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Approximate derivations and isomorphisms in algebras of unbounded operators

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Abstract. There are considered mappings between unbounded standard operator algebras. If these mappings are approximate isomorphisms or derivations, then they are actually isomorphisms or derivations. The proofs are quite similar to those of Šemrl for corresponding results for algebras of bounded operators.

1. Introduction and statement of the results

In recent years there can be observed some interest in the following kind of questions. Given an algebraic structure $\mathcal{A}$ (usually a ring or an algebra) and a mapping $G : \mathcal{A} \to \mathcal{A}$, what conditions are sufficient to identify G as an isomorphism/automorphism or a derivation?

Quite different kinds of algebras are considered: function algebras, Banach- or C*-algebras, operator algebras etc. However, there are only few papers investigating such questions for algebras of unbounded operators (e.g. [5] [6]).

One type of conditions is connected with the notion of locality. Another type of conditions is obtained if one drops or weakens the defining properties of a derivation or isomorphism to get in some sense approximate derivations or isomorphisms. This issue is well-known as stability of

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functional equations for some kind of mappings of algebraic structures; see for example [2] and the references therein.

There two papers by P. Šemrl [7], [8] discussing these questions in the context of operator algebras. To formulate the results let us collect some notions and notation.

Let $\mathcal{A} \subset \mathcal{B}$ be two algebras (over $\mathbb{C}$, for simplicity). A mapping $D : \mathcal{A} \to \mathcal{B}$ is called a linear derivation (or simply a derivation) if

i) $D(\lambda A) = \lambda D(A)$ for all $\lambda \in \mathbb{C}, A \in \mathcal{A}$

ii) $D(A + B) = D(A) + D(B)$ for all $A, B \in \mathcal{A}$

iii) $D(AB) = AD(B) + D(A)B$ for all $A, B \in \mathcal{A}$.

If one drops i) D is called an additive derivation or ring derivation; if only iii) is fulfilled, D is a multiplicative derivation.

For a Banach space $\mathcal{X}$ let us denote by $\mathcal{B}(\mathcal{X})$ the Banach algebra of all bounded operators on $\mathcal{X}$ and let $\mathcal{F}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X})$ be the ideal of all finite rank operators. An algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$ is said to be a standard operator algebra if it contains $\mathcal{F}(\mathcal{X})$. The results of Šemrl read as follows:

Theorem A ([7])**.** *Let $\mathcal{X}$ be an infinite-dimensional Banach space and $\mathcal{A}$ a standard operator algebra on $\mathcal{X}$. Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying*

$$\lim_{t\to\infty} \frac{f(t)}{t} = 0.$$

Suppose that $D : \mathcal{A} \to \mathcal{B}(\mathcal{X})$ is a mapping such that

$$\|D(AB) - AD(B) - D(A)B\| < f(\|A\| \cdot \|B\|)$$

for all $A, B \in \mathcal{A}$. Then there exists $T \in \mathcal{B}(\mathcal{X})$ such that $D(A) = AT - TA$ for all $A \in \mathcal{A}$, i.e. D is a spatial derivation.

Theorem B ([8])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, dim $\mathcal{X} = \infty$, and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $\epsilon > 0$, and assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective mapping satisfying $\|\Phi(AB) - \Phi(A)\Phi(B)\| \le \epsilon$ for all $A, B \in \mathcal{A}$. Then Φ is of the form $\Phi(A) = TAT^{-1}$, $A \in \mathcal{A}$, where $T : \mathcal{X} \to \mathcal{Y}$ is either a bounded linear bijective operator or a bounded conjugate linear bijective operator (i.e. Φ is a so-called spatial isomorphism in case T is linear).*

Let us emphasize that neither in Theorem A nor in Theorem B the mappings are assumed to be linear or additive! The mapping D in Theorem A could be called an approximate multiplicative derivation.

Now we turn to algebras of unbounded operators and start with the necessary notation (a standard reference for algebras of unbounded operators is [4]).

