{1}{q^*(k)} + \frac{1}{q^*(k)^2} + \cdots + \frac{1}{q^*(k)^k} = \\ &= \left(\frac{1}{q^*(k)} + \frac{1}{q^*(k)^2} + \cdots + \frac{1}{q^*(k)^{k-1}}\right)\left(1 + \frac{1}{q^*(k)} + \frac{1}{q^*(k)^2} + \cdots\right). \end{aligned}$$

From this, we infer that the sequence

$$\alpha := (\underbrace{1,1,\dots,1}_{k-1},0,\underbrace{1,1,\dots,1}_{k-1},0,\dots)$$

is not univoque with respect to $q^*(k)$. At the same time $\alpha \in H_k$, and by Lemma 5.2 this implies that $\alpha \in U(q)$ for $q \in]q^*(k), 2]$. This will say, however, that $q^*(k)$ is not stable. $\square$

Theorem 5.4. *If $q^* < q < q(k)$ for some $k \geq 2$, then q is a stable number and*

$$U(q) = U(q(k)) = H_k. \tag{5.3}$$

Moreover, the number $q(k)$ is not stable.

Proof. By Lemma 5.2, $H_k \subset U(q(k))$. Suppose that, contrarily to our assertion, there exists $\varepsilon \in U(q(k))$ satisfying $\varepsilon \notin H_k$. Then, there are two possibilities :

(1) there exists a smallest natural number p such that $\varepsilon_p = 0$ and $\varepsilon_{p+1} + \varepsilon_{p+2} + \dots + \varepsilon_{p+k} = k$, or else

(2) there exists a smallest natural number r such that $\varepsilon_r = 1$ and $\varepsilon_{r+1} + \varepsilon_{r+2} + \dots + \varepsilon_{r+k} = 0$.

The two cases can be dealt with in an analogous manner. Let us therefore suppose that the first case materializes (in the contrary case, we simply replace ε by $(\underline{1} - \varepsilon)$).

In case (1), $\varepsilon_{p+1} + \varepsilon_{p+2} + \dots + \varepsilon_{p+k} = 1$ necessarily holds. Hence, by the inequality (2.7), we have

$$\frac{1}{q(k)} + \frac{1}{q(k)^2} + \dots + \frac{1}{q(k)^k} + \sum_{j=k+1}^{\infty} \frac{\varepsilon_{p+j}}{q(k)^j} < 1 = \\ = \frac{1}{q(k)} + \frac{1}{q(k)^2} + \dots + \frac{1}{q(k)^k} + \frac{1}{q(k)^{2k}}.$$

This implies that necessarily $\varepsilon_{p+k+1} = \varepsilon_{p+k+2} = \dots = \varepsilon_{p+2k}$. Now, let $r := p + k$, then by $\varepsilon_r = 1$ and $\varepsilon_{r+1} = 0$ the inequality (2.8) yields

$$\frac{1}{q(k)} + \frac{1}{q(k)^2} + \dots + \frac{1}{q(k)^k} + \sum_{j=k+1}^{\infty} \frac{1 - \varepsilon_{r+j}}{q(k)^j} < 1 = \\ = \frac{1}{q(k)} + \frac{1}{q(k)^2} + \dots + \frac{1}{q(k)^k} + \frac{1}{q(k)^{2k}}.$$

From this, we infer that $1 - \varepsilon_{r+k+1} = 1 - \varepsilon_{r+k+2} = \dots = 1 - \varepsilon_{r+2k} = 0$, i.e., $\varepsilon_{r+k+1} = \varepsilon_{r+k+2} = \dots = \varepsilon_{r+2k} = 1$. Hence, by $r = p + k$, we get

$\varepsilon_{p+2k+1} = \varepsilon_{p+2k+2} = \cdots = \varepsilon_{p+3k} = 1$. Continuing this process, we see that in this case our sequence is of the form

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p-1}, \overbrace{0}^{p}, \underbrace{1, 1, \ldots, 1}_{k}, 0, \underbrace{1, 1, \ldots, 1}_{k}, 0, \ldots).$$

In view of $\varepsilon \in U(q(k))$, the sequence

$$T^p\varepsilon = (\underbrace{1, 1, \ldots, 1}_{k}, 0, \underbrace{1, 1, \ldots, 1}_{k}, 0, \ldots)$$

is also univoque with respect to $q(k)$. However, this is impossible since

$$1 = \langle T^p\varepsilon, q(k)\rangle = \langle \delta, q(k)\rangle,$$

where

$$\delta := (\underbrace{1, 1, \ldots, 1}_{k}, \underbrace{0, 0, \ldots, 0}_{k-1}, \overbrace{1}^{2k}, 0, 0, \ldots),$$

i.e., $T^p\varepsilon$ is not univoque with respect to $q(k)$. At the same time $T^p\varepsilon \in U(q)$ for any $q \in]q(k), 2]$, and this means that $q(k)$ is not stable. $\square$

Remark. From this theorem it follows that, in case $q(1) = q^*(2) < q \leq q(2)$, one has

$$U(q) = U(q(2)) = H_2.$$

Here any element ε of H_2 with $\varepsilon \neq \underline{0}$ and $\varepsilon \neq \underline{1}$ has the property that there exists an integer $p \geq 0$ such that

$$T^p\varepsilon = (1, 0, 1, 0, \ldots).$$

We have already remarked this fact in our paper [3]. The set H_2 is clearly countable, while the sets H_k, for $k \geq 3$, have already the power of the continuum.

Our investigations carried out so far can be summarized as follows : The elements of the set

$$S := \bigcup_{k=1}^{\infty}]q^*(k), q(k)[\subset]1, 2[$$

are stable numbers, while the numbers $q(k)$, for $k = 1, 2, \ldots,$ and the numbers $q^*(k)$, for $k = 2, 3, \ldots,$ are not stable.

If $q \in S$, then there exists $k \in \mathbb{N}$ such that $q^*(k) < q < q(k)$, and moreover

$$U(q) = H_k \qquad \text{if} \qquad k = 2, 3, \ldots,$$

and $U(q) = H_1 := \{\underline{0}, \underline{1}\}$.

Our investigations do not exactly clarify the stability behaviour of the elements of the open interval

$$]q(k), q^*(k+1)[\qquad \text{for} \qquad k = 2, 3, \dots .$$

We shall return to this question in a subsequent paper.

References

[1] Z. Daróczy and I. Kátai, Additive functions, *Analysis Mathematica* **12** (1986), 85–96.

[2] Z. Daróczy and I. Kátai, Continuous additive functions and difference equations of infinite order, *Analysis Mathematica* **12** (1986), 237–249.

[3] Z. Daróczy, A. Járai and I. Kátai, Intervallfüllende Folgen und volladditive Funktionen, *Acta Sci. Math. (Szeged)* **50** (1986), 337–350.

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(Received February 11, 1993)