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Some examples of semi-nodal perfect 4-polytopes

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Abstract. Existence of a semi-nodal perfect polytope means that there is a perfect polytope P such that both P and its polar P^* has vertices of not only zero degree of freedom. Yet, it is perfect, i.e. its shape cannot be changed without changing the action of its symmetry group on its face-lattice. In this paper three new 4-dimensional examples of such polytopes are constructed. A peculiar feature of them is that all types of their facets are non-perfect 3-polytopes.

1. Introduction and preliminaries

The notion of a perfect polytope, introduced by S. A. Robertson [16], generalizes that of a regular (convex) polytope. Intuitively, a polytope is perfect if it cannot be deformed to a polytope of different shape without altering its symmetry properties. The classification problem of perfect polytopes is settled in dimension 2 and 3, but it is still open from 4 on. Namely, the perfect 2-polytopes are just the regular (convex) polygons, and the class of perfect 3-polytopes includes the Platonic solids, the cuboctahedron and icosidodecahedron along with their polars, the rhombic dodecahedron and rhombic triacontahedron. There was an attempt to classify the perfect 4-polytopes, the main point of which seemed giving a proof for Rostami's conjecture [12]. However, some errors in the proof was

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pointed out in [8], and in fact the conjecture is by now disproved by the existence of several classes of counter-examples [8], [9]. Without going into the details, which can be found in the two cited papers of the first author, we only remark that the perfect 4-polytopes allowed by the conjecture form a rather limited class of Wythoffian polytopes. Contrary to that, now it is known that not only the class of Wythoffian perfect 4-polytopes is wider, but there exist even non-Wythoffian examples as well. In the gradual extension of the set of perfect polytopes, the notion of a semi-nodal perfect polytope represents an even greater conceptual distance from the original one.

Now we briefly summarize the most important definitions. For more details, see [8] and the references therein.

Let $G(P)$ and $F(P)$ denote the symmetry group and the face-lattice of a polytope P, respectively. Two (convex) n-polytopes P and Q are *symmetry equivalent* if and only if there exists an isometry φ of $\mathbb{E}^n$ and a lattice isomorphism $\lambda : F(P) \to F(Q)$ such that for each $g \in G(P)$ and each $A \in F(P)$, $\lambda\big(g(A)\big) = (\varphi g \varphi^{-1})\big(\lambda(A)\big)$. Each symmetry equivalence class is called a *symmetry type*.

A polytope P is said to be *perfect* if and only if all polytopes symmetry equivalent to P are similar to P.

Let G be a finite group of isometries of $\mathbb{E}^n$ and denote e the identity in G. Then the *symmetry scaffolding* of G is the union of the fixed point sets of all transformations in $G\backslash\{e\}$ and is denoted by $\operatorname{scaf} G$. Here we prefer using the same term (and notation) for the intersection of this set with the unit sphere $\mathbb{S}^{n-1}$ (centered at the origin); however, when the distinction is important, the attribute *spherical* will be used for the latter.

Likewise, we will also use the spherical variant of a polytope as follows. For a given n-polytope P, take a unit sphere $\mathbb{S}^{n-1}$ centered at the centroid of P. Then project P radially to $\mathbb{S}^{n-1}$. The image of the set of facets of P under this projection forms a tessellation of $\mathbb{S}^{n-1}$, which we shall refer to as the *spherical image* of P.

For a given group G and a point A in $\operatorname{scaf} G$, the *fixed point set* of A is defined as the set $\operatorname{fix}_A = \{x \in \mathbb{E}^n : g(x) = x,\ \forall g \in G_A\}$, where G_A is the stabilizer of A in G. Then $\dim(\operatorname{fix}_A)$, the dimension of fix_A, is called the *degree of freedom* of A. A point in the spherical symmetry scaffolding of G is called a *node* in exactly the case it has zero degree of freedom. A vertex of a polytope P is called *nodal* if in the spherical image of P it coincides with a node in scaf $G(P)$. A *nodal polytope* is a polytope whose vertices are all nodal.

A nodal polytope need not be perfect (for an instance, see the deltoidal dodecahedron used in Section 5), but in a special case we have

Theorem 1.1. *Every vertex-transitive nodal polytope is perfect.*

Proof. [8, p. 245]. □

Thus transitivity properties are important in investigating perfection of polytopes. Hence a useful notion is the the *orbit vector* of a polytope P which is $\theta(P) = \big(\theta_0, \dots, \theta_{n-1}\big)$, where θ_i is the number of orbits of i-faces of P, for each $i = 0, \dots, n-1$, under the action of $G(P)$.