Let $\mathcal{D}$ be a dense linear manifold in a Hilbert space $\mathcal{H}$ with scalar product $\langle\ ,\ \rangle$ (which is supposed to be conjugate linear in the first and linear in the second component). The set of linear operators $\mathcal{L}^+(\mathcal{D}) = \{A : A\mathcal{D} \subset \mathcal{D},\ A^*\mathcal{D} \subset \mathcal{D}\}$ is a $*$-algebra with respect to the natural operations and the involution $A \to A^+ = A^* \mid \mathcal{D}$. The graph topology t on $\mathcal{D}$ induced by $\mathcal{L}^+(\mathcal{D})$ is generated by the directed family of seminorms $\phi \to \|\phi\|_A = \|A\phi\|,\ \forall\ A \in \mathcal{L}^+(\mathcal{D}),\ \phi \in \mathcal{D}$. $\mathcal{D}$ is called an (F)-domain, if $(\mathcal{D}, t)$ is an (F)-space. Remark that in this case the graph topology t can be given by a system of seminorms $\{\|\cdot\|_n = \|A_n \cdot\|,\ n \in \mathbb{N},\ A_n \in \mathcal{L}^+(\mathcal{D})\}$ with:

$$A_1 = I,\ A_n = A_n^+,\ \|A_n\phi\| \le \|A_{n+1}\phi\| \quad \text{for all } \phi \in \mathcal{D},\ n \in \mathbb{N}.$$

A standard operator algebra is a $*$-subalgebra $\mathcal{A}(\mathcal{D}) \subset \mathcal{L}^+(\mathcal{D})$ containing the ideal $\mathcal{F}(\mathcal{D}) \subset \mathcal{L}^+(\mathcal{D})$ of all finite rank operators on $\mathcal{D}$.

In what follows we consider domains $\mathcal{D}$ with additional properties. One of those will be the following:

(B) There exists an infinite orthonormal system (ϕ_n) in $\mathcal{D}$ with the following two properties:
i) there is a sequence (t_n), $t_n \neq 0$, $t_n \in \mathbb{C}$ such that $\sum t_n\phi_n \in \mathcal{D}$,
ii) for all (s_n), $s_n \in \mathbb{C}$ and $|s_n| \le |t_n|$, $\sum s_n\phi_n$ belongs also to $\mathcal{D}$.

Let us remark that this property is clearly fulfilled in the case when $\mathcal{D} = \mathcal{H}$ is an infinite dimensional Hilbert space. In a Banach space (ϕ_n) must be replaced by an appropriate basic sequence. In our context of algebras of unbounded operators this property holds if $(\mathcal{D}, t)$ is an (F)-space or a (QF)-space (for this notion see [4]) which contains at least one bounded set $\mathcal{M}$ which spans an infinite dimensional (F)-subspace of $(\mathcal{D}, t)$.

The proofs of the main results in the next two sections are adaptions of those given by Šemrl for algebras of bounded operators. Nevertheless, it seems worthwhile to carry out the proofs for algebras of unbounded operators in detail. While the algebraic part of the original proofs cause no problems, we must be careful when we are concerned with domain questions. Here several modifications of the assumptions or the proofs are necessary.

2. Approximate derivations

To prove the main result of this section we need the following two results concerning derivations. For this let $\mathcal{A}(\mathcal{D})$ be a standard operator algebra.

i) Let $D : \mathcal{A}(\mathcal{D}) \to \mathcal{L}^+(\mathcal{D})$ be a derivation. Then D is spatial, i.e. there is a $T \in \mathcal{L}^+(\mathcal{D})$ such that $D(A) = [T, A] = TA - TA$ for all $A \in \mathcal{A}(\mathcal{D})$, (see [4]).

ii) Let $\mathcal{D}$ fulfil condition (B) (see Section 1) and let $D : \mathcal{A}(\mathcal{D}) \to \mathcal{L}^+(\mathcal{D})$ be an additive derivation. Then D is spatial (see [6]).

Theorem 2.1. *Let $\mathcal{A} \subset \mathcal{L}^+(\mathcal{D})$ be a standard operator algebra on $\mathcal{D}$ with property (B). Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying*

$$f(0) = 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{f(t)}{t} = 0.$$

Suppose $D : \mathcal{A} \to \mathcal{L}^+(\mathcal{D})$ is a mapping with

$$|\langle \phi, (D(AB) - AD(B) - D(A)B)\psi\rangle| \leq r(A, B; \phi, \psi) \quad \forall \phi, \psi \in \mathcal{D}; A, B \in \mathcal{A}$$

where $r(A, B; \phi, \psi)$ denotes one of the following three expressions: $f(\|A\phi\| \cdot \|B\psi\|)$, $f(|\langle A\phi, B\psi\rangle|)$ or $f(|\langle A^+\phi, B\psi\rangle|)$. Then D is a spatial derivation, i.e. there is an $S \in \mathcal{L}^+(\mathcal{D})$ such that $D(A) = [A, S] = AS - SA$.

Proof. The idea consists in proving that D is a multiplicative derivation on $\mathcal{A}$ mapping $\mathcal{F}(\mathcal{D})$ into itself. Then a result of Daif [1] implies that D is an additive derivation and by [6] the proof will be completed. Define $F : \mathcal{A} \to \mathcal{L}(\mathcal{D} \oplus \mathcal{D})$ by

$$F(A) = \begin{pmatrix} A & D(A) \\ 0 & A \end{pmatrix}.$$

Put $\Phi = \binom{\phi}{0}, \Psi = \binom{0}{\psi} \in \mathcal{D} \oplus \mathcal{D}$ with $\phi, \psi \in \mathcal{D}$. Then

$$\begin{aligned}
&|\langle \Phi, (F(AB) - F(A)F(B))\Psi\rangle| \\
&\quad = \left| \left\langle \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \begin{pmatrix} AB & D(AB) \\ 0 & AB \end{pmatrix} \begin{pmatrix} 0 \\ \psi \end{pmatrix} - \begin{pmatrix} A & D(A) \\ 0 & A \end{pmatrix} \begin{pmatrix} B & D(B) \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\rangle \right| \\
&\quad = |\langle \phi, (D(AB) - AD(B) + D(A)B)\psi\rangle| \le r(A, B; \phi, \psi).
\end{aligned} \tag{1}$$

Let $A, B, C \in \mathcal{A}$ be arbitrary. A simple computation shows

$$\begin{aligned}
&\langle \phi, (D(AB) - AD(B) - D(A)B)C\psi\rangle \\
&\quad = \left\langle \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & (D(AB) - AD(B) - D(A)B)C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\rangle \\
&\quad = \langle \Phi, (F(AB) - F(A)F(B))F(C)\Psi\rangle.
\end{aligned} \tag{2}$$

Now let us estimate the right-hand side of (2) using (1).

$$\begin{aligned}
&|\langle \Phi, (F(AB) - F(A)F(B))F(C)\Psi\rangle| \\
&\quad = |\langle \Phi, (F(AB)F(C) - F(ABC))\Psi + (F(ABC) - F(A)F(BC))\Psi \\
&\qquad + F(A)(F(BC) - F(B)F(C))\Psi\rangle| \\
&\quad \le |\langle \Phi, (F(AB)F(C) - F(ABC))\Psi\rangle| + |\langle \Phi, (F(ABC) - F(A)F(BC))\Psi\rangle| \\
&\qquad + |\langle F(A)^+\Phi, (F(BC) - F(B)F(C))\Psi\rangle| \\
&\quad \le r(AB, C; \phi, \psi) + r(A, BC; \phi, \psi) + r(B, C; A^+\phi, \psi).
\end{aligned}$$

Replace C by tC, $t \in \mathbb{R}_+$ and let t go to infinity. Using the properties of f and the possible expressions for r we get $\langle \phi, (D(AB) - AD(B) - D(A)B)C\psi\rangle = 0$ for all $\phi, \psi \in \mathcal{D}$, i.e. $(D(AB) - AD(B) - D(A)B)C = 0$ for all $C \in \mathcal{A}$, hence $D(AB) - AD(B) - D(A)B = 0$. That means, D is a multiplicative derivation. We show that D maps $\mathcal{F}(\mathcal{D})$ into itself. Let $F \in \mathcal{F}(\mathcal{D})$, and choose a projection $P \in \mathcal{F}(\mathcal{D})$ with $PF = F$. Then $D(PF) = D(F) = D(P)F + PD(F)$ implies $D(F) \in \mathcal{F}(\mathcal{D})$. But $\mathcal{F}(\mathcal{D})$ has the following properties:

$A\mathcal{F}(\mathcal{D}) = 0$ implies $A = 0$; there is a nontrivial idempotent $P \in \mathcal{F}(\mathcal{D})$ such that $P\mathcal{F}(\mathcal{D})A = 0$ implies $A = 0$, $PAP\mathcal{F}(\mathcal{D})(I - P) = 0$ implies $PAP = 0$. Therefore the assumptions of a theorem of Daif [1] are satisfied and D is a ring derivation on $\mathcal{F}(\mathcal{D})$. By [6] D is a spatial derivation, i.e.

there is an $S \in \mathcal{L}^+(\mathcal{D})$ with $D(A) = AS - SA$ for all $A \in \mathcal{F}(\mathcal{D})$. Now let $B \in \mathcal{L}^+(\mathcal{D}), A \in \mathcal{F}(\mathcal{D})$ be arbitrary, then

$$\begin{aligned}
&|\langle \phi, (A(BS - SB - D(B))\psi\rangle| \\
&\quad = |\langle \phi, (ABS - SAB - AD(B) - ASB + SAB)\psi\rangle| \\
&\quad = |\langle \phi, (D(AB) - AD(B) - D(A)B)\psi\rangle| \leq r(A, B; \phi, \psi).
\end{aligned}$$

Now the same argument as above works. Replace A by tA, divide by t and let t tend to infinity. This implies the desired relation: $D(B) = BS - SB$ for all $B \in \mathcal{L}^+(\mathcal{D})$. The proof is complete. □

Remark 2.2. In contrast to Theorem A we (must) suppose that $f(0)=0$, because it may happen that one of the arguments of f indicated in Theorem 2.1 is zero but A or B is not the zero operator. This cannot happen in Theorem A, because there the arguments are $\|A\|, \|B\|$ resp.

3. Approximate isomorphisms

In this section we prove a variant of Theorem B above for algebras of unbounded operators in Hilbert space.

Theorem 3.1. *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{L}^+(\mathcal{D})$ be standard operator algebras on an (F)-domain $\mathcal{D}$. Suppose $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective mapping such that for every $\psi \in \mathcal{D}$ there is a $\delta_\psi > 0$ with*

$$\|(\Phi(AB) - \Phi(A)\Phi(B))\psi\| \leq \delta_\psi \quad \forall A, B \in \mathcal{A}.$$

Then there exists a $T : \mathcal{D} \to \mathcal{D}$, bijective, linear or bijective, conjugate linear such that

$$\Phi(A) = TAT^{-1} \quad \forall A \in \mathcal{A}.$$

Proof. The proof is an adaption of [8] and is divided into four steps.

Step 1: Φ is a ring isomorphism.

Let $A, B \in \mathcal{A}$, $\phi, \psi \in \mathcal{D}$, $C \in \mathcal{F}(\mathcal{D}) \subset \mathcal{B}$. Then there is a $D \in \mathcal{A}$ such that $\Phi(D) = C$. Next

$$|\langle \phi, (\Phi(AB) - \Phi(A)\Phi(B))\Phi(D)\psi\rangle|$$

$$
\begin{aligned}
&= |\langle \phi, (\Phi(AB)\Phi(D) - \Phi(ABD) + \Phi(ABD) - \Phi(A)\Phi(BD) \\
&\quad + \Phi(A)\Phi(BD) - \Phi(A)\Phi(B)\Phi(D))\psi\rangle| \\
&\leq \|\phi\| \cdot \|(\Phi(AB)\Phi(D) - \Phi(ABD))\psi\| \\
&\quad + \|\phi\| \cdot \|(\Phi(ABD) - \Phi(A)\Phi(BD))\psi\| \\
&\quad + \|\Phi(A)^{+}\phi\| \cdot \|(\Phi(BD) - \Phi(B)\Phi(D))\psi\| \leq \delta_\psi(2\|\phi\| + \|\Phi(A)^{+}\phi\|).
\end{aligned}
$$

As in the proof of Theorem 2.1, replace C by tC, divide by t and let $t \to \infty$. Then the left-hand side of the estimation above is equal to zero. Since $\phi, \psi \in \mathcal{D}, C \in \mathcal{F}(\mathcal{D})$ are arbitrary, it follows that $\Phi(AB) = \Phi(A)\Phi(B)$, i.e. Φ is multiplicative. Using the fact that every standard operator algebra is a prime ring, a result by Martindale [3] implies that Φ is a ring isomorphism.

Step 2: Φ maps rank one projections onto rank one projections.

Let $\phi_0, \psi_0 \in \mathcal{D}$ such that $\langle \psi_0, \phi_0\rangle = 1$, i.e. $\langle \psi_0, \cdot\rangle\phi_0$ is a rank one projection. Notice that we will use these ψ_0, ϕ_0 in the next step, too. $\Phi(\langle \psi_0, \cdot\rangle\phi_0)$ is a projection. Suppose it is not of rank one. Then there are projections Q_1, Q_2 such that $\Phi(\langle \psi_0, \cdot\rangle\phi_0) = Q_1 + Q_2$, Q_1 is of rank one and in $\mathcal{F}(\mathcal{D}) \subset \mathcal{B}$, hence $Q_2 = \Phi(\langle \psi_0, \cdot\rangle\phi_0) - Q_1 \in \mathcal{B}$. Since Φ is a ring isomorphism there are non-zero projections $P_1, P_2 \in \mathcal{A}$ such that $\Phi(P_i) = Q_i$. This implies $\langle \psi_0, \cdot\rangle\phi_0 = P_1 + P_2$, a contradiction. Hence $\Phi(\langle \psi_0, \cdot\rangle\phi_0)$ is one-dimensional, i.e. equal to $\langle \psi_1, \cdot\rangle\phi_1$ with $\phi_1, \psi_1 \in \mathcal{D}, \langle \psi_1, \phi_1\rangle = 1$. These vectors ϕ_1, ψ_1 will also be used in the next step.

Step 3: Define an additive operator $T : \mathcal{D} \to \mathcal{D}$ by

$$T\phi = \Phi(\langle \psi_0, \cdot\rangle\phi)\phi_1$$

with ϕ_1, ψ_0 as above. Then for all $\phi \in \mathcal{D}$, $A \in \mathcal{A}$ it follows that $TA\phi = \Phi(A\langle \psi_0, \cdot\rangle\phi)\phi_1 = \Phi(A) \cdot \Phi(\langle \psi_0, \cdot\rangle\phi)\phi_1 = \Phi(A)T\phi$. Hence

$$TA = \Phi(A)T \quad \text{on } \mathcal{D} \tag{3}$$

It remains to prove that T is bijective and homogenous, which is the more complicated part.

i) T is injective: Let $T\theta = 0$ for some $\theta \in \mathcal{D}$. Choose $\chi \in \mathcal{D}$ with $\langle \theta, \chi\rangle = \langle \chi, \theta\rangle = 1$. Then

$0 = \Phi(\langle\chi,\cdot\rangle\phi_0)T\theta = \Phi(\langle\chi,\cdot\rangle\phi_0)\cdot\Phi(\langle\psi_0,\cdot\rangle\theta)\phi_1 = \Phi(\langle\chi,\cdot\rangle\phi_0\cdot\langle\psi_0,\cdot\rangle\theta)\phi_1 = \Phi(\langle\psi_0,\cdot\rangle\phi_0)\phi_1 = \langle\psi_1,\phi_1\rangle\phi_1 = \phi_1$.

This is in contradiction with $\phi_1 \neq 0$.

ii) T is surjective: Let $\rho \in \mathcal{D}$. It must be shown that there is $\chi \in \mathcal{D}$ such that $T\chi = \rho$. Choose a nonzero $\sigma \in \mathcal{D}$. Since $T\sigma \neq o$ there is a $\phi \in \mathcal{D}$ with $\langle\phi, T\sigma\rangle = 1$. Moreover, the surjectivity of Φ implies the existence of an $A \in \mathcal{A}$ such that $\Phi(A) = \langle\phi,\cdot\rangle\rho$. Equation (3) gives the desired result: $TA\sigma = \Phi(A)T\sigma = \langle\phi, T\sigma\rangle\rho = \rho$.

Hence $\chi = A\sigma$ is the desired vector from $\mathcal{D}$.

Step 4: T is homogeneous.

This tricky step uses property (B) of the domain $\mathcal{D}$.

Let $\phi,\psi \in \mathcal{D}$ with $\langle\psi,\phi\rangle = 1$, i.e. $\langle\psi,\cdot\rangle\phi$ is a projection. From Step 2, $\Phi(\langle\psi,\cdot\rangle\phi) = \langle\rho,\cdot\rangle\chi$ for appropriate $\chi,\rho \in \mathcal{D}$. Use equation (3) with $A = \langle\psi,\cdot\rangle\phi$, then on the one hand $T\phi = (T\cdot\langle\psi,\cdot\rangle\phi)\phi = \Phi(\langle\psi,\cdot\rangle\phi)T\phi = \langle\rho,\cdot\rangle\chi\cdot T\phi = \langle\rho, T\phi\rangle\chi$. On the other hand, for arbitrary nonzero $\lambda \in \mathbb{C}$:
$0 \neq T(\lambda\phi) = T\cdot(\langle\psi,\cdot\rangle\phi)(\lambda\phi) = \Phi(\langle\psi,\cdot\rangle\phi)\ T(\lambda\phi) = \langle\rho,\cdot\rangle\chi\cdot T(\lambda\phi) = \langle\rho, T(\lambda\phi)\rangle\chi$.

This means that $T(\lambda\phi)$ is in the span of $T\phi$ for all $\lambda \in \mathbb{C}, \phi \in \mathcal{D}$. Hence for any nonzero $\phi \in \mathcal{D}$ there is an additive mapping $\tau_\phi : \mathbb{C} \to \mathbb{C}$ such that

$$T(\lambda\phi) = \tau_\phi(\lambda)T\phi \quad \text{for all } \lambda \in \mathbb{C}.$$

It is quite standard (cf. e.g. [8]) to prove that τ_ϕ is actually independent of ϕ. Hence there is an additive mapping $\tau : \mathbb{C} \to \mathbb{C}$ such that $T(\lambda\phi) = \tau(\lambda)T\phi$ for all $\phi \in \mathcal{D}$, $\lambda \in \mathbb{C}$.

From $\tau(\lambda\mu)T\phi = T(\lambda\mu\phi) = \tau(\lambda)T(\mu\phi) = \tau(\lambda)\tau(\mu)T\phi$ it is seen that τ is a ring homomorphism of $\mathbb{C}$. Since $\tau(1) = 1$, $\tau(0) = 0$, the range of τ contains the rationals $\mathbb{Q}$ and also $\mathbb{Q} + i\,\mathbb{Q}$, i.e. the range of τ is dense in $\mathbb{C}$.

Moreover

$$T^{-1}(p\phi) = pT^{-1}\phi \quad \text{for all } p \in \mathbb{Q},\ \phi \in \mathcal{D}. \tag{4}$$

We prove that τ is continuous. Suppose this is not the case, then τ is unbounded on every neighbourhood of zero. Now we will construct two sequences $(\phi_n), (\psi_n) \subset \mathcal{D}$ with the properties:

(i) $\|A_k\psi_k\| < 2^{-k}$ for all $k \in \mathbb{N}$;

(ii) $\|\phi_k\| < 2^{-k}$ for all $k \in \mathbb{N}$;

(iii) $\langle \psi_i, T^{-1}\phi_k \rangle = 0$ for all $i \neq k$;

(iv) $|\tau(\langle \psi_n, T^{-1}\phi_n \rangle)| > n + \sum_{i=1}^{n-1} |\tau(\langle \psi_i, T^{-1}\phi_i \rangle)|$.

The operators A_k are supposed to define the topology t as described in Section 1.

Let $\phi_1, \psi_1 \in \mathcal{D}$ be arbitrary with $\|\phi_1\| < 2^{-1}$, $\|\psi_1\| < 2^{-1}$. Suppose the vectors $\phi_1, \cdots, \phi_n$, $\psi_1, \cdots, \psi_n$ are already chosen with properties (i)–(iv). Form the following spaces: $\mathcal{K}_n = \mathrm{lin}\{T^{-1}\phi_i, \psi_i : 1 \leq i \leq n\} \subset \mathcal{D}$, $\mathcal{D}_n = \mathcal{K}_n^{\perp} \cap \mathcal{D}$. Let P_n be the orthoprojector on $\mathcal{K}_n$. Since $\mathcal{K}_n$ is finite-dimensional, $P_n, I - P_n \in \mathcal{L}^+(\mathcal{D})$, hence $(I - P_n)\mathcal{D} \subset \mathcal{D}_n$, that means $\mathcal{D}_n \neq \{0\}$. Using the surjectivity of T and equation (4) there exists $\phi_{n+1} \in \mathcal{D}$ such that $T^{-1}\phi_{n+1} \in \mathcal{D}_n$ with $\|\phi_{n+1}\| < 2^{-n-1}$. Next choose $\widehat{\psi}_{n+1} \in \mathrm{lin}\{T^{-1}\phi_1, \cdots, T^{-1}\phi_n\}^{\perp} \cap \mathcal{D}$ such that

$$\langle \widehat{\psi}_{n+1}, T^{-1}\phi_{n+1} \rangle \neq 0, \quad \|A_{n+1}\widehat{\psi}_{n+1}\| < 2^{-n-1}.$$

As τ is unbounded on the neighbourhood of zero $\{\lambda\langle \widehat{\psi}_{n+1}, T^{-1}\phi_{n+1}\rangle : |\lambda| < 1, \lambda \in \mathbb{C}\}$, there is $\psi_{n+1} = \lambda\widehat{\psi}_{n+1}$, $|\lambda| < 1$ such that $|\tau(\langle \psi_{n+1}, T^{-1}\phi_{n+1} \rangle)|$ fulfils condition (iv). Moreover, by construction (iii) is also fulfilled. The sequence (χ_n) with $\chi_n = \sum_{i=1}^{n} \phi_i$ is $\| \; \|$-bounded and as a consequence of (i) $\psi = \sum_{i=1}^{\infty} \psi_i$ belongs to $\mathcal{D}$. Choose $\phi \in \mathcal{D}$ such that $\langle \phi, \psi \rangle = 1$, i.e. $\langle \psi, \cdot \, \rangle\phi$ is a projection in $\mathcal{L}^+(\mathcal{D})$ and so $T(\langle \psi, \cdot \, \rangle\phi)T^{-1} = \Phi(\langle \psi, \cdot \, \rangle\phi)$ is also a projection in $\mathcal{L}^+(\mathcal{D})$, hence a bounded operator. But

$$\begin{aligned}
\|\Phi(\langle \psi, \cdot \, \rangle\phi)\chi_n\| &= \left\| T(\langle \psi, \cdot \, \rangle\phi)T^{-1}\left(\sum_{i=1}^{n} \phi_i\right)\right\| = \left\| T\left(\left\langle \psi, T^{-1}\sum_{i=1}^{n} \phi_i \right\rangle\right)\phi \right\| \\
&= \left| \tau\left(\left\langle \psi, T^{-1}\sum_{i=1}^{n} \phi_i \right\rangle\right)\right| \|T\phi\| = \left| \sum_{i=1}^{n} \tau(\langle \psi, T^{-1}\phi_i \rangle)\right| \|T\phi\| \\
&= \left| \sum_{i=1}^{n} \tau(\langle \psi_i, T^{-1}\phi_i \rangle)\right| \|T\phi\| \\
&\geq \left(|\tau(\langle \psi_n, T^{-1}\phi_n \rangle)| - \sum_{i=1}^{n} |\tau(\langle \psi_i, T^{-1}\phi_i \rangle)| \right) \|T\phi\| \geq n\|T\phi\|.
\end{aligned}$$

This contradiction implies that τ is continuous. Consequently $\tau(\lambda) = \lambda$ or $\tau(\lambda) = \bar{\lambda}$. Therefore, T is linear or conjugate linear. Now we proved among other things that Φ preserves rank one projections into both directions (i.e. $\Phi(P)$ is of rank one if and only if P is of rank one). In case T is linear this implies that T belongs to $\mathcal{L}^+(\mathcal{D})$ [5]. This completes the proof. $\square$

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