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On solutions of a conditional generalization of the Gołąb–Schinzel equation

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Abstract. We determine the functions $f:\mathbb{R}_+\to\mathbb{R}$ satisfying equation (1), continuous at a point $a\in\mathbb{R}_+$ such that $f(a)\neq 0$. As a consequence we obtain a solution of a problem of P. Kahlig and J. Matkowski and a partial solution of a problem of J. Brzdęk.

Let $\mathbb{N}$ and $\mathbb{R}$ denote, as usual, the sets of positive integers and reals. Motivated by a problem of P. Kahlig, arising from meteorology and fluid mechanics (cf. [14]), J. Aczél and J. Schwaiger [3] have determined the continuous solutions $f:\mathbb{R}\to\mathbb{R}$ of the following conditional version of the well known Gołąb–Schinzel functional equation

$$f(x+f(x)y)=f(x)f(y)\quad \text{for } x\geq 0,\ y\geq 0.$$

Some further conditional generalizations of the Gołąb–Schinzel equation have been considered in [9], [17] and [18].

In connection with those results, at the 38th International Symposium on Functional Equations (Noszvaj, Hungary, June 11–17, 2000), J. Brzdęk (see [8]) raised, among others, the problem of solving the conditional equation

$$f(x+f(x)y)=f(x)f(y)\quad \text{whenever } x,y,x+f(x)y\in\mathbb{R}_+,\tag{1}$$

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in the class of functions $f : \mathbb{R}_+ \to \mathbb{R}$ that are continuous at a point, where $\mathbb{R}_+ = (0, \infty)$. A first partial answer to the problem has been given in [15], where equation (1) has been solved in the class of functions $f : \mathbb{R}_+ \to [0, \infty)$, continuous at a point $a \in \mathbb{R}_+$ such that $f(a) > 0$. In this paper we improve that outcome by solving equation (1) in the class of functions $f : \mathbb{R}_+ \to \mathbb{R}$ that are continuous at a point $a \in \mathbb{R}_+$ such that $f(a) \neq 0$. Thus we also give an answer to Problem 1 in [14] (see Remark 2) and generalize the results in [3], [9], [17] and (to some extent) [18]. Let us mention that our result is closely related to that of [17], where L. Reich has determined the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the conditional equation

$$f(x + f(x)y) = f(x)f(y) \quad \text{whenever } x, y, x + f(x)y \geq 0. \tag{2}$$

For more information on the Gołąb–Schinzel functional equation, some recent results, applications, generalizations and further references see also [1], [2], [4]–[7], [10]–[13] and [16].

From now on we assume that $f : \mathbb{R}_+ \to \mathbb{R}$ is a solution of equation (1) and $\lim_{x\to 0^+} f(x) = 1$, unless explicitly stated otherwise.

Let us start with some lemmas.

Lemma 1. *Suppose that $f(y_2) = f(y_1) \neq 0$ for some $y_2 > y_1 > 0$. Then there exists $x_0 > 0$ such that $f(x_0) = 1$ and $f(t + x_0) = f(t)$ for $t > 0$.*

Proof. First assume that $f(y_2) = f(y_1) < 0$. Then there exists a point $x_0 > 0$ such that $y_1 = y_2 + x_0 f(y_2)$. Thus

$$f(y_1) = f(y_2 + x_0 f(y_2)) = f(y_2)f(x_0) = f(y_1)f(x_0),$$

whence $f(x_0) = 1$.

Further, in the case $f(y_1) = f(y_2) > 0$, there exists a point $x_0 > 0$ such that $y_2 = y_1 + x_0 f(y_1)$. Since

$$f(y_2) = f(y_1 + x_0 f(y_1)) = f(y_1)f(x_0) = f(y_2)f(x_0),$$

again we have $f(x_0) = 1$.

Consequently, in either of the cases, by (1) we have

$$f(t + x_0) = f(x_0 + t) = f(x_0 + f(x_0)t) = f(x_0)f(t) = f(t) \quad \text{for } t > 0. \ \square$$

Lemma 2. *Let $y_1, y_2 \in \mathbb{R}$, $y_2 > y_1 > 0$ and $f(y_1) = f(y_2) > 0$. Then $f(t + (y_2 - y_1)) = f(t)$ for $t > 0$.*

Proof. On account of (1) we have

$$\begin{aligned} f(t + (y_2 - y_1)) &= f\left(y_2 + \frac{t - y_1}{f(y_1)} f(y_1)\right) \\ &= f\left(y_2 + \frac{t - y_1}{f(y_1)} f(y_2)\right) = f(y_2) f\left(\frac{t - y_1}{f(y_1)}\right) \\ &= f(y_1) f\left(\frac{t - y_1}{f(y_1)}\right) = f\left(y_1 + \frac{t - y_1}{f(y_1)} f(y_1)\right) = f(t) \\ &\text{for } t > y_1. \end{aligned} \tag{3}$$

Fix $t_0 > 0$. According to Lemma 1 there exists $x_0 \in \mathbb{R}_+$ with $f(x_0) = 1$. Take $n \in \mathbb{N}$ such that $t_0 + nx_0 > y_1$. Then, in view of (3), $f(t_0 + nx_0) = f(t_0 + nx_0 + (y_2 - y_1))$. This and Lemma 1 imply

$$f(t_0) = f(t_0 + nx_0) = f(t_0 + (y_2 - y_1) + nx_0) = f(t_0 + (y_2 - y_1)). \quad \square$$

Lemma 3. *Suppose that there exists $y_1, y_2 \in \mathbb{R}$ with $y_2 > y_1 > 0$ and $f(y_1) = f(y_2) > 0$. Then there exists $x_0 > 0$ such that*

(a) *$f(t + f(z)x_0) = f(t)$ for $t > 0$, $z > 0$ with $f(z) > 0$;*

(b) *if z_1, $z_2 > 0$ and $f(z_2) > f(z_1) > 0$, then*

$$f(t + (f(z_2) - f(z_1))x_0) = f(t) \quad \text{for } t > 0.$$

Proof. (a) According to Lemma 1 there exists $x_0 > 0$ with $f(x_0) = 1$. Since $f(z + f(z)x_0) = f(z)f(x_0) = f(z) > 0$, Lemma 2 yields

$$f(t) = f(t + z + x_0 f(z) - z) = f(t + x_0 f(z)) \quad \text{for } t > 0.$$

(b) Note that $t + (f(z_2) - f(z_1))x_0 > 0$ for $t > 0$. Thus using (a) twice, for $z = z_1$ and $z = z_2$, for every $t > 0$ we have

$$\begin{aligned} f(t + (f(z_2) - f(z_1))x_0) &= f(t + (f(z_2) - f(z_1))x_0 + f(z_1)x_0) \\ &= f(t + f(z_2)x_0) = f(t). \end{aligned} \quad \square$$

Lemma 4. *Suppose that there exist $y_1, y_2 \in \mathbb{R}$ with $y_2 > y_1 > 0$ and $f(y_1) = f(y_2) \neq 0$. Then, for every $d > 0$, there exists $c \in (0, d)$ with $f(t + c) = f(t)$ for $t > 0$.*

Proof. First suppose that there exists $\delta > 0$ such that $f(x) = \text{const}$ for $x \in U := (0, \delta]$. Since $\lim_{x\to 0^+} f(x) = 1$, $f(x) > 0$ for $x \in U$. Hence, in view of Lemma 2, we have

$$f(t + (\delta - x)) = f(t) \quad \text{for } t > 0, x \in U.$$

Now assume that there does not exist any $\delta > 0$ such that $f(x) = \text{const}$ for $x \in U := (0, \delta]$. Take $\varepsilon \in (0, 1)$. Since $\lim_{x\to 0^+} f(x) = 1$, there exists $\delta > 0$ such that $f(x) \in (1 - \varepsilon, 1 + \varepsilon)$ for $x \in U_1 := (0, \delta)$. Take $x_1, x_2 \in U_1$ with $f(x_1) < f(x_2)$. Then $f(x_2) - f(x_1) < 2\varepsilon$ and $f(x_1) > 0$. Moreover, according to Lemma 3(b), there is $x_0 > 0$ with

$$f(t + (f(x_2) - f(x_1))x_0) = f(t) \quad \text{for } t > 0.$$

To complete the proof it is enough to observe that the point x_0 may chosen independently of the values of x_1 and x_2 and therefore, by a suitable choice of ε, the value $c := (f(x_2) - f(x_1))x_0$ can be made arbitrarily small. □

Lemma 5. *If there exist $y_1, y_2 \in \mathbb{R}$ such that $y_2 > y_1 > 0$ and $f(y_2) = f(y_1) \neq 0$, then $f \equiv 1$.*

Proof. For the proof by contradiction suppose that there exists $t > 0$ with $f(t) \neq 1$. Put $\varepsilon := |f(t) - 1|$. Since $\lim_{x\to 0^+} f(x) = 1$, there exists $\delta > 0$ such that $|f(x) - 1| < \varepsilon$ for $x \in (0, \delta)$. From Lemma 4 we infer that there is $x_1 \in (0, \delta)$ with $f(x_1) = f(t)$, which means that $|f(t) - 1| < \varepsilon$, contrary to the definition of ε. □

Lemma 6. *There is $c \in \mathbb{R}$ such that $f(x) \in \{cx + 1, 0\}$ for all $x > 0$.*

Proof. The case where $f \equiv 1$ is trivial. Therefore assume that $f(x) \neq 1$ for some $x > 0$. First we show that there exists $c \in \mathbb{R}$ with

$$\frac{f(x) - 1}{x} = c \quad \text{for } x > 0 \quad \text{with } f(x) > 0. \tag{4}$$

For the proof by contradiction suppose that $x > y > 0$, $f(x), f(y) > 0$ and

$$\frac{f(x) - 1}{x} \neq \frac{f(y) - 1}{y}.$$

Then

$$x + yf(x) \neq y + xf(y),$$

and

$$f(x+yf(x)) = f(x)f(y) = f(y+xf(y)) \neq 0.$$

Thus, by Lemma 5, $f \equiv 1$, a contradiction.

Now suppose that there exists $x > 0$ with $f(x) < 0$ and

$$\frac{f(x)-1}{x} \neq c. \tag{5}$$

Since $\lim_{x\to 0^+} f(x) = 1$, there exists $d > 0$ with $f(d) > 0$ and $x+df(x) > 0$. Next, by (4) $\frac{f(d)-1}{d} = c$. This and (5) imply that

$$\frac{f(x)-1}{x} \neq \frac{f(d)-1}{d}.$$

Thus

$$x + df(x) \neq d + xf(d),$$

and

$$f(x+df(x)) = f(x)f(d) = f(d+xf(d)) \neq 0.$$

Hence on account of Lemma 5, $f \equiv 1$, a contradiction.

In this way we have shown that there is $c \in \mathbb{R}$ such that $\frac{f(x)-1}{x} = c$ for $x > 0$ with $f(x) \neq 0$, which implies the statement. □

Lemma 7. *Suppose that there exists $y > 0$ with $f(y) \neq 0$ and $c := \frac{f(y)-1}{y} \neq 0$. Then the following statements are valid:*

I) (a) *In the case $c < 0$, $f(x) = cx + 1$ for $x \in (0, -\frac{1}{c})$.*

(b) *In the case $c > 0$, $f(x) = cx + 1$ for $x > 0$.*

II) *In the case $c < 0$, either $f(x) = cx + 1$ for $x \geq -\frac{1}{c}$ or $f(x) = 0$ for $x \geq -\frac{1}{c}$.*

Proof. I) Since $\lim_{x\to 0^+} f(x) = 1$, there exists $h_1 > 0$ such that

$$f(x) > 0 \quad \text{for } x \in (0, h_1]. \tag{6}$$

Define a sequence $\{h_n\}$ by

$$h_{n+1} = h_1 + f(h_1)h_n \quad \text{for } n \in \mathbb{N}$$

and let $U_n := (0, h_n]$. Note that $f(h_{n+1}) = f(h_1 + f(h_1)h_n) = f(h_1)f(h_n)$ for $n \in \mathbb{N}$. Thus, by induction, we get

$$f(h_n) = (f(h_1))^n \quad \text{for } n \in \mathbb{N}. \tag{7}$$

Next we prove that

$$f(x) > 0 \quad \text{for } x \in U_n. \tag{8}$$

So fix $n \in \mathbb{N}$ and assume (8). Define a function $g : U_n \to U_{n+1}$ by

$$g(x) = h_1 + f(h_1)x \quad \text{for } x \in U_n.$$

Then $g(U_n) = g((0, h_n]) = (h_1, h_{n+1}] =: V_{n+1}$ and

$$f(V_{n+1}) = f(g(U_n)) = f(h_1)f(U_n) \subset \mathbb{R}_+.$$

Since $U_{n+1} = U_1 \cup V_{n+1}$, $f(U_{n+1}) \subset \mathbb{R}_+$. Consequently, in view of (6), (8) holds for every $n \in \mathbb{N}$.

Observe that (6) and Lemma 6 imply $f(h_1) = ch_1 + 1$. Moreover $c \neq 0$ and $h_1 \neq 0$; whence $f(h_1) \neq 1$. Two cases may occur:

1) $f(h_1) < 1$ (then $c < 0$);
2) $f(h_1) > 1$ (then $c > 0$).

In the first case, by (7), we have

$$\lim_{n\to\infty} f(h_n) = \lim_{n\to\infty} (f(h_1))^n = 0. \tag{9}$$

Further, on account of (8), $f(h_n) > 0$ for $n \in \mathbb{N}$. Thus, according to Lemma 6, $f(h_n) = ch_n + 1$. This and (9) imply $\lim_{n\to\infty} ch_n + 1 = 0$. Therefore $\lim_{n\to\infty} h_n = -\frac{1}{c}$ and consequently from (8) and Lemma 6 we infer that $f(x) = cx + 1$ for $x \in (0, -\frac{1}{c})$.

Now consider case 2). Then, by (7), we get

$$\lim_{n\to\infty} f(h_n) = \lim_{n\to\infty} (f(h_1))^n = \infty. \tag{10}$$

On the other hand $f(h_n) = ch_n + 1$. Hence from (10) we derive $\lim_{n\to\infty} ch_n + 1 = \infty$, which means that $\lim_{n\to\infty} h_n = \infty$. Consequently (8) and Lemma 6 yield $f(x) = cx + 1$ for $x > 0$.

II) According to I) f is continuous on the interval $(0,-\frac{1}{c})$ and, by Lemma 6, $f(-\frac{1}{c})=0$. Suppose that there is a point $b_2>-\frac{1}{c}$ with $f(b_2)=0$. Take $b_1>-\frac{1}{c}$ and consider first the case where $b_1<b_2$. Let

$$g(x)=x+b_2f(x) \quad \text{for } x\in\left(0,-\frac{1}{c}\right).$$

Since $f(x)=cx+1$ for $x\in(0,-\frac{1}{c})$, we get

$$\lim_{x\to 0^+} g(x)=b_2 \quad \text{and} \quad \lim_{x\to -\frac{1}{c}} g(x)=-\frac{1}{c}. \tag{11}$$

Moreover by the continuity of f on $(0,-\frac{1}{c})$, g is continuous. This and (11) imply that there exists $x_1\in(0,-\frac{1}{c})$ with $g(x_1)=b_1$. Consequently

$$f(b_1)=f(g(x_1))=f(x_1+b_2f(x_1))=f(x_1)f(b_2)=0.$$

If $b_1>b_2$ we put $g(x)=x+b_1f(x)$ for $x\in(0,-\frac{1}{c})$ and obtain, in a similar way, $g(x_2)=b_2$ for some $x_2\in(0,-\frac{1}{c})$. Hence $0=f(b_2)=f(x_2)f(b_1)$, which implies $f(b_1)=0$.

Thus we have shown that either $f(x)=0$ for $x>-\frac{1}{c}$ or $f(x)\neq 0$ for $x>-\frac{1}{c}$. In the latter case, in view of Lemma 6, we get $f(x)=cx+1$ for $x>-\frac{1}{c}$. This completes the proof. □

Lemma 8. *Suppose that* $f(x)\in\{0,1\}$ *for* $x>0$. *Then* $f(x)=1$ *for* $x>0$.

PROOF. Since $\lim_{x\to 0^+} f(x)=1$, there exists $\delta>0$ such that $f(x)\neq 0$ for $x\in(0,\delta)$, which means that $f(x)=1$ for $x\in(0,\delta)$. Hence, according to Lemma 5, we get $f(x)=1$ for $x>0$. □

Finally we have the following.

Theorem 1. *Suppose that a function* $f:\mathbb{R}_+\to\mathbb{R}$ *satisfies* (1) *and one of the subsequent three conditions holds.*

(a) $\lim_{x\to a^+} f(x)=f(a)$ *for some* $a\in\mathbb{R}_+$ *with* $f(a)>0$

(b) $\lim_{x\to a^-} f(x)=f(a)$ *for some* $a\in\mathbb{R}_+$ *with* $f(a)<0$

(c) $\lim_{x\to 0^+} f(x)=1$.

Then

$$f(x) = \max\{cx+1, 0\} \quad \textit{for every } x \in \mathbb{R}_+,$$

or

$$f(x) = cx+1 \quad \textit{for every } x \in \mathbb{R}_+.$$

Proof. Assume (a) ((b), respectively) and fix $\varepsilon > 0$. Then there exists $\delta \in (0, a)$ such that $|f(t) - f(a)| < \varepsilon|f(a)|$ for $t \in (a, a+\delta)$ ($t \in (a-\delta, a)$, respectively). Let $\delta_1 := \frac{\delta}{|f(a)|}$ and take $x_1 \in (0, \delta_1)$. Notice that $x_1|f(a)| < \delta < a$, which means $-a < f(a)x_1$ and consequently $x := a + f(a)x_1 > 0$. Since

$$|x - a| = |a + f(a)x_1 - a| = |f(a)x_1| = |f(a)|x_1 < |f(a)|\delta_1 = \delta,$$

so

$$|f(x) - f(a)| < \varepsilon|f(a)|. \tag{12}$$

From (1) and (12) we have

$$|f(a)f(x_1) - f(a)| = |f(a + f(a)x_1) - f(a)| = |f(x) - f(a)| < \varepsilon|f(a)|.$$

Hence

$$|f(x_1) - 1| < \varepsilon.$$

Thus we have proved that, for every $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|f(x_1) - 1| < \varepsilon$ for $x_1 \in (0, \delta_1)$. This means that $\lim_{x\to 0^+} f(x) = 1$. Now from Lemmas 6, 7 and 8 we get the statement. □

Remark 1. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be given by: $f(x) = 1$ for $x \in \mathbb{N}$ and $f(x) = 0$ for $x \in \mathbb{R}_+ \setminus \mathbb{N}$. Then it is easily seen that f satisfies (1). This example shows that continuity at a point $a \in \mathbb{R}_+$ does not need to imply continuity of a solution $f : \mathbb{R}_+ \to \mathbb{R}$ of (1), unless $f(a) \neq 0$.

Remark 2. P. Kahling and J. Matkowski (see [14], Problem 1) have raised the problem to determine all functions $f : [0, \infty) \to [0, \infty)$, satisfying the following conditional Gołąb–Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y) \quad \text{for } x, y \geq 0, \tag{13}$$

that are differentiable at some point $y_0 \geq 0$ with $f(y_0) \neq 0$. A solution to the problem can be easily derived from Theorem 1. Namely let f :

$[0, \infty) \to [0, \infty)$ satisfy (13) and be differentiable at a point $y_0 \geq 0$ with $f(y_0) \neq 0$. Then f is continuous at y_0. Next, with $x = y = 0$, from (13) we get $f(0) = (f(0))^2$, which means that $f(0) \in \{0, 1\}$. Now it is easily seen that in the case $y_0 = 0$ we have $\lim_{x \to 0^+} f(x) = 1$. Therefore one of conditions (a)–(c) of Theorem (1) are fulfilled, whence we obtain the form of f.

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