The *polar* of a polytope P is $P^* = \{\mathbf{y} \in \mathbb{E}^n : \forall\, \mathbf{x} \in P,\ \langle \mathbf{x}, \mathbf{y} \rangle \leq \mathbf{1}\}$, provided that the origin coincides with the centroid of P. Note that this is a restriction of a more general notion of polarity of polytopes, for which usually only the condition is required that the origin be contained in the interior of P. In our case $G(P) = G(P^*)$ holds and perfection of P implies perfection of P^*.

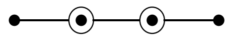
Now we are ready to define the following notion introduced in [8, p. 258]:

Definition 1.2. A polytope P such that both P and P^* has non-nodal vertex is called semi-nodal.

Numerous examples, including that mentioned above, show that perfection can be perished even by the existence of more than one transitivity classes of *nodal* vertices. Thus it is somewhat counter-intuitive to suppose for a perfect polytope P the existence of vertices having non-zero degree of freedom both in P and P^*. Yet, such strange example was found as a by-product through the construction of a series of non-Wythoffian perfect 4-polytopes in [9].

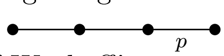
Here we report on 3 new examples. Our starting point is the perfect (and uniform) 10-cell, since knowledge of its facet structure and the symmetry scaffolding of its symmetry group is needed for the following constructions.

2. The structure and symmetry of the perfect 10-cell

The perfect 10-cell is described by Coxeter as a uniform 4-polytope [5] which is denoted by him as $\mathrm{t}_{1,2}\alpha_4 =$ . It can be obtained by Wythoff's construction, however, there is no Coxeter group such that a vertex of its fundamental domain would serve as an initial point of the construction. For this reason, it was one of the first counter-examples to Rostami's conjecture which were described by the first author [8]. It has already been used for obtaining new, non-Wythoffian 4-polytopes as well [9]. Here we recall some of its properties described in these papers completed with further data. For its notation, we continue using the symbol X introduced in [9].

X is bounded by 10 congruent Archimedean truncated tetrahedra (in what follows the abbreviation ATT will be used). Its f-vector and orbit vector is $f = (30, 60, 40, 10)$ and $\theta = (1, 1, 2, 1)$, respectively.

Its face structure is represented in Figure 1, where identical numbers denote identical vertices. Using an appropriate positioning in 5-space of the regular 4-simplex from which it can be derived, cf. [5, p. 573], the coordinates of its vertices take the simple form $(1, 1, 0, -1, -1)^P$, where the superscript means all permutations. Assignation of coordinates to the vertices is given in an earlier paper [9, Table 2].

Its symmetry group can be obtained from the symmetry group $[3, 3, 3]$ of the regular 4-simplex as follows. The spherical fundamental domain D of $[3, 3, 3]$ is symmetrical by a half-turn ρ about the join of the midpoints of two opposite edges. These are the edges A_0A_3 and A_1A_2, where A_i are the vertices of D indexed according to the opposite mirror walls which are associated to the generating reflections (note that we use the convention of numbering the generators represented by the nodes of a Coxeter diagram of form  from 0 to 3 from left to right). Now the initial point of Wythoff's construction for obtaining X is the midpoint of A_1A_2. Hence the symmetry group of X is an extension of $[3, 3, 3]$ by the group of order 2 generated by ρ, namely, it is $[[3, 3, 3]] \cong [3, 3, 3] \rtimes \langle\rho\rangle$ (split extension, semi-direct product).

