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Functional equations and Gelfand measures

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Abstract. Let G be a locally compact Hausdorff topological group, let σ be a continuous involution of G and let μ be a generalized Gelfand measure which is σ-invariant.

In this paper we introduce and study properties of solutions of selected functional equations of the form

$$\int_G f(xty)d\mu(t) \pm \int_G f(xt\sigma(y))d\mu(t) = 2\sum_{i=1}^{n} g_i(x)h_i(y),$$

where the functions f, $\{g_i\}$, $\{h_i\} : G \to \mathbb{C}$ to be determined are bounded and continuous functions on G.

When G is a compact group and μ is a real invariant generalized Gelfand measure, the explicit solution formulas are found by Fourier analysis; this extends the well-known results on d'Alembert-type equations on abelian groups.

Furthermore, we also give a generalization of the Chojnacki, Stetkær and Niechwiej results for the operator cosine function and the K-spherical operator function to the case of the operator cosine function on Gelfand measure.

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1. Introduction

In previous papers (see [5], [12]), the authors have introduced a generalized d'Alembert functional equation

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)f(y). \tag{1}$$

When μ is a generalized Gelfand measure which is σ-invariant, the solutions of (1) are completely determined.

There is an important classical example of equation (1): $\sigma(x) = x^{-1}$ and $\mu = \delta_e$, where δ_e denotes the Dirac measure concentrated at the identity element of G. In this setting G is an abelian group and (1) reduces to the d'Alembert functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G. \tag{2}$$

In this paper we study functional equations of the form

$$\int_G f(xty)d\mu(t) \pm \int_G f(xt\sigma(y))d\mu(t) = 2\sum_{i=1}^{n} g_i(x)h_i(y), \tag{3}$$

where μ is a generalized Gelfand measure which is σ-invariant and f, g_i, $h_i \in C_b(G)$ are unknown functions that we want to determine. In the particular case, when μ is a compactly supported measure on G, the solutions may be sought in $C(G)$.

A number of results have been obtained for the functional equation (2), when $\mu = \delta_e$. Penny and Rukhin [17] and Rukhin [18] found the form of f in the functional equation $f(xy) \pm f(xy^{-1}) = 2\sum_{i=1}^{i=n} g_i(x)h_i(y)$, $x, y \in G$.

The most comprehensive recent study is by Stetkær [20] and by Friis and Stetkær [14], who found explicit solution formulas for the following functional equations: $f(x+y) \pm f(x+\sigma(y)) = 2\sum_{i=1}^{i=2} g_i(x)h_i(y)$, $x, y \in G$ ([20]) and $f(x+y) + f(x+\sigma(y)) = 2(f(x)g(y) + g(x)f(y) + h(x)h(y))$, $x, y \in G$ ([14]). Other references and information on detailed discussions can be found in the monographs by Aczél [1] and by Aczél and Dhombres [2].

The results of the present paper are organized as follows:

In Section 3 we construct to each solution $\{f, \{g_i\}, \{h_i\}\}$ of (3) a map $\Psi \in M_{(n,n)}(\mathbb{C})$ such that

$$\begin{cases} \int_G \Psi(xty)d\mu(t) + \int_G \Psi(xt\sigma(y))d\mu(t) = 2\Psi(x)\Psi(y) \\ \Psi(e) = I_n, \end{cases} \tag{4}$$

$$\left\{\int_G g_j(x_i tz)d\mu(t) + \int_G g_j(x_i t\sigma(z))d\mu(t)\right\} = 2\{g_j(x_i)\}\Psi(z) \tag{5}$$

and

$$\left\{\int_G h_i(ztx_j)d\mu(t) \mp \int_G h_i(zt\sigma(x_j))d\mu(t)\right\} = 2\Psi(z)\{h_i(x_j)\} \tag{6}$$

for all $x_1, \dots, x_n, x, y, z \in G$. (Theorem 3.1).

In Section 4, G is a compact group and μ is a real invariant generalized Gelfand measure. By diagonalizing such generalized cosine functions $\Psi(x)$, we solve various functional equations and we give solution formulas for equation (3) in Theorem 4.3.

The above results can be viewed as a continuation of the recent ones of Stetkær [19] on relations between solutions of the functional equation

$$\int_K f(x\ k.y)\bar{\chi}(k)dk = \sum_{i=1}^{n} g_i(x)h_i(y) \tag{7}$$

where K is a compact subgroup of the automorphism group of G and $\chi : K \to \{z \in \mathbb{C} \text{ such that } |z| = 1\}$ is a continuous homomorphism. His methods and ideas are pursued in Section 3 and Section 4.

In Section 5 we study the spectral representation of the generalized operator cosine function $\Gamma : G \to \pounds(\mathcal{H})$ such that

$$\begin{cases} \int_G \Gamma(xty)d\mu(t) + \int_G \Gamma(xty^{-1})d\mu(t) = 2\Gamma(x)\Gamma(y),\ x, y \in G, \\ \Gamma(e) = I, \end{cases} \tag{8}$$

where μ is an invariant generalized Gelfand measure and $\mathcal{H}$ is a Hilbert space. Two cases are considered: Uniformly bounded generalized cosine functions (Theorem 5.1 and Theorem 5.3) and bounded generalized cosine functions with exponential type of growth (Theorem 5.4).

This extends results of Chojnacki [6] and Niechwiej [16] on cosine operator function on locally compact abelian groups.

2. Notations and terminology

Throughout this paper, G will be a Hausdorff topological locally compact group. $M(G)$ denotes the Banach algebra of complex bounded measures, it is the dual of $C_0(G)$, the Banach space of continuous functions vanishing at infinity.

$C(G)$ (resp. $C_b(G)$) designates the Banach space of continuous (resp. continuous and bounded) complex valued functions.

We let $M_{(n,n)}(\mathbb{C})$ denote the algebra of all complex $n\times n$ matrices and A^t the transpose, where $A \in M_{(n,n)}(\mathbb{C})$. If $A \in GL_n(\mathbb{C})$ i.e. A is invertible, we let A^{-1} denote the inverse of A.

Let σ be a continuous involution of G (i.e. $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma \circ \sigma(x) = \sigma(x)$ for all $x, y \in G$) and let $\mu \in M(G)$. We say that μ is σ-invariant if $\prec f \circ \sigma,\, \mu \succ = \prec f,\, \mu\succ,\ \forall\ f \in C_b(G)$, where $\prec f,\, \mu\succ = \int_G f$ $d\mu$. When $\sigma(x) = x^{-1}$ if μ is σ-invariant, we will say that μ is invariant.

For all $\mu, \nu \in M(G)$, we recall that the convolution $\mu*\nu$ is the measure given by

$$\prec f, \mu * \nu\succ := \int_G \int_G f(ts)d\mu(t)d\nu(s), \quad f \in C_b(G).$$

For every $\mu \in M(G)$ and every continuous and bounded function $f : G \to M_{(n,n)}(\mathbb{C})$, (resp. $\pounds(\mathcal{H})$) we set

$$f_\mu(x) = \int_G \int_G f(txs)d\mu(t)d\mu(s), \quad x \in G,$$

where $\mathcal{H}$ is a Hilbert space and $\pounds(\mathcal{H})$ denotes the Banach space of bounded operators on $\mathcal{H}$.

Definition 2.1. Let $\mu \in M(G)$. μ is called a generalized Gelfand measure if $\mu * \mu = \mu$ and the Banach algebra $\mu * M(G) * \mu$ is commutative under convolutions.

If K is a compact subgroup on G and $\mu = dk$ (the normalized Haar measure of K), then dk is a generalized Gelfand measure if and only if (G, K) is a Gelfand pair (see [11]).

For the notion of Gelfand measure see [4].

Definition 2.2. Let $\mu \in M(G)$. A nonzero function $\Phi \in C_b(G)$ is a μ-spherical function if it satisfies the functional equation $\int_G \Phi(xty)d\mu(t) = \Phi(x)\Phi(y)$, for all $x, y \in G$.

A function $\Phi \in \pounds(\mathcal{H})$, ($\mathcal{H} \notin \{\mathbb{C}, \mathbb{R}\}$) is a μ-spherical operator function if $\Phi(e) = I$ and Φ satisfies the functional equation $\int_G \Phi(xty)d\mu(t) = \Phi(x)\Phi(y)$ for all $x, y \in G$, where I denotes the identity operator.

3. Generalized d'Alembert-type functional equations

This section is devoted to the study of general properties of the functional equation

$$\int_G f(xty)d\mu(t) \pm \int_G f(xt\sigma(y))d\mu(t) = 2\sum_{i=1}^{n} g_i(x)h_i(y), \quad x, y \in G, \quad (3)$$

where μ is a generalized Gelfand measure which is σ-invariant and $f, g_i, h_i \in C_b(G)$.

Throughout the paper we will assume that the solutions f of (3) are μ-biinvariant. This means that $f_\mu = f$.

Notice that in this case, by [12] (Proposition 1.1) f satisfies the following conditions:

$$\int_G \int_G f(xtysz)d\mu(t)d\mu(s) = \int_G \int_G f(ytxsz)d\mu(t)d\mu(s), \quad (9)$$

$$\int_G \int_G f(ztxsy)d\mu(t)d\mu(s) = \int_G \int_G f(ztysx)d\mu(t)d\mu(s), \quad (10)$$

$$\int_G f(xty)d\mu(t) = \int_G f(ytx)d\mu(t), \quad (11)$$

for all $x, y, z \in G$

Proposition 3.1 (Necessary condition for (3) to have a solution)**.** *If* $(f,\{g_i\},\{h_i\})$ *are solutions of the functional equation* (3) *then*

$$\begin{aligned}&2\sum_{i=1}^{n}\Big[\int_G g_i(xty)d\mu(t)+\int_G g_i(xt\sigma(y))d\mu(t)\Big]h_i(z)\\&\qquad=2\sum_{i=1}^{n}\Big[\int_G h_i(ytz)d\mu(t)\mp\int_G h_i(yt\sigma(z))d\mu(t)\Big]g_i(x),\end{aligned}$$

for all $x,y,z\in G$.

Proof. Let $x,y,z\in G$. Since μ is a σ-invariant measure and f satisfies the condition (10), by using equation (3) we get

$$\begin{aligned}&2\sum_{i=1}^{n}\Big[\int_G g_i(xty)d\mu(t)+\int_G g_i(xt\sigma(y))d\mu(t)\Big]h_i(z)\\&\quad=\int_G\Big[\int_G f(xtysz)d\mu(s)\mp\int_G f(xtys\sigma(z))d\mu(s)\Big]d\mu(t)\\&\qquad+\int_G\Big[\int_G f(xt\sigma(y)sz)d\mu(s)\mp\int_G f(xt\sigma(y)s\sigma(z))d\mu(s)\Big]d\mu(t)\\&\quad=\Big[\int_G\int_G f(xtysz)d\mu(t)d\mu(s)\mp\int_G\int_G f(xt\sigma(ysz))d\mu(t)d\mu(s)\Big]\\&\qquad\mp\Big[\int_G\int_G f(xtys\sigma(z))d\mu(t)d\mu(s)\mp\int_G\int_G f(xt\sigma(ys\sigma(z)))d\mu(t)d\mu(s)\Big]\\&\quad=2\sum_{i=1}^{n}\Big[\int_G h_i(ysz)d\mu(s)\mp\int_G h_i(ys\sigma(z))d\mu(s)\Big]g_i(x).\end{aligned}$$

This ends the proof. □

From now on, we assume that each of the two sets of functions $\{g_i\}_{i\in\{1,\dots,n\}}$ and $\{h_i\}_{i\in\{1,\dots,n\}}$ is linearly independent.

This implies that there exist $\{x_i^0\}_{i\in\{1,\dots,n\}}\in G\times\cdots\times G$ and $\{y_i^0\}_{i\in\{1,\dots,n\}}\in G\times\cdots\times G$ such that the matrices $\{g_j(x_i^0)\}$, $\{h_i(y_j^0)\}$ are invertible (see [2]).

Using a similar argument we get that the solutions $\{g_i\}$ and $\{h_i\}$ of equation (3) satisfy the conditions (9)–(11).

In Theorem 3.1 we produce the relationships which exist between the solutions of (3) and some generalized cosine matrix function (4).

Theorem 3.1. *Let* $(f, \{g_i\}, \{h_i\})$ *be a solution of* (3). *Then*

(i)
$$2\left\{\int_G g_j(x_itz)d\mu(t) + \int_G g_j(x_it\sigma(z))d\mu(t)\}\{h_i(y_j)\right\}$$
$$= 2\{g_j(x_i)\}\left\{\int_G h_i(zty_j)d\mu(t) \mp \int_G h_i(zt\sigma(y_j))d\mu(t)\right\},$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n, z \in G$.

(ii) *The matrix function* $z \to \Psi(z)$ *defined by*

$$\Psi(z) = \frac{1}{2}\{g_j(x_i^0)\}^{-1}\left\{\int_G g_j(x_i^0tz)d\mu(t) + \int_G g_j(x_i^0t\sigma(z))d\mu(t)\right\}$$
$$= \frac{1}{2}\left\{\int_G h_i(zty_j^0)d\mu(t) \mp \int_G h_i(zt\sigma(y_j^0))d\mu(t)\right\}\{h_i(y_j^0)\}^{-1}$$

satisfies the functional equations

$$\begin{cases} \int_G \Psi(xty)d\mu(t) + \int_G \Psi(xt\sigma(y))d\mu(t) = 2\Psi(x)\Psi(y), \\ \Psi(e) = I_n, \end{cases} \tag{4}$$

$$\left\{\int_G g_j(x_itz)d\mu(t) + \int_G g_j(x_it\sigma(z))d\mu(t)\right\} = 2\{g_j(x_i)\}\Psi(z) \tag{5}$$

and

$$\left\{\int_G h_i(ztx_j)d\mu(t) \mp \int_G h_i(zt\sigma(x_j))d\mu(t)\right\} = 2\Psi(z)\{h_i(x_j)\}, \tag{6}$$

for all $x_1, \ldots, x_n, x, y, z \in G$.

In the particular case of (3):

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2\sum_{i=1}^n g_i(x)h_i(y), \tag{12}$$

we have

$$h_k(x) = \sum_{j=1}^{n} \Psi_{k,j}(x) h_j(e) \tag{13}$$

and

$$f(x) = \sum_{i=1}^{n} h_i(e) g_i(x) \tag{14}$$

for all $x \in G$.

Proof. (i) Follows directly from Proposition 3.1.

(ii) First we prove (5). Let $x_1, \ldots, x_n, z \in G$

$$2\{g_j(x_i)\}\Psi(z)$$

$$= \{g_j(x_i)\}\left[\left\{\int_G h_i(zty_j^0)d\mu(t) \mp \int_G h_i(zt\sigma(y_j^0))d\mu(t)\right\}\{h_i(y_j^0)\}^{-1}\right].$$

In view of (i) we get

$$2\{g_j(x_i)\}\Psi(z)$$

$$= \left\{\int_G g_j(x_itz)d\mu(t) + \int_G g_j(x_it\sigma(z))d\mu(t)\right\}\{h_i(y_j^0)\}\{h_i(y_j^0)\}^{-1}$$

$$= \left\{\int_G g_j(x_itz)d\mu(t) + \int_G g_j(x_it\sigma(z))d\mu(t)\right\}.$$

This proves equation (5).

We shall now explore this equation (5) to establish equations (4) and (6).

Let $x, y \in G$. In virtue of the definition of Ψ, we have

$$\int_G \Psi(xsy)d\mu(s) + \int_G \Psi(xs\sigma(y))d\mu(s)$$

$$= \frac{1}{2}\{g_j(x_i^0)\}^{-1}\left\{\int_G\int_G g_j(x_i^0txsy)d\mu(t)d\mu(s)\right.$$

$$\left. + \int_G\int_G g_j(x_i^0t\sigma(y)s\sigma(x))d\mu(t)d\mu(s)\right\}$$

$$+\frac{1}{2}\{g_j(x_i^0)\}^{-1}\Big\{\int_G\int_G g_j(x_i^0txs\sigma(y))d\mu(s)d\mu(t)$$
$$+\int_G\int_G g_j(x_i^0tys\sigma(x))d\mu(s)d\mu(t)\Big\}$$
$$=\frac{1}{2}\{g_j(x_i^0)\}^{-1}\Big\{\int_G\int_G g_j(x_i^0txsy)d\mu(s)d\mu(t)$$
$$+\int_G\int_G g_j(x_i^0txs\sigma(y))d\mu(s)d\mu(t)\Big\}$$
$$+\frac{1}{2}\{g_j(x_i^0)\}^{-1}\Big\{\int_G\int_G g_j(x_i^0t\sigma(x)sy)d\mu(s)d\mu(t)$$
$$+\int_G\int_G g_j(x_i^0t\sigma(x)s\sigma(y))d\mu(s)d\mu(t)\Big\}$$
$$=\frac{1}{2}\{g_j(x_i^0)\}^{-1}\Big\{2\int_G g_j(x_i^0tx)d\mu(t)\Psi(y)\Big\}$$
$$+\frac{1}{2}\{g_j(x_i^0)\}^{-1}\Big\{2\int_G g_j(x_i^0t\sigma(x))d\mu(s)\Psi(y)\Big\}$$
$$=2\Psi(x)\Psi(y).$$

This proves equation (4). By a small computation we also get (6), and the theorem is proved. □

Corollary 3.1. *Consider the functional equation*

$$\int_G f(xty)d\mu(t)-\int_G f(xt\sigma(y))d\mu(t)=2g(x)h(y),\quad x,y\in G. \tag{15}$$

With the notation
$\mathcal{N}_\mu=\{f\in C_b(G)\mid f_\mu=f$ *and* $\int_G f(xty)d\mu(t)=\int_G f(xt\sigma(y))d\mu(t)\ \forall x,y\in G\}$, *we have*

(1) $g=0$, *h is arbitrary,* $f\in\mathcal{N}_\mu$,

(2) $h=0$, *g is arbitrary,* $f\in\mathcal{N}_\mu$,

(3) *there exists a μ-spherical function ω for which* $\omega\neq\omega\circ\sigma$, $\alpha\in\mathbb{C}^*$, $\beta,\gamma\in\mathbb{C}$ *and* $l_0\in\mathcal{N}_\mu$ *such that*

$$g=\beta\frac{\omega+\omega\circ\sigma}{2}+\gamma\frac{\omega-\omega\circ\sigma}{2}, \tag{16}$$

$$h = \alpha \frac{\omega - \omega \circ \sigma}{2} \tag{17}$$

and

$$f = \alpha\gamma \frac{\omega + \omega \circ \sigma}{2} + \alpha\beta \frac{\omega - \omega \circ \sigma}{2} + l_0. \tag{18}$$

(4) *There exists a μ-spherical function ω for which $\omega = \omega \circ \sigma$, $\alpha, \beta \in \mathbb{C}$ and $l_1 \in C_b(G)$, $l_0 \in \mathcal{N}_\mu$ such that h is a solution of the functional equation*

$$\int_G h(xty)d\mu(t) = h(x)\omega(y) + \omega(x)h(y), \quad x, y \in G, \tag{19}$$

l_1 is a solution of the functional equation

$$\int_G l_1(xty)d\mu(t) - \int_G l_1(xt\sigma(y))d\mu(t) = 2\beta h(x)h(y), \quad x, y \in G, \tag{20}$$

$$g = \alpha\omega + \beta h, \tag{21}$$

$$f = \alpha h + l_1 + l_0. \tag{22}$$

Proof. The first two cases are trivial. From now on we may assume that f, g are nonzero functions. By Theorem 3.1 there exists a nonzero solution $\Psi \in C_b(G)$ of the generalized d'Alembert functional equation (1) such that $\int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) = 2g(x)\Psi(y)$ and $\int_G h(xty)d\mu(t) = \Psi(x)h(y) + \Psi(y)h(x)$, $h(\sigma(x)) = -h(x)$, for all $x, y \in G$. Ψ satisfies the generalized d'Alembert functional equation (1) and so has the desired form by ([12] Theorem 2.2) i.e. there exists a μ-spherical function ω such that $\Psi = \frac{\omega+\omega\circ\sigma}{2}$. If $\omega \neq \omega \circ \sigma$ then by ([12] Theorem 3.1 and Proposition 3.2) there exist $\alpha \in \mathbb{C}^*$, β, $\gamma \in \mathbb{C}$ and $l_0 \in \mathcal{N}_\mu$ such that

$$g = \beta \frac{\omega + \omega \circ \sigma}{2} + \gamma \frac{\omega - \omega \circ \sigma}{2}$$

$$h = \alpha \frac{\omega - \omega \circ \sigma}{2}$$

$$f = \alpha\gamma \frac{\omega + \omega \circ \sigma}{2} + \alpha\beta \frac{\omega - \omega \circ \sigma}{2} + l_0$$

i.e. case (3). If $\omega = \omega \circ \sigma$ then $\int_G h(xty)d\mu(t) = \omega(x)h(y) + \omega(y)h(x)$, and from Theorem 3.1 and Proposition 3.1 of [12] we see that each solution f, g of equation (15) has the stated form. This finishes the proof. $\square$

Remark (On Kannappan's condition). In [15], Kannappan has solved the d'Alembert functional equation (2) under the famous condition $f(zxy) = f(zyx)$, $x, y \in G$. Later this condition was used by several mathematicians in order to obtain the general solutions of some equations like (2), with $\mu = \delta_e$ and $\sigma(x) = x^{-1}$, see for example [3], [7] and [8].

In our situation, in Section 3, the requirement that μ is a σ-invariant generalized Gelfand measure and f is μ-biinvariant can be relaxed. Instead we may assume that μ is a σ-invariant measure and f satisfies the Kannappan-type condition

$$\int_G \int_G f(ztxsy)d\mu(t)d\mu(s) = \int_G \int_G f(ztysx)d\mu(t)d\mu(s),$$

$$\int_G f(xty)d\mu(t) = \int_G f(ytx)d\mu(t), \quad \text{for all } x, y, z \in G.$$

4. Solutions of (3) for compact groups

In this section G is a compact group endowed with the fixed normalized Haar measure dx, $\sigma(x) = x^{-1}$ and μ is a real invariant generalized Gelfand measure. For such a measure, we let Σ_μ denote the set of all μ-spherical functions on G, it is the Gelfand spectrum of $L_1^\mu(G) = \mu * L_1(G, dx) * \mu$. We recall that in compact groups a μ-spherical function ω is also a positive definite function, in particular $\breve{\omega}(x) = \omega(x^{-1}) = \bar{\omega}(x)$ for all $x \in G$, (see [4]).

We denote by $\mathcal{F}$ the standard Fourier transform on $L_1^\mu(G)$; $\mathcal{F} : L_1^\mu(G) \to C_0(\Sigma_\mu)$ given by $\mathcal{F}f(\omega) = \int_G f(x)\omega(x)dx$.

Finally the solutions of (3) are supposed to be continuous.

Theorem 4.1. *Let $\Psi : G \to M_n(\mathbb{C})$ be a continuous function such that Ψ is a solution of the generalized cosine matrix function*

$$\begin{cases} \int_G \Psi(xty)d\mu(t) + \int_G \Psi(xty^{-1})d\mu(t) = 2\Psi(x)\Psi(y), \quad x, y \in G, \\ \Psi(e) = I_n. \end{cases} \tag{4}$$

Then there exist $\omega_1, \dots, \omega_n \in \Sigma_\mu$ and $A \in GL_n(\mathbb{C})$ such that

$$A\Psi A^{-1} = \mathrm{Diag}\left(\frac{\omega_i + \bar{\omega}_i}{2}\right)_{i \in \{1,\dots,n\}}.$$

Proof. We shall develop a method used in [19] to solve equation (4). Let $\hat{G}$ denote the set of irreducible characters of G and consider the operator function $\hat{\Psi} : \hat{G} \to M_n(\mathbb{C})$ defined by $\hat{\Psi}(\chi_\pi) = \int_G \Psi(x)\chi_\pi(x)dx$.

On using $\int_G \Psi(xt^{-1})d\mu(t) = \Psi(x)$, we see that

$$\begin{aligned}\int_G \Psi(x)\chi_\pi(x)dx &= \int_G\int_G \Psi(xt^{-1})\chi_\pi(x)d\mu(t)dx\\ &= \int_G \Psi(x)(\int_G \chi_\pi(xt)d\mu(t))dx\\ &= \int_G \Psi(x)\omega_\pi(x)dx,\end{aligned}$$

where

$$\omega_\pi(x) = \int_G \chi_\pi(xt)d\mu(t). \tag{23}$$

It was shown in [4] that the μ-spherical functions in compact groups are given by (23).

For an arbitrary $y \in G$, we obtain

$$\begin{aligned}\Psi(y)\hat{\Psi}(\chi_\pi) &= \int_G \Psi(y)\Psi(x)\omega_\pi(x)dx\\ &= \frac{1}{2}\int_G\int_G \Psi(ytx)\omega_\pi(x)d\mu(t)dx\\ &\quad + \frac{1}{2}\int_G\int_G \Psi(ytx^{-1})\omega_\pi(x)d\mu(t)dx.\end{aligned}$$

In view of $\int_G \Psi(k^{-1}x)d\mu(k) = \Psi(x)$ (see [12] Lemma 2.3), $\check{\Psi} = \Psi$ and G unimodular, we have

$$\begin{aligned}\Psi(y)\hat{\Psi}(\chi_\pi) &= \frac{1}{2}\int_G\int_G\int_G \Psi(k^{-1}ytx)\omega_\pi(x)d\mu(k)d\mu(t)dx\\ &\quad + \frac{1}{2}\int_G\int_G\int_G \Psi(k^{-1}ytx)\check{\omega}_\pi(x)d\mu(k)d\mu(t)dx\\ &= \frac{1}{2}\int_G\int_G\int_G \Psi(x)\omega_\pi(t^{-1}y^{-1}kx)d\mu(k)d\mu(t)dx\\ &\quad + \frac{1}{2}\int_G\int_G\int_G \Psi(x)\check{\omega}_\pi(t^{-1}y^{-1}kx)d\mu(k)d\mu(t)dx\end{aligned}$$

$$= \frac{1}{2}\bar{\omega}_\pi(y)\int_G \Psi(x)\omega_\pi(x)dx + \frac{1}{2}\omega_\pi(y)\int_G \Psi(x)\omega_\pi(x)dx$$

$$= \frac{1}{2}(\omega_\pi(y) + \bar{\omega}_\pi(y))\hat{\Psi}(\chi_\pi). \tag{24}$$

On the other hand, the irreducible characters of the compact group G form a basis of the Banach space $L^2(G, dx)$, then Span$\{\hat{\Psi}(\chi_\pi)\varsigma,\ \chi_\pi \in \hat{G},\ \varsigma \in \mathbb{C}^n\} = \mathbb{C}^n$. Consequently there exist $\chi_{\pi_1}, \chi_{\pi_2}, \dots, \chi_{\pi_n} \in \hat{G}$ and $\varsigma_1, \varsigma_2, \dots$ $\dots, \varsigma_n \in \mathbb{C}^n$ such that $\{\hat{\Psi}(\chi_{\pi_i})\varsigma_i,\ i = 1, \dots, n\}$ is a basis of $\mathbb{C}^n$. □

Combining this result with (24), Theorem 4.1 follows immediately.

Corollary 4.1 ([5], [12]). *Let $f : G \to \mathbb{C}^*$ be a continuous solution of the generalized d'Alembert functional equation* (1). *Then there exists a positive definite μ-spherical function ω such that*

$$f(x) = \frac{\omega(x) + \bar{\omega}(x)}{2}, \quad \textit{for all } x \in G.$$

Theorem 4.2. *If f, g, h are solutions of the functional equation*

$$\int_G f(xty)d\mu(t) \pm \int_G f(xty^{-1})d\mu(t) = 2g(x)h(y), \quad x, y \in G, \tag{25}$$

such that neither g nor h is identically zero, then there exist a positive definite μ-spherical function $\omega \in \Sigma_\mu$ and $\alpha, \beta, \in \mathbb{C}$, $\gamma \in \mathbb{C} \setminus \{0\}$ such that

$$h = \frac{\gamma}{2}(\omega \mp \bar{\omega}), \tag{26}$$

$$g = \frac{\alpha\omega + \beta\bar{\omega}}{2}, \tag{27}$$

$$f = \frac{\gamma}{2}(\alpha\omega \mp \beta\bar{\omega}) + \mathcal{N}_\mu, \tag{28}$$

where $\mathcal{N}_\mu =$
$\{f \in C(G)/f_\mu = f$ *and* $\int_G f(xty)d\mu(t) = -(\pm)\int_G f(xty^{-1})d\mu(t)\ \forall\, x, y \in G\}$.

Conversely, the triple (f, g, h) given by (28), (27) *and* (26) *solves* (25).

Proof. The central technical point of the proof can be managed by Corollary 3.1, ([12] Proposition 3.2) and the following lemma. □

Lemma 4.1. *Let f be a continuous μ-biinvariant function on G and let $\omega \in \Sigma_\mu$. If f is a solution of the functional equation*

$$\int_G f(xty)d\mu(t) = f(x)\omega(y) + f(y)\omega(x), \quad x, y \in G, \tag{29}$$

then $f(x) = 0$, $x \in G$.

Proof. Multiplying (29) by $\check{\omega}(x)$ and integrating the result over G with respect to x, we get

$$\int_G \int_G f(xty)\check{\omega}(x)d\mu(t)dx = f(y)\int_G |\omega(x)|^2 dx + \mathcal{F}(f^{-1})(\omega)\omega(y),$$

since

$$\begin{aligned}\int_G \int_G f(xty)\check{\omega}(x)d\mu(t)dx &= \int_G \int_G \int_G f(xtyk)\check{\omega}(x)d\mu(t)d\mu(k)dx \\ &= \int_G \int_G \int_G f(x)\check{\omega}(xk^{-1}y^{-1}t^{-1})d\mu(t)dx \\ &= \omega(y)\mathcal{F}(f^{-1})(\omega).\end{aligned}$$

Consequently $f(y)\int_G |\omega(x)|^2 dx = 0$. In view of $\int_G |\omega(x)|^2 dx \neq 0$ (see [4]), we obtain $f(y) = 0$, $y \in G$. □

Corollary 4.2. *Let (f, g) be a solution of the functional equation*

$$\int_G f(xty)d\mu(t) + \int_G f(xty^{-1})d\mu(t) = 2f(x)g(y), \quad x, y \in G, \tag{30}$$

such that $f \neq 0$. Then g is a solution of the generalized d'Alembert functional equation (1) and there exist a positive definite μ-spherical function $\omega \in \Sigma_\mu$ and $\alpha, \beta \in \mathbb{C}$ such that

$$g = \frac{\omega + \bar{\omega}}{2}, \tag{31}$$

$$f = \frac{\alpha\omega + \beta\bar{\omega}}{2}. \tag{32}$$

Corollary 4.3. *Let (f, g) be a solution of the functional equation*

$$\int_G f(xty)d\mu(t) \pm \int_G f(xty^{-1})d\mu(t) = 2g(x)f(y), \quad x, y \in G, \tag{33}$$

such that neither f nor g is identically zero. Then there exist a positive definite μ-spherical function $\omega \in \Sigma_\mu$ and $\alpha \in \mathbb{C} \setminus \{0\}$ such that

$$g(x) = \frac{\omega(x) + \bar{\omega}(x)}{2} \tag{34}$$

$$f(x) = \frac{\alpha}{2}(\omega(x) \mp \bar{\omega}(x)) + \mathcal{N}_\mu. \tag{35}$$

Now, by applying the above results (Theorem 4.1, Theorem 4.2,…), we can derive the solutions of (3).

Theorem 4.3. *Let $(f, \{g_i\}, \{h_i\})$ be a solution of* (3). *Then there exist $A \in GL_n(\mathbb{C})$, $\omega_i \in \Sigma_\mu$ and $\alpha_i, \beta_i \in \mathbb{C}$, $\gamma_i \in \mathbb{C} \setminus \{0\}$ such that*

$$\{g_i\} = \left\{ \frac{\alpha_i\omega_i + \beta_i\bar{\omega}_i}{2} \right\} A, \tag{36}$$

$$\{h_i\} = \left\{ \frac{\gamma_i}{2}(\omega_i \mp \bar{\omega}_i) \right\} (A^t)^{-1} \tag{37}$$

and

$$f = \sum_{i=1}^{n} \frac{\gamma_i}{2}(\alpha_i\omega_i \mp \beta_i\bar{\omega}_i) + \mathcal{N}_\mu. \tag{38}$$

Conversely, the formulas in (36), (37) *and* (38) *define a solution of* (3).

Proof. From Theorem 3.1, we know that there exists $\Psi : G \to M_n(\mathbb{C})$ which satisfies

$$\begin{cases} \displaystyle\int_G \Psi(xty)d\mu(t) + \int_G \Psi(xty^{-1})d\mu(t) = 2\Psi(x)\Psi(y), \\ \Psi(e) = I_n, \end{cases} \tag{4}$$

$$\left\{ \int_G g_j(x_itz)d\mu(t) + \int_G g_j(x_itz^{-1})d\mu(t) \right\} = 2\{g_j(x_i)\}\Psi(z), \tag{5}$$

and

$$\left\{ \int_G h_i(ztx_j)d\mu(t) \mp \int_G h_i(ztx_j^{-1})d\mu(t) \right\} = 2\Psi(z)\{h_i(x_j)\} \tag{6}$$

for all $x_1, \dots, x_n, x, y, z \in G$.

Now, by Theorem 4.1 there exist $A \in GL_n(\mathbb{C})$ and $\omega_i \in \Sigma_\mu$ such that

$$A\Psi A^{-1} = \mathrm{Diag}\left(\frac{\omega_i + \bar{\omega}_i}{2}\right)_{i\in\{1,\dots,n\}}. \tag{39}$$

Furthermore $\{g_i\}A^{-1}$ and $\{h_i\}A^t$ are linearly independent sets satisfying the following functional equations:

$$\sum_{i=1}^{n} g_i(x)h_i(y) = \sum_{i=1}^{n}(\{g_j\}A^{-1})_i(x)(\{h_j\}A^t)_i(y), \tag{40}$$

$$\begin{aligned} &A\Big\{\int_G h_i(zty_j)d\mu(t) \mp \int_G h_i(zty_j^{-1})d\mu(t)\Big\} \\ &\quad = \Big\{\int_G (\{h_r\}A^t)_i(zty_j)d\mu(t) \mp \int_G (\{h_r\}A^t)_i(zty_j^{-1})d\mu(t)\Big\}, \end{aligned} \tag{41}$$

$$\begin{aligned} &\int_G (\{g_j\}A^{-1})_i(xty)d\mu(t) + \int_G (\{g_j\}A^{-1})_i(xty^{-1}) \\ &\quad = 2(\{g_j\}A^{-1})_i(x)\left(\frac{\omega_i(y)+\bar{\omega}_i(y)}{2}\right), \end{aligned} \tag{42}$$

and

$$\begin{aligned} &\int_G (\{h_j\}A^t)_i(xty)d\mu(t) \mp \int_G (\{h_j\}A^t)_i(xty^{-1})d\mu(t) \\ &\quad = 2\frac{\omega_i(x)+\bar{\omega}_i(x)}{2}(\{h_j\}A^t)_i(y) \end{aligned} \tag{43}$$

for all $i = 1, \dots, n$ and $x, y, z \in G$.

Now, by using Theorem 4.2, Corollary 4.1 and Corollary 4.2, we obtain the proof of the theorem. □

5. Spectral representation of the generalized cosine functions in Hilbert spaces

In what follows we are interested in the spectral representation of the generalized cosine operator function (8).

A classical example of (8) is the cosine operator function

$$\begin{cases} \Gamma(x+y)+\Gamma(x-y)=2\Gamma(x)\Gamma(y), \quad x,y\in G, \\ \Gamma(e)=I. \end{cases} \tag{44}$$

In recent years a good deal of interest has been manifested for the study of the operator cosine functions on locally compact abelian groups.

The cosine functions on the real line are related to the Cauchy problem for 2nd order differential equations, see FATTORINI [13].

For matrices this problem was solved by SZÉKELYHIDI [21] and RUKHIN [18] for Abelian groups which are divisible by 2.

CHOJNACKI [6] and STETKAER [19] (for the K-spherical operator function) have proved that any uniformly bounded, normal, weakly continuous cosine function Γ on a locally compact Abelian group G has the form

$$\Gamma(x)=\frac{1}{2}(U(x)+U(-x)), \quad x\in G,$$

where U is a strongly continuous unitary representation of G on the Hilbert space $\mathcal{H}$.

In the recent study [16], NIECHWIEJ obtained the spectral representation theorem for the case of not uniformly bounded cosine operators and unbounded operators.

One purpose of this section is to extend these investigations to the integral equation (8).

First Case: Uniformly bounded generalized cosine functions.

Theorem 5.1. *Assume that in a given Hilbert space $\mathcal{H}$ we have a uniformly bounded, normal and continuous function $\Gamma : G \to \pounds(\mathcal{H})$, that is a solution of the integral functional equation* (8)*. Then there exists a spectral Borel measure E on Σ_μ such that*

$$\Gamma(x)=\frac{1}{2}\int_{\Sigma_\mu}(\omega(x)+\omega(x^{-1}))dE(\omega), \quad x\in G.$$

The proof uses the CHOJNACKI method that appears in [6] and the generalized Fourier analysis that was studied in [4].

We will now formulate a theorem on spectral measure that we will use later. The proof can be derived from part of the one given in [6].

Theorem 5.2. *Let N be a Borel regular measure on Σ_μ. Denote by $\text{ß}(\Sigma_\mu)\check{}$ the set of all symmetric Borel subsets of Σ_μ, that is a Borel subset Ω of Σ_μ such that $\Omega = \check{\Omega} = \{\check{\omega},\, \omega \in \Omega\}$.*

If N is a spectral measure on $\text{ß}(\Sigma_\mu)\check{}$ then there exists a spectral measure E on Σ_μ, that extends N from symmetric subsets by the equality

$$2N(\Omega) = E(\Omega) + E(\check{\Omega}), \quad \Omega \in \text{ß}(\Sigma_\mu).$$

First we are going to recall the following lemma of that result used in the proof of Theorem 5.1.

Lemma 5.1 ([12])**.** *Let $\Gamma : G \to \pounds(\mathcal{H})$ be a uniformly bounded and continuous solution of the integral equation* (8)*. Then*

i) $\Gamma(x^{-1}) = \Gamma(x)$,

ii) $\Gamma(x)\Gamma(y) = \Gamma(y)\Gamma(x)$, *for all $x, y \in G$.*

Proof of Theorem 5.1. Let the operator function Λ be defined on $\mathcal{F}(L_1^\mu(G))$ by

$$\Lambda : \mathcal{F}(L_1^\mu(G)) \to \pounds(\mathcal{H}), \quad \Lambda(\mathcal{F}(f)) = \int_G f(t)\Gamma(t)dt, \quad \text{for } f \in L_1^\mu(G).$$

Consider the subset of $L_1^\mu(G)$ defined by

$$P_1 L_1^\mu(G) = \left\{ f \in L_1^\mu(G) \mid P_1 f = \frac{f + \check{f}}{2} = f \right\},$$

i.e. the set of even functions on $L_1^\mu(G)$, since G is unimodular and $\Gamma(t^{-1}) = \Gamma(t)$, then we deduce

$$\Lambda \circ \mathcal{F}(f) = \Lambda \circ \mathcal{F}(P_1 f), \quad \text{for all } f \in L_1^\mu(G).$$

In what follows we will prove that the mapping $\Lambda \circ \mathcal{F}$ is an algebraic homomorphism on $P_1 L_1^\mu(G)$. Let $f, g \in L_1^\mu(G)$ be such that $P_1 f = f$, then we have

$$\begin{aligned} \Lambda \circ \mathcal{F}(f * g) &= \int_G (f * g)(t)\Gamma(t)dt = \int_G \int_G f(s)g(s^{-1}t)\Gamma(t)dtds \\ &= \int_G \int_G \int_G f(s)g(k^{-1}s^{-1}t)\Gamma(t)d\mu(k)dtds \end{aligned}$$

$$= \frac{1}{2}\Big[\int_G\int_G\int_G f(s)g(t)\Gamma(tks)d\mu(k)dtds\Big]$$

$$+ \frac{1}{2}\Big[\int_G\int_G\int_G f(s)g(t)\Gamma(tks^{-1})d\mu(k)dtds\Big]$$

$$= \int_G\int_G g(t)f(s)\Gamma(t)\Gamma(s)dtds = \Lambda\circ\mathcal{F}(f)\Lambda\circ\mathcal{F}(g).$$

On the other hand, the range of $\Lambda\circ\mathcal{F}$ is included in a commuting Von Neumann algebra generated by the operators $\Gamma(x)$, $x\in G$, denoted by Θ. Therefore the operators $\Lambda\circ\mathcal{F}(f)$, $f\in L_1^\mu(G)$ belonging to this algebra are normal, hence we have for $f\in L_1^\mu(G)$:

$$\|\Lambda\circ\mathcal{F}(f)\| = \|\Lambda\circ\mathcal{F}(P_1f)\| = \lim_{n\to\infty}\|(\Lambda\circ\mathcal{F}(P_1f))^n\|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\|\Lambda\circ\mathcal{F}(P_1f)^n\|^{\frac{1}{n}} \preceq \lim_{n\to\infty}\|\Lambda\circ\mathcal{F}\|^{\frac{1}{n}}\|((P_1f)^n\|_1^{\frac{1}{n}}$$

$$= \|\mathcal{F}(P_1(f))\|_\infty \preceq \|\mathcal{F}f\|_\infty.$$

The Stone–Weierstrass theorem makes the image of $\mathcal{F}$ to be dense in $C_0(\Sigma_\mu)$ (see [4]), hence by the preceding inequality we immediately see that the operator Λ can be extended by continuity to the contracting operator on Σ_μ.

Now define the projections on symmetric functions in $C_0(\Sigma_\mu)$ by

$$(P_2f)(\omega) = \frac{1}{2}(f(\omega)+f(\check{\omega})),$$

since

$$P_2\mathcal{F}(L_1^\mu(G)) = \mathcal{F}P_1(L_1^\mu(G)),$$

the extended operator is an algebraic homomorphism on the algebra $P_2(C_0(\Sigma_\mu))$.

Let ß(Σ_μ) be the space of all bounded and measurable, complex functions on Σ_μ, and let $M_1(\Sigma_\mu)$ denote the space of all Borel bounded measures on Σ_μ. By considering in ß(Σ_μ) the topology $\sigma($ß$(\Sigma_\mu), M_1(\Sigma_\mu))$ and in Θ the weak operator topology, we see that the representation Λ can be extended by continuity on ß(Σ_μ), it will preserve the representation property on the set of all Borel symmetric functions on Σ_μ and its image will still be contained in Θ.

If we define the mapping

$$N : ß(\Sigma_\mu) \to \Theta \quad \text{by } N(\Omega) = \Lambda(1_\Omega),$$

then we will obtain a semi-spectral measure on Σ_μ that is spectral on the symmetric subsets of Σ_μ and satisfies

$$\int_{\Sigma_\mu} f(\Phi) dN(\Phi) = \Lambda(f).$$

By applying Theorem 5.2 we obtain a spectral measure E on Σ_μ such that

$$2N(\Omega) = E(\Omega) + E(\check{\Omega}), \quad \text{where } \check{\Omega} = \{\check{\omega}, \omega \in \Omega\}.$$

The definition of N gives us

$$\int_G f(t)\Gamma(t)\ dt = \Lambda(\mathcal{F}f) = \int_{\Sigma_\mu} \mathcal{F}f(\omega)\ dN(\omega) = \int_G \int_{\Sigma_\mu} \omega(t) f(t)\ dt dN(\omega)$$

and this implies

$$\Gamma(x) = \int_{\Sigma_\mu} \omega(x) dN(\omega) = \frac{1}{2} \int_{\Sigma_\mu} (\omega(x) + \check{\omega}(x)) dE(\omega), \quad x \in G.$$

This completes the proof. □

Now we are going to make explicit the uniformly bounded, normal and continuous solution of the integral equation (8).

Theorem 5.3. *Let Γ be a normal, uniformly bounded and continuous solution of the integral equation* (8). *Then there exists a strongly continuous and uniformly bounded μ-spherical operator function $\Psi : G \to £(\mathcal{H})$ such that*

$$\Gamma(x) = \frac{\Psi(x) + \Psi(x^{-1})}{2}, \quad x \in G.$$

Proof. By Theorem 5.1, there exists a spectral measure E on Σ_μ, such that $\Gamma(t) = \frac{1}{2}\int_{\Sigma_\mu} (\omega(t) + \check{\omega}(t)) dE(\omega)$. Let $\Psi(x) = \int_{\Sigma_\mu} \omega(x) dE(\omega)$. $\int_G \Psi(xty) d\mu(t) = \int_{\Sigma_\mu} \omega(x)\omega(y) dE(\omega)$. Since $\omega(x)$ and $\omega(y)$ are measurable functions on Σ_μ, by [10] (13.9.12), (13.8.4) there exist two sequences g_n^x and g_p^y of simple measurable functions with compact support such that

$g_n^x \to \omega(x)$, $g_p^y \to \omega(y)$ (p.p), $|g_n^x| \preceq 1$, $|g_p^y| \preceq 1$ and by [9] (II 4. Theorem 1) we get

$$\begin{aligned}\int_{\Sigma_\mu} \omega(x)\omega(y)dE(\omega) &= \int_{\Sigma_\mu} \lim_{n,p} g_n^x g_p^y \, dE(\omega)\\ &= \lim_{n,p} \int_{\Sigma_\mu} g_n^x g_p^y dE(\omega) = \lim_{n,p} \int_{\Sigma_\mu} g_n^x dE(\omega) \int_{\Sigma_\mu} g_p^y dE(\omega)\\ &= \int_{\Sigma_\mu} \omega(x)dE(\omega) \int_{\Sigma_\mu} \omega(y)dE(\omega).\end{aligned}$$

Consequently $\int_G \Psi(xty)d\mu(t) = \Psi(x)\Psi(y)$, for all $x, y \in G$.

Since $\Psi(e) = \int_{\Sigma_\mu} \omega(e)dE(\omega) = E(\Sigma_\mu) = I$, this ends the proof. □

In particular we have the following corollary which extends the result obtained in [12].

Corollary 5.1. *Let $f : G \to \mathbb{C}^*$ be a measurable and essentially bounded solution of the integral equation* (1)*. Then*

(i) *(Existence) There exists a μ-spherical function Φ such that*

$$f(x) = \frac{\Phi(x) + \Phi(x^{-1})}{2}, \quad \textit{for all } x \in G.$$

(ii) *(Uniqueness) If $f = \frac{\Phi+\check{\Phi}}{2}$ and $f = \frac{\Psi+\check{\Psi}}{2}$ where Φ, Ψ are μ-spherical functions, then $\Phi = \Psi$ or $\Phi = \check{\Psi}$.*

Proof. Since f is a solution of equation (1), we have $\check{f} = f$. Hence for all $g \in \mu * K(G) * \mu$ and for almost all $y \in G$, we have

$$\begin{aligned}2\prec g, f\succ f(y) &= 2\int_G f(y)\check{f}(x)g(x)dx\\ &= \int_G \int_G g(x)f(x^{-1}ty)d\mu(t)dx + \int_G \int_G g(x)f(x^{-1}ty^{-1})d\mu(t)dx\\ &= \int_G (g * f)(ty)d\mu(t) + \int_G (g * f)(ty^{-1})d\mu(t)\\ &= (g * f)(y) + (g \check{*} f)(y).\end{aligned}$$

Consequently, f is a bounded and continuous function on G.

Now, by Theorem 5.1, the bounded continuous function f can be written in the form

$$f(x) = \frac{1}{2}\int_{\Sigma_\mu} (\omega(x) + \check{\omega}(x))dE(\omega), \quad x \in G.$$

Considering that the space $\mathbb{C}$ is a Hilbert space of one dimension, it is clear that the spectral measure E associated to the function f is carried by a single μ-spherical function Ψ on Σ_μ.

ii) μ-spherical functions are linearly independent (see [12]). This ends the proof. $\square$

Second Case: Bounded generalized cosine functions with exponential type of growth.

In the present paragraph we assume that $\mu = dk$, where dk is a normalized Haar measure of a subcompact group K such that (G, K) is a Gelfand pair, and we let $L_1(K\backslash G/K) = \{f \in L_1(G),\ f(kxk') = f(x)$ for all $x \in G, k,$ $k' \in K\}$ denote the Banach algebra of L_1, K-invariant functions on G.

A continuous K-invariant function $\varpi : G \to [1, +\infty[$ is called a subspherical function if it is pair $(\varpi(x^{-1}) = \varpi(x))$, and satisfies the inequality $\int_K \varpi(xky)dk \preceq \varpi(x)\varpi(y)$, for all $x, y \in G$.

We consider the following subset of spherical functions:
$\Sigma_{dk}^{\varpi} = \{\omega$ continuous complex spherical functions, and $|\omega(x)| \prec \varpi(x)\}$.
Σ_{dk}^{ϖ} turns out to be the Gelfand spectrum of the Banach algebra
$L_1^{\varpi}(K\backslash G/K) = \{f \in L_1(K\backslash G/K)$ such that $\int_G |f(x)|\varpi(x)dx < +\infty\}$.
Note that $\check{\Sigma_{dk}^{\varpi}} = \{\check{\omega},\ \omega \in \Sigma_{dk}^{\varpi}\} \subset \Sigma_{dk}^{\varpi}$ and $\Sigma_{dk} \subset \Sigma_{dk}^{\varpi}$.

Theorem 5.4. *Let Γ be a continuous generalized cosine function, such that $\Gamma(x)$ is a normal operator and satisfies $\|\Gamma(x)\| \preceq \varpi(x)$.*

Then there exists a Borel spectral measure E on Σ_{dk}^{ϖ} such that

$$\Gamma(x) = \frac{1}{2}\int_{\Sigma_{dk}^{\varpi}} \left(\omega(x) + \omega(x^{-1})\right) dE(\omega), \quad x \in G.$$

Proof. Let $\mathcal{F}_\varpi$ be the ϖ-Fourier transform $\mathcal{F}_\varpi : L_1^{\varpi}(K\backslash G/K) \to C_0(\Sigma_{dk}^{\varpi})$ and $\mathcal{F}_\varpi f(\omega) = \int_G f(x)\omega(x)dx$.

Obviously, we have $\mathcal{F}_\varpi f/\Sigma_{dk} = \mathcal{F} f$ for $f \in L_1^{\varpi}(K\backslash G/K) \subset L_1(K\backslash G/K)$. Similarly we define the mapping $\Lambda \circ \mathcal{F}_\varpi : L_1^{\varpi}(K\backslash G/K) \to \pounds(H)$ by
$\Lambda \circ \mathcal{F}_\varpi(f) = \int_G f(x)\Gamma(x)dx$.

With the help of the proof of Theorem 5.1, we can derive very simply the rest of the proof. □

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