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Note on a Jensen type functional equation

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Abstract. We look for solutions $f : M \to S$ and examine the stability of the functional equation

$$3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z)$$
$$= 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right],$$

where M is an Abelian semigroup in which the division by 2 and 3 is performable and S is an abstract convex cone. Some applications to a multivalued version of this equation are given.

1. Introduction

Let $(S,+)$ be an Abelian semigroup, written additively. Suppose that S contains the identity element 0 and for each $\lambda \geq 0$ and $s \in S$, an element λs in S is defined, for which the following axioms hold

$$1s = s, \quad \lambda(\mu s) = (\lambda\mu)s, \quad \lambda(s+t) = \lambda s + \lambda t, \quad (\lambda+\mu)s = \lambda s + \mu s,$$

where $s,t \in S$ and $\lambda, \mu \geq 0$. Then S is said to be an *abstract convex cone.*

If $s,t,t' \in S$, $t+s = t'+s$ always implies that $t = t'$, then S is said to satisfy the *cancellation law.*

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Suppose that an invariant with respect to translations and positively homogeneous metric ϱ is given in S, i.e.,

$$\varrho(t+s, t'+s) = \varrho(t, t')$$

and

$$\varrho(\lambda s, \lambda t) = \lambda \varrho(s, t)$$

for $\lambda > 0$ and $s, t, t' \in S$.

It is easy to see that the mappings $[0, \infty) \times S \ni (\lambda, s) \longmapsto \lambda s \in S$ and $S \times S \ni (s, t) \longmapsto s + t \in S$ are continuous in the metric topology.

Let $(M, +)$ be an Abelian semigroup with the identity element 0 in which the division by 2 and 3 is performable.

We are going to look for all solutions $f : M \to S$ of the functional equation

$$\begin{aligned} &3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \\ &= 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]. \end{aligned} \tag{1}$$

The inequality

$$\begin{aligned} &3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \\ &\geq 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right] \end{aligned}$$

appeared in T. Popoviciu's paper [3] in connection with the following theorem: The real continuous function f defined on an interval I is convex (i.e. the second divided differences of f are non-negative) if and only if the above inequality holds true for every triples x, y, z in I.

Tiberiu Trif [7] solved equation (1) in the class of functions $f : X \to Y$, where X, Y are real vector spaces. His considerationes cannot be applied in our reality as subtraction in vector spaces was used.

The main objective of this note is to find all solutions $f : M \to S$ of (1) and to examine its stability. The natural range of equation (1) is a commutative semigroup. If we consider its stability then the semigroup ought to be endowed with a metric. Restrictions of the second part of the paper (the range of f is an abstract convex cone with the cancellation law and endowed with a complete metric invariant under translations and positive homogeneous) enable us to prove Theorem 1. The family $\mathrm{clb}(X)$ of all non-empty convex closed and bounded subsets of a real Banach space fulfils these conditions and we can apply Theorem 1 to study equation (1) in the multivalued case. The similar results associated with the Jensen and Pexider functional equations were obtained in [5].

2. Solutions and Hyers–Ulam stability of (1)

We shall assume that

(i) M is a commutative semigroup with zero in which the division by 2 and 3 is performable;

(ii) S is an abstract cone satisfying the cancellation law;

(iii) (S,ρ) is a complete metric space and ρ is invariant with respect to translations and positively homogeneous.

Let $a : M \to S$ be an additive function which means $a(x+y) = a(x)+a(y)$ for all $x,y \in M$. It is easily seen that for every $b \in S$, the function $f(x) = a(x)+b$, $x \in M$, satisfies (1). The converse follows from the following

Theorem 1. *Assume that conditions* (i)–(iii) *are fulfilled. If* $\varepsilon \geq 0$ *and if* $f : M \to S$ *satisfies*

$$\begin{gathered}\varrho\left(3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z),\right.\\ \left.2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]\right)\leq\varepsilon\end{gathered} \tag{2}$$

for all $x,y,z \in M$*, then there exists a unique additive function* $a : M \to S$ *such that*

$$\varrho\left(f(x),a(x)+f(0)\right)\leq\varepsilon \tag{3}$$

for $x \in M$*.*

Proof. Setting in (2) $y = x$ and $z = 0$ we obtain

$$\varrho\left(3f\left(\frac{2}{3}x\right)+2f(x)+f(0),\ 2f(x)+4f\left(\frac{1}{2}x\right)\right)\leq\varepsilon.$$

Since the metric ϱ is invariant with respect to translation and positively homogeneous, we have

$$\varrho\left(\frac{3}{4}f\left(\frac{4}{3}x\right)+\frac{1}{4}f(0),f(x)\right)\leq\frac{1}{4}\varepsilon. \tag{4}$$

Replacing x by $\frac{4}{3}x$, multiplying by $\frac{3}{4}$ both the sides of (4) we infer

$$\varrho\left(\left(\frac{3}{4}\right)^2 f\left(\left(\frac{4}{3}\right)^2 x\right)+\frac{3}{4^2}f(0),\frac{3}{4}f\left(\frac{4}{3}x\right)\right)\leq\frac{3}{4^2}\varepsilon,$$

whence

$$\varrho\left(\left(\frac{3}{4}\right)^2 f\left(\left(\frac{4}{3}\right)^2 x\right)+\frac{1}{4}\left(1+\frac{3}{4}\right)f(0),\frac{3}{4}f\left(\frac{4}{3}x\right)+\frac{1}{4}f(0)\right)\leq\frac{3}{4^2}\varepsilon.$$

Hence in virtue of (4) follows

$$\varrho\left(\left(\frac{3}{4}\right)^2 f\left(\left(\frac{4}{3}\right)^2 x\right)+\frac{1}{4}\left(1+\frac{3}{4}\right)f(0),f(x)\right)\leq\frac{1}{4}\left(1+\frac{3}{4}\right)\varepsilon.$$

By induction we can show that for every positive integer n the inequality

$$\begin{aligned}\varrho\left(\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n x\right)+\frac{1}{4}\left[1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{n-1}\right]f(0),f(x)\right)&\\ \leq\frac{1}{4}\left(1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{n-1}\right)\varepsilon&\end{aligned} \tag{5}$$

holds. Write

$$f_n(x):=\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n x\right),\quad x\in M,\ n\in\mathbb{N}. \tag{6}$$

It follows by (5) that for arbitrary $n,m\in\mathbb{N}$, and $x\in M$ we have

$$\begin{aligned}\varrho\left(f_{n+m}(x),f_n(x)\right)&=\varrho\left(\left(\frac{3}{4}\right)^{n+m} f\left(\left(\frac{4}{3}\right)^{n+m} x\right),\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n x\right)\right)\\ &=\left(\frac{3}{4}\right)^n\varrho\left(\left(\frac{3}{4}\right)^m f\left(\left(\frac{4}{3}\right)^{n+m} x\right)+\frac{1}{4}\left[1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{m-1}\right]f(0),\right.\end{aligned}$$

$$f\left(\left(\frac{4}{3}\right)^n x\right)+\frac{1}{4}\left[1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{m-1}\right]f(0)\right)$$

$$\leq\left(\frac{3}{4}\right)^n\varrho\left(\left(\frac{3}{4}\right)^m f\left(\left(\frac{4}{3}\right)^m\left(\frac{4}{3}\right)^n x\right)\right.$$

$$\left.+\frac{1}{4}\left[1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{m-1}\right]f(0), f\left(\left(\frac{4}{3}\right)^n x\right)\right)$$

$$+\left(\frac{3}{4}\right)^n\varrho\left(f\left(\left(\frac{4}{3}\right)^n x\right), f\left(\left(\frac{4}{3}\right)^n x\right)\right.$$

$$\left.+\frac{1}{4}\left[1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{m-1}\right]f(0)\right)$$

$$\leq\left(\frac{3}{4}\right)^n\frac{1}{4}\left[1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{m-1}\right]\varepsilon$$

$$+\left(\frac{3}{4}\right)^n\frac{1}{4}\left[1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{m-1}\right]\varrho\left(0, f(0)\right)$$

$$<\left(\frac{3}{4}\right)^n\left[\varepsilon+\varrho\left(0, f(0)\right)\right].$$

Thus for every $x \in M$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. Let

$$a(x)=\lim_{n\to\infty} f_n(x), \quad x\in M. \tag{7}$$

Letting $n\to\infty$, we obtain by (5)

$$\varrho\left(a(x)+f(0), f(x)\right)\leq\varepsilon.$$

We have by (2)

$$\varrho\left(3\left(\frac{3}{4}\right)^n f\left(\frac{1}{3}\left[\left(\frac{4}{3}\right)^n x+\left(\frac{4}{3}\right)^n y+\left(\frac{4}{3}\right)^n z\right]\right)\right.$$

$$+\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n x\right)+\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n y\right)+\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n z\right),$$

$$2\left[\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n\frac{x+y}{2}\right)+\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n\frac{y+z}{2}\right)\right.$$

$$\left.\left.+\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n\frac{z+x}{2}\right)\right]\right)\leq\left(\frac{3}{4}\right)^n\varepsilon,$$

i.e.,

$$\varrho\left(3f_n\left(\frac{x+y+z}{3}\right)+f_n(x)+f_n(y)+f_n(z),\right.$$
$$\left.2\left[f_n\left(\frac{x+y}{2}\right)+f_n\left(\frac{y+z}{2}\right)+f_n\left(\frac{z+x}{2}\right)\right]\right)\le\left(\frac{3}{4}\right)^n\varepsilon.$$

Passing to the limit as $n\to\infty$ we get

$$\begin{aligned}&3a\left(\frac{x+y+z}{3}\right)+a(x)+a(y)+a(z)\\&=2\left[a\left(\frac{x+y}{2}\right)+a\left(\frac{y+z}{2}\right)+a\left(\frac{z+x}{2}\right)\right]\end{aligned}\tag{8}$$

for $x,y,z\in M$, which means that a satisfies equation (1). Since $a(0)=\lim_{n\to\infty}(3/4)^n f(0)$ by (7) and (6), we have $a(0)=0$. Now we shall prove that a is an additive function. Putting in (8) $y=x$ we get

$$3a\left(\frac{2x+z}{3}\right)+a(z)=4a\left(\frac{x+z}{2}\right).\tag{9}$$

If we put $u=\frac{2x+z}{3}$, (9) turns into

$$3a(u)+a(z)=4a\left(\frac{3u+z}{4}\right),\quad u,z\in M.\tag{10}$$

Substitute $z=0$ in (10). Then $a\left(\frac{3}{4}u\right)=\frac{3}{4}a(u)$ or

$$a(3u)=\frac{3}{4}a(4u).\tag{11}$$

Setting $u=0$ in (10) we obtain $a(z)=4a\left(\frac{1}{4}z\right)$ or $a(4z)=4a(z)$. Hence and by (11), $a(3u)=3a(u)$. Now formula (10) may be rewritten in the form $a(3u)+a(z)=a(3u+z)$, whence the additivity of a follows.

To end the proof we have to show the uniqueness of a.

Suppose that (3) holds with an additive function $\tilde{a}:M\to S$. We have for arbitrary $n\in\mathbb{N}$

$$\varrho\left(a(x),\tilde{a}(x)\right)=\left(\frac{3}{4}\right)^n\varrho\left(\left(\frac{4}{3}\right)^n a(x),\left(\frac{4}{3}\right)^n\tilde{a}(x)\right)$$

$$= \left(\frac{3}{4}\right)^n \varrho\left(a\left(\left(\frac{4}{3}\right)^n x\right), \tilde{a}\left(\left(\frac{4}{3}\right)^n x\right)\right)$$
$$= \left(\frac{3}{4}\right)^n \varrho\left(a\left(\left(\frac{4}{3}\right)^n x\right) + f(0), \tilde{a}\left(\left(\frac{4}{3}\right)^n x\right) + f(0)\right)$$
$$\leq \left(\frac{3}{4}\right)^n \varrho\left(a\left(\left(\frac{4}{3}\right)^n x\right) + f(0), f\left(\left(\frac{4}{3}\right)^n x\right)\right)$$
$$+ \left(\frac{3}{4}\right)^n \varrho\left(f\left(\left(\frac{4}{3}\right)^n x\right), \tilde{a}\left(\left(\frac{4}{3}\right)^n x\right) + f(0)\right) \leq 2\left(\frac{3}{4}\right)^n \varepsilon,$$

whence $a(x) = \tilde{a}(x)$, $x \in M$. □

Taking $\varepsilon = 0$ in Theorem 1 we obtain the following

Theorem 2. *Assume that conditions* (i)–(iii) *are fulfilled. If* $f : M \to S$ *satisfies* (1), *then there exists an additive function* $a : M \to S$ *and* $b \in S$ *such that* $f(x) = a(x) + b$, $x \in M$.

3. Multivalued solutions of (1)

Let X be a real Banach space and let $\mathrm{clb}(X)$ denote the set of all non-empty convex closed and bounded subsets of X. Introduce a binary operation $\overset{*}{+}$ in $\mathrm{clb}(X)$ by the formula

$$A \overset{*}{+} B = \mathrm{cl}(A + B) = \mathrm{cl}(\mathrm{cl}\, A + \mathrm{cl}\, B),$$

where $A + B$ denotes the usual Minkowski sum of A and B while $\mathrm{cl}\, A$ denotes the closedness of the set A. The second operation in $\mathrm{clb}(X)$ is given by

$$\lambda A = \{\lambda a : a \in A\}$$

for all $\lambda \geq 0$ and $A \in \mathrm{clb}(X)$. It is easily seen that $\mathrm{clb}(X)$ is an abstract convex cone with the identity element $0 := \{0\}$.

The proof of the following generalization of the Rådström lemma (cf. [4]) can be found in [6].

Lemma 1. *If a set* $B \subset X$ *is a non-empty and bounded and* $C \subset X$ *is convex and closed, then for every* $A \subset X$,

$$A + B \subset C \overset{*}{+} B \Longrightarrow A \subset C.$$

From Lemma 1, we derive that the cancellation law holds in the abstract convex cone clb(X).

The set clb(X) is a metric space with the Hausdorff distance h defined as follows

$$h(A,B) = \max\big\{\sup\{d(a,B) : a \in A\},\ \sup\{d(b,A) : b \in B\}\big\},$$

where $d(a,B) = \inf\{\|a-b\| : b \in B\}$. The metric space $(\mathrm{clb}(X), h)$ is complete (cf. e.g. [1]).

Lemma 2. *If $A, B, C \in \mathrm{clb}(X)$ and $\lambda \geq 0$, then*

$$h(A \overset{*}{+} B, C \overset{*}{+} B) = h(A+B, C+B) = h(A,C), \tag{12}$$

and

$$h(\lambda A, \lambda B) = \lambda h(A,B). \tag{13}$$

The first equality in (12) is easy to verify, the proof of the second one can be found in [2]. Formula (13) is well known. Thus the abstract cone clb(X) satisfy assumptions (ii) and (iii).

A multifunction $F_0 : M \to \mathrm{clb}(X)$ is said to be $(*)$-*additive* if

$$F_0(x+y) = F_0(x) \overset{*}{+} F_0(y)$$

for all $x, y \in M$.

From Theorem 1 we derive the following result.

Theorem 3. *Let $(M,+)$ satisfy condition* (i) *and let X be a real Banach space. We assume that $\varepsilon \geq 0$ and that $F : M \to \mathrm{clb}(X)$ satisfies the inequality*

$$h\left(3F\left(\frac{x+y+z}{3}\right) \overset{*}{+} F(x) \overset{*}{+} F(y) \overset{*}{+} F(z),\right.$$
$$\left. 2\left[F\left(\frac{x+y}{2}\right) \overset{*}{+} F\left(\frac{y+z}{2}\right) \overset{*}{+} F\left(\frac{z+x}{2}\right)\right]\right) \leq \varepsilon,$$

then there exists a unique $(*)$-additive multifunction $F_0 : M \to \mathrm{clb}(X)$ such that

$$h\left(F(x), F_0(x) \overset{*}{+} F(0)\right) \leq \varepsilon$$

for all $x, y \in M$.

In particular, putting $\varepsilon = 0$, we have the following result.

Theorem 4. *Let* $(M,+)$ *satisfy assumption* (i) *and let* X *be a real Banach space. If* $F : M \to \mathrm{clb}(X)$ *satisfies the functional equation*

$$\begin{aligned} &3F\left(\frac{x+y+z}{3}\right) \overset{*}{+} F(x) \overset{*}{+} F(y) \overset{*}{+} F(z) \\ &= 2\left[F\left(\frac{x+y}{2}\right) \overset{*}{+} F\left(\frac{y+z}{2}\right) \overset{*}{+} F\left(\frac{z+x}{2}\right)\right], \end{aligned} \tag{14}$$

then there exists a $(*)$*-additive multifunction* $F_0 : M \to \mathrm{clb}(X)$ *such that*

$$F(x) = F_0(x) \overset{*}{+} F(0)$$

for all $x \in M$*. Conversely, every multifunction* $F(x) = F_0(x) \overset{*}{+} B$*, where* $F_0 : M \to \mathrm{clb}(X)$ *is a* $(*)$*-additive multifunction and* $B \in \mathrm{clb}(X)$ *is an arbitrary set, actually satisfies* (14).

Remark 1. Every additive multifunction $F_0 : M \to \mathrm{clb}(X)$ which means

$$F_0(x+y) = F_0(x) + F_0(y) \quad \text{for all } x, y \in M \tag{15}$$

is $(*)$-additive. In fact by (15), $F_0(x+y) = \mathrm{cl}(F_0(x+y)) = \mathrm{cl}(F_0(x) + F_0(y)) = F_0(x) \overset{*}{+} F_0(y)$.

Remark 2. A $(*)$-additive multifunction $F_0 : M \to \mathrm{clb}(X)$ does not have to be additive. To see that take $A, B \in \mathrm{clb}(X)$ such that $\mathrm{cl}(A+B) \neq A+B$. The authoress believes that an example such sets A, B is known but we will construct one below for convenience of a reader. The multifunction $F : [0,\infty)^2 \to \mathrm{clb}(X)$ given by the formula

$$F(t_1, t_2) = \mathrm{cl}(t_1 A + t_2 B)$$

is $(*)$-additive. Indeed,

$$\begin{aligned} F((t_1,t_2)+(s_1,s_2)) &= F(t_1+s_1, t_2+s_2) = \mathrm{cl}\left[(t_1+s_1)A + (t_2+s_2)B\right] \\ &= \mathrm{cl}\left[t_1A + t_2B + s_1A + s_2B\right] = \mathrm{cl}\left[\mathrm{cl}(t_1A+t_2B) + \mathrm{cl}(s_1A+s_2B)\right] \\ &= F(t_1,t_2) \overset{*}{+} F(s_1,s_2) \end{aligned}$$

for all $t_1, t_2, s_1, s_2 \in [0, \infty)$. However, F is not additive, as

$$F(1,0) + F(0,1) = \operatorname{cl} A + \operatorname{cl} B = A + B$$

and

$$F((1,0) + (0,1)) = F(1,1) = \operatorname{cl}(A+B) \neq A + B = F(1,0) + F(0,1).$$

The following example has been suggested by Dr Anna Kucia (Katowice), the authoress wish to thank her for that in this place. Let $X = l_1$ denote the space of all summable sequences real numbers. For each $i \in \mathbb{N}$, let e_i be the vector in l_1 with zeros in all its coordinates except the i^{th} coordinate which is equal to one. Define

$$A_1 = \left\{ \left(1 + \frac{1}{i}\right) e_i : i \in \mathbb{N} \right\}, \quad B_1 = \left\{ \left(-1 + \frac{1}{i}\right) e_i : i \in \mathbb{N} \right\},$$

and

$$A = \overline{\operatorname{co}} A_1, \quad B = \overline{\operatorname{co}} B_1,$$

where $\overline{\operatorname{co}} A_1$ denotes the intersection of all convex closed sets containing A_1. At first we observe that

$$A_0 := \left\{ \left(2p_1, \frac{3}{2}p_2, \frac{4}{3}p_3, \ldots, \frac{i+1}{i}p_i, \ldots\right) : \sum_{i=1}^{\infty} p_i = 1,\ p_i \geq 0 \right\}$$
$$\subset A = \operatorname{cl}(\operatorname{co} A_1);$$

for every element of A_0 is a limit of some sequence of points belonging to $\operatorname{co} A_1$.

Next, we shall show that $A \subset A_0$. Take an arbitrary $a = \left(2p_1, \frac{3}{2}p_2, \frac{4}{3}p_3, \ldots, \frac{i+1}{i}p_i, \ldots\right) \in A$. Of course $p_i \geq 0$, $i \in \mathbb{N}$. It is enough to prove that $\sum_{i=1}^{\infty} p_i = 1$. We can find $a^n = \left(2p_1^n, \frac{3}{2}p_2^n, \ldots, \frac{i+1}{i}p_i^n, \ldots\right) \in \operatorname{co} A_1$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} a^n = a$ and $r_m^n := 1 - \sum_{i=1}^{m} p_i^n \longrightarrow 0$ as $m \to \infty$ for each $n \in \mathbb{N}$. Since

$$\frac{i+1}{i}\,|p_i^n - p_i| \leq \sum_{i=1}^{\infty} \frac{i+1}{i}\,|p_i^n - p_i| = \|a^n - a\|$$

and $\lim_{n\to\infty} \|a^n - a\| = 0$,

$$\sum_{i=1}^{m} p_i^n \longrightarrow \sum_{i=1}^{m} p_i \quad \text{as } n \to \infty$$

for every $m \in \mathbb{N}$. Thus

$$\lim_{n\to\infty} r_m^n = 1 - \sum_{i=1}^{m} p_i =: r_m \tag{16}$$

exists for each $m \in \mathbb{N}$. Now we shall show that the sequence $(r_m^n)_{n\in\mathbb{N}}$ satisfies the Cauchy condition uniformly with respect to m. We have for all $m, k, n \in \mathbb{N}$,

$$\left|r_m^n - r_m^k\right| = \left|\sum_{i=1}^{m} p_i^k - \sum_{i=1}^{m} p_i^n\right| \le \sum_{i=1}^{m} \left|p_i^k - p_i^n\right| \le \sum_{i=1}^{\infty} \left|p_i^k - p_i^n\right|.$$

Let us fix $\varepsilon > 0$. Since (a^n) is convergent, there exists a positive number α such that

$$\sum_{i=1}^{\infty} \frac{i+1}{i} \left|p_i^k - p_i^n\right| < \varepsilon$$

for all $n, k > \alpha$. Hence

$$\left|r_m^n - r_m^k\right| < \varepsilon, \quad n, k > \alpha \quad \text{and} \quad m \in \mathbb{N}.$$

Now, when $k \to \infty$, the sequence $(r_m^k)_{k\in\mathbb{N}}$ tends to r_m and

$$|r_m^n - r_m| \le \varepsilon, \quad \text{for } n > \alpha \quad \text{and } m \in \mathbb{N}.$$

Fix arbitrarily $n > \alpha$. Since for every $m \in \mathbb{N}$,

$$0 \le r_m \le |r_m - r_m^n| + r_m^n,$$

letting $m \to \infty$, we obtain $\limsup_{m\to\infty} r_m \le \varepsilon$. Consequently, $\lim_{m\to\infty} r_m = 0$ and by (16), $\sum_{i=1}^{\infty} p_i = 1$. We have proved that $A = A_0$. Similarly we can show that

$$B = \left\{ \left(0, -\frac{1}{2}q_2, -\frac{2}{3}q_3, \ldots, -\frac{i-1}{i}q_i, \ldots\right) : \sum_{i=2}^{\infty} q_i = 1,\ q_i \ge 0 \right\}.$$

We observe that $0 \in \mathrm{cl}(A+B)$. Indeed the set $A+B$ contains points of the form $2\left(p_1, \frac{1}{2}p_2, \ldots, \frac{1}{i}p_i, \ldots\right)$, where $p_i \ge 0$ and $\sum_{i=1}^{\infty} p_i = 1$. Thus this set has elements of arbitrarily small norms. To show that $0 \notin A+B$ we argue by contradiction: if for each $i \in \mathbb{N}$, $\frac{i+1}{i}p_i - \frac{i-1}{i}q_i = 0$ and $\sum_{i=1}^{\infty} p_i = 1$, $\sum_{i=1}^{\infty} q_i = 1$, we would have then

$$1 = \sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{i-1}{i+1} q_i = 1 - 2\sum_{i=1}^{\infty} \frac{1}{i+1} q_i < 1.$$

References

[1] C. Castaing and M. Valadier, Convex analysis and measurable multifunctions, Lecture Notes in Math. 580, *Springer-Verlag, Berlin, Heidelberg, New York*, 1977.

[2] F. S. De Blasi, On differentiability of multifunctions, *Pacific J. Math.* **66** (1976), 67–81.

[3] T. Popoviciu, Sur certaines inégalités qui caractérisentles fonctions convexes, *An. Ştiinţt. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.)* **11** (1965), 155–164.

[4] H. Rådström, An embedding theorem for spaces of convex sets, *Proc. Amer. Math. Soc.* **3** (1952), 165–169.

[5] W. Smajdor, Note on Jensen and Pexider functional equation, *Demonstratio Math.* **32** (1999), 363–376.

[6] R. Urbański, A generalization of Minkowski–Rådström–Hörmander theorem, *Bull. Acad. Pol. Sci.* **24**, 9 (1976), 709–715.

[7] T. Trif, Hyers–Ulam–Rassias stability of a Jensen type functional equation, *J. Math. Anal. Appl.* **250** (2000), 579–588.

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