For a better understanding of its symmetry scaffolding, we show that this group of order 240 can be obtained in the following way as well. Let π

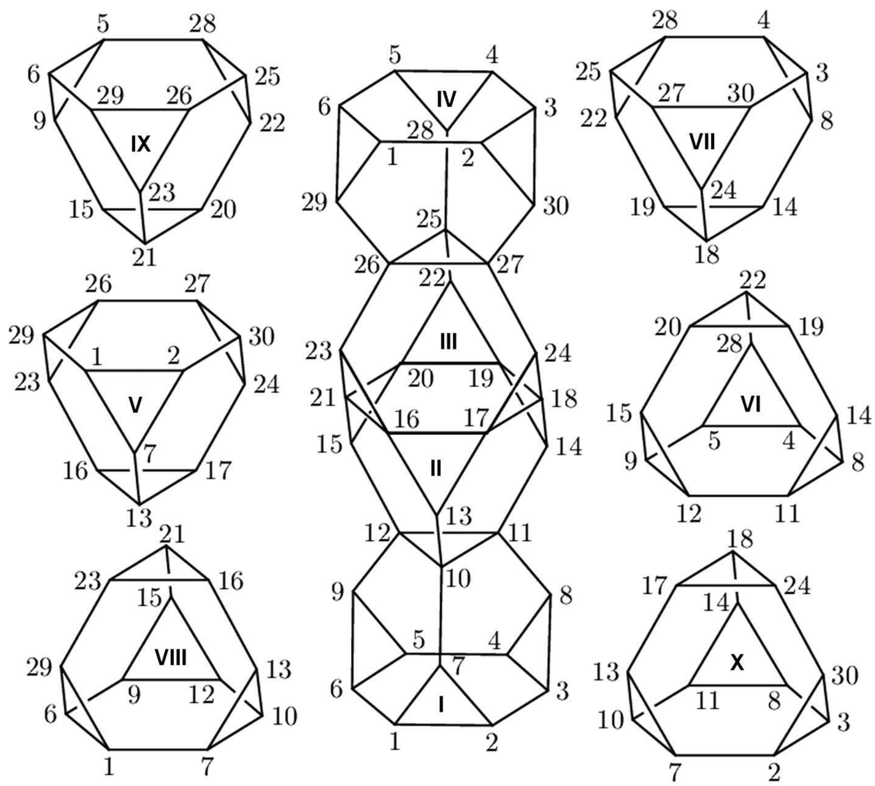


Figure 1

be a symmetry operation that cyclically permutes the vertices (or equivalently, the cells) of the regular 5-cell. Let γ denote the central inversion in the origin, then the product $\delta = \pi\gamma$ is a "square root" of π, thus it is a symmetry operation of order 10. δ cyclically permutes the 10 cells of X. Accordingly, either of the cells of X may serve as a fundamental domain for the cyclic group $\langle\delta\rangle$ (considering in the spherical image). Taking the symmetry group $[3,3] \cong S_4$ of such a cell, not only that cell but the whole spherical image of X is invariant to this group (S_4 denotes the symmetric group of degree 4). From purely algebraic point of view, this is a consequence of the following fact. The group $\langle\delta\rangle \cong C_{10}$ is generated by δ, δ^3, δ^7 and δ^9, and the action of S_4 on the set of these generators induces an action of S_4 on C_{10}. Thus $[3,3]$ acts as a group of (outer) automorphisms of $\langle\delta\rangle$. Hence we have the semi-direct decomposition $[\![3,3,3]\!] \cong \langle\delta\rangle \rtimes [3,3]$.

Thus a cell of X can be subdivided to $\big|[3,3]\big| = 24$ congruent pieces such that either of them may serve as a fundamental domain D_1 for $[\![3,3,3]\!]$ (in the spherical image). The Euclidean variant of D_1 and its location in an ATT cell is shown in Figure 2. We note that in accordance with our former decomposition for $[\![3,3,3]\!]$, D_1 can also be obtained by a dissection of D. The dissecting plane is lying on the axis of the half-turn ρ and is intersecting exactly 4 edges of the spherical tetrahedron D. In the figure the axis is indicated by a broken line. We remark that, conversely, the ATT cells of X can be obtained from the fundamental tessellation belonging to the group $[\![3,3,3]\!]$, in accordance with the factor tessellation method described in [7].

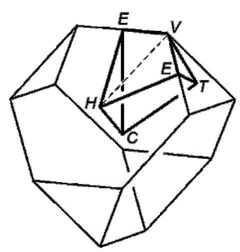


Figure 2

Now we have an overview on the symmetry scaffolding of the group $[\![3,3,3]\!]$ and its node structure. There are altogether 4 transitivity classes of nodes. They coincide with certain special points of X and will be referred to as various *types.* The stabilizer in $[\![3,3,3]\!]$ of the node of a particular type can easily be established from the structure of $[3,3,3]$. Namely, we have:

- type V: vertices of X, with stabilizer isomorphic to $[4,2^+] \cong D_{2d}$;
- type T: centroids of the triangular faces, with stabilizer isomorphic to $[3,2] \cong D_{3h}$;
- type H: centroids of the hexagonal faces, with stabilizer isomorphic to $[6,2^+] \cong D_{3d}$;
- type C: centroids of the cells, with stabilizer isomorphic to $[3,3] \cong T_d$.

A representative of each class is indicated in Figure 2. Note that the point indicated by C coincides with the vertex A_0 of the fundamental domain D, and that indicated by T with vertex A_1. We remark that we shall use a further type of points, namely type E consisting of the midpoints of edges, although these points are not nodes.

3. A scalenohedral polytope derived from the 10-cell

Take the 2 triangular faces incident to a chosen vertex of X, say V_{27}. Let their centroids be denoted by T_{34} and T_{57}, where the subscript indicates the cells which have the respective triangular face in common. Let G_T be the centroid of the pair (T_{34}, T_{57}). Take the 4 edges incident to V_{27}. Let their midpoints be denoted by E_{345}, E_{347}, E_{357} and E_{457}, where the subscripts obey a similar rule as before. Let G_E be the centroid of this quadruple of points. It is checked by some calculation that G_T and G_E do not coincide. Both of them lie, however, on the straight line connecting the centroid O of X and the vertex V_{27}. This is a consequence of the local symmetry around V_{27} which corresponds to the stabilizer $[4, 2^+] \cong D_{2d}$ of the vertex. To bring them together, we shift each point E_{ijk} along a line segment of type EC (i.e. a line segment with endpoints of type E and C) to a suitable extent, so that each get in the interior of a cell into a new position E'_{ijk}. For a point E_{ijk} the cell is chosen so that it be the i-th cell which does not have the point T_{jk} on its boundary. Besides, the extent of shifting must be the same for all the four points in order to preserve symmetry.

Now it is clear that the quadruple (E_{345}, E_{347}, E_{357}, E_{457}) forms the set of vertices of a tetrahedron, actually, a tetragonal disphenoid bounded by congruent isosceles triangles. The same is true for the quadruple (E'_{345}, E'_{347}, E'_{357}, E'_{457}). Furthermore, this latter, new tetrahedron and the points T_{34} and T_{57} are lying in a common hyperplane. In fact, this is the hyperplane that is invariant with respect to the stabilizer of V_{27} and passes through G_T as well as the new centroid G'_E coinciding with G_T. It is checked again by calculation that each of the points E'_{345}, E'_{347}, E'_{357} and

E'_{457} fulfilling this condition of coincidence are in fact located in the interior of an appopriate cell. On the other hand, we obtain as well that T_{34} and T_{57} are located outside of our new tetrahedron.

This latter condition, together with that the arrangement of these six points exhibits a symmetry corresponding to the group D_{2d}, implies that the convex hull of this sextuple of points is a facet-transitive polyhedron called a *tetragonal scalenohedron*. This polyhedron is bounded by 8 congruent scalene triangles and its symmetry type is well-known in geometric crystallography as a crystal form [2], [11]. Its shape, obtained by calculation, is shown in Figure 3. Observe that it is combinatorially equivalent to a regular octahedron, or to a tetragonal dipyamid. Referring to this latter equivalence, we shall call its edges starting from a vertex of type T *apical*, and the edges connecting its vertices of type E' *medial*.

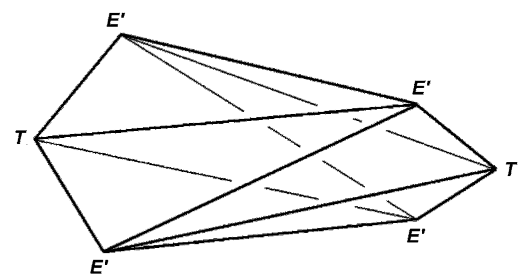


Figure 3

Perform now the construction of a scalenohedron in the same way around each vertex of X and take the convex hull of the whole figure that is obtained. This results in a new polytope, of which the scalenohedra are in fact facets, as we shall see below. Hence we denote this "scalenohedral polytope" obtained from the 10-cell by ScP_{10}. In what follows we determine its facet structure.

First consider the points of type E' obtained by shifting those of type E as described above. Take 6 of them that are obtained from the midpoints of the sides of a hexagonal face of X. Since the stabilizer of such a hexagon is isomorphic to the group D_{3d}, it is seen that these six points form the vertices of a trigonal antiprism. Such antiprisms can be constructed in the vicinity of the centroids of all hexagonal faces and we shall see as well that they form in fact the next type of facets of ScP_{10}.

Take now 6 points of type E' and 4 points of type T that all belong to a cell of X. The symmetry of the cell implies that the shape of the convex hull of these points is obtained in the following way. Consider the regular octahedron the vertices of which are the points of type E' and take alternately four of its faces. Take such a face as a base and erect a small trigonal pyramid onto it such that the apex of the pyramid coincides with a suitable point of type T. Perform this on all of the four alternating faces. The polyhedron obtained this way will provide the third type of facets of ScP_{10}. Now the polyhedron that is obtained by erecting small pyramids onto its all the eight faces in the same way, provided that convexity is preserved, is called a *triakisoctahedron*. Hence our new figure can be called a *semi-triakisoctahedron*.

Now it is seen that all the three kinds of these polyhedra have a triangular face which is not a common face with any other of them. In fact, the scalenohedron is bounded by scalene triangles of type $E'E'T$, the side faces of the antiprism are isosceles triangles of type $E'E'E'$ and the triangular faces of the semi-triakisoctahedron are isosceles triangles of type $E'E'T$ (where the type is denoted by using the type of the vertices). Hence there are gaps among these polyhedra, and the facets of the fourth type can be considered as just "to fill in" these gaps. In fact, take e.g. the convex hull of the following quadruple of points: (E'_{357}, E'_{359}, E'_{345}, T_{34}). This is a tetrahedron with mirror symmetry and it can directly be checked in Figure 1 that it has just the faces mentioned before. It is adjacent to 2 distinct scalenohedra through its scalene triangular faces. The hyperplane spanned by it cannot coincide with a hyperplane containing either of these scalenohedra. For otherwise, due to symmetry, all the three hyperplanes would coincide, which is impossible. It has its face of type $E'E'E'$ and $E'E'T$ in common with an antiprism and a semi-triakisoctahedron, respectively. All these three latter polyhedra belong to distinct hyperplanes, since the degree of their common edge of type $E'E'$ cannot be reduced, for convexity and symmetry reasons. (By the degree of an edge we mean the number of facets meeting in that edge).

Thus we have polyhedra of altogether four distinct types. We obtained that any two of them belonging to distinct types are contained in no common hyperplane. On the other hand, any two of them do not overlap, apart from their possibly common 2-face. It follows that each of these polyhedra is in fact a facet of our polytope ScP_{10}, as we promised above. Moreover, it is directly seen that these types coincide just with the transitivity classes of facets of ScP_{10}. The number of facets of the various types can be obtained from the f-vector of X. Namely, there are 10 semi-triakisoctahedron, 20 antiprism, 30 scalenohedron and 120 tetrahedron facets.

As a result of the arguments above, taking into consideration the transitivity properties of the individual types of polyhedra, we obtain the transitivity classes of the 2-faces as well. These are just the 3 types of triangles considered above, together with a fourth one that consists of the regular hexagonal faces of the semi-triakisoctahedra.

It is found that the edges form 4 transitivity classes as well. Their types and degrees can be seen in Figure 4 which shows the Schlegel diagram of a tetrahedron facet with typical vertices. Further data are listed in Table 1. We note that for calculating the edge lengths the coordinates of the vertices were chosen as mentioned above. On the other hand, the incidences, i.e. the number and types of facets meeting in a given edge are checked using Figure 1.

Hence we found that the orbit vector is $\theta(ScP_{10}) = (2, 4, 4, 4)$. The f-vector is: $f(ScP_{10}) = (80, 420, 520, 180)$.

Note that our construction of ScP_{10} from X ensures that the symmetry group of ScP_{10} remains to be $[\![3, 3, 3]\!]$.

Proposition 3.1. *ScP_{10} is perfect and semi-nodal.*

Proof. To see that our polytope is perfect, first take a polytope P_0 such that it is the convex hull of the set of all vertices of type T of ScP_{10}. Since P_0 is vertex-transitive and nodal, it is perfect, by Theorem 1.1. Hence this system of points is fixed in the sense that it is an arrangement that cannot be changed without changing its symmetry. Adding new vertices, of type E', ScP_{10} is obtained, whose "characteristic facets" the scalenohedra may be considered. The shape of a scalenohedron can change while preserving its symmetry type. Considered within ScP_{10}, its apexes being fixed, the only way of changing its shape is by displacing its medial vertices. These latter, being of type E', are located each on a line segment connecting a point of type C and a point of type E. Therefore, their degree of freedom is 1 (considered in the spherical image), i.e. they can only be displaced within that line segments. But such a displacement results

TYPE	DEGRE	INCIDENCES	NUMBER OF OCCURENCE	LENGTH
TE'	4	2 scalenohedra 2 tetrahedra	60	$\frac{\sqrt{70}}{18} \approx 0.4648$
TE'	4	1 scalenohedron 1 semi-triakisoctahedron 2 tetrahedra	120	$\frac{5\sqrt{22}}{18} \approx 1.3029$
$E'E'$	4	1 scalenohedron 1 antiprism 2 tetrahedra	120	$\frac{\sqrt{105}}{9} \approx 1.1386$
$E'E'$	3	1 semi-triakisoctahedron 1 antiprism 1 tetrahedron	120	$\frac{25\sqrt{2}}{18} \approx 1.9642$

Table 1

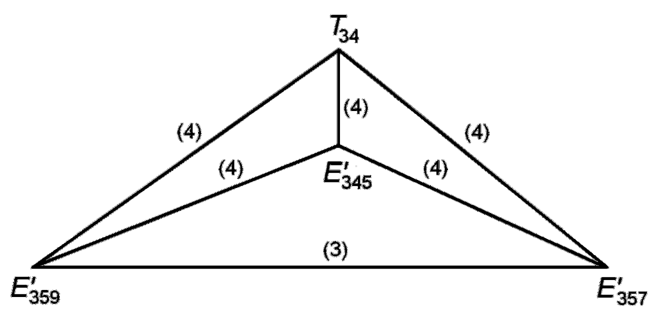


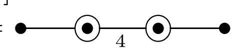
Figure 4

in that the centroid of a quadruple of points of type E' in the vicinity of a point of type V will not coincide with the centroid of the two apexes in the same neighbourhood. That is, the four points of type E' and the two points of type T would not form a scalenohedron, cf. our method for the construction of ScP_{10}. Hence such a displacement is impossible as well.

We note that in Euclidean space the points of type E' can move towards the origin as well. In fact, scalenohedra can also be constructed in the way that shifting the point of type E towards the centroid of a cell is associated with a simultanous shifting towards the origin to a suitable extent, producing points of type E''. But such points of type E'' within an ATT cell form a sextuple the centroid of which do not coincide with the four points of type T in the same cell. In other words, we do not obtain semi-triakisoctahedron facets in this case.

Our polytope is semi-nodal, since both ScP_{10} itself and its polar has non-nodal vertex. In fact, the vertices of type E' are non-nodal. On the other hand, a vertex of the polar P^* of a polytope P is nodal if and only if the normal vector of the facet of P corresponding to that vertex points to a node. Now it is directly seen that the normal vector of a tetrahedron facet of ScP_{10} can be parallel to neither of a vector pointing to some node. $\square$

4. An analogous construction from the perfect 48-cell

The perfect 48-cell is an analogue of the perfect 10-cell, whose existence is due to a symmetry property of the Coxeter group $[3,4,3]$ like that of $[3,3,3]$. It is likewise a uniform 4-polytope, denoted by Coxeter as $\mathrm{t}_{1,2}\{3,4,3\}$ =  (for its description, see again [5], [8]). It is bounded by 48 Archimedean truncated cubes (ATC-s). A characteristic detail of its face structure is represented in Figure 5, where, just as in Figure 1, identical numbers denote identical vertices.

The coordinates of its 288 vertices can be given as follows:

$$\big(\pm(1+\sqrt{2}\,), \pm(-1+\sqrt{2}\,), \pm1, \pm1\big)^P, \quad (\pm2, \pm\sqrt{2}, \pm\sqrt{2}, 0)^P.$$

An assignation of these coordinates to the vertices in Figure 5 is given in the Appendix, Table 4.

Now it is directly seen that here a construction analogous to that in the former section can be performed. We denote the polytope obtained this way by ScP_{48} Without the details, being almost the same, we mention only the results. We obtain 288 tetragonal scalenohedron facets with centroids of type V. Furthermore, here are regular octagons instead of hexagons. Hence the facets of the next type are tetragonal antiprisms, of which there are 144. The 48 facets with centroids of type C can be obtained as follows. The points of type E' within an Archimedean truncated cube cell provide the set of vertices of a cuboctahedron. Onto the

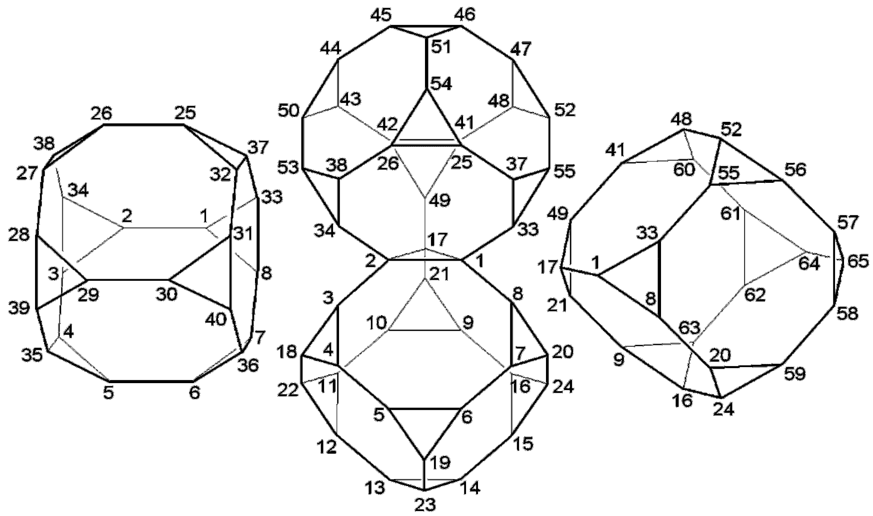


Figure 5

TYPE	DEGREE	INCIDENCES	NUMBER OF OCCURENCE	LENGTH
TE'	4	2 scalenohedra 2 tetrahedra	576	$\frac{\sqrt{14-8\sqrt{2}}}{6} \approx 0.2732$
TE'	4	1 scalenohedron 1 augmented cuboctahedron 2 tetrahedra	1152	$\frac{\sqrt{14+8\sqrt{2}}}{6} \approx 0.8386$
$E'E'$	4	1 scalenohedron 1 antiprism 2 tetrahedra	1152	$\frac{\sqrt{5}}{3} \approx 0.7454$
$E'E'$	3	1 augmented cuboctahedron 1 antiprism 1 tetrahedron	1152	$\frac{\sqrt{4+3\sqrt{2}}}{6} \approx 1.3738$

Table 2

triangular faces of such a cuboctahedron small pyramids are built, whose apices coincide with the centroids of the triangular faces of the ATC cell. In an analogy of Johnson's terminology [10], we may call such a polyhedron an *augmented cuboctahedron.* Moreover, the apices of the augmenting pyramids may form the set of vertices of either a cube or an octahedron depending on that they are built onto the triangular or square faces, respectively. In our specific case, the polyhedra in question are *cubically augmented cuboctahedra.* The facets of the last type are tetrahedra here as well, which exhibit mirror symmetry. Their number is 1152. The number of the transitivity classes of the 2-faces is four as well, and the only difference worth to mention is that the bases of antiprisms are squares instead of triangles. The edge structure is also completely analogous. Its data are given in Table 2.

The orbit vector is equal to that of ScP_{10}: $\theta(ScP_{48}) = (2, 4, 4, 4)$. The f-vector is: $f(ScP_{48}) = (768, 4032, 4896, 1632)$. Since the symmetry group is preserved through the construction here as well, it is equal to $[\![3, 4, 3]\!]$.

Proposition 4.1. *ScP_{48} is perfect and semi-nodal.*

Proof. Analogous to that of Proposition 3.1. □

5. A deltoid-dodecahedral perfect 4-polytope

A *deltoidal dodecahedron* is a facet-transitive 3-polytope bounded by 12 deltoids (or kites) (Figure 6). Its symmetry group is equal to that of a regular tetrahedron. It is well-known in geometric cystallography as a crystal form [2], [11]. Furthermore, it occurs in the image of central projections of the 4-cube [15]. Its set of vertices decomposes to 3 transitivity classes as follows:

- type 1, consisting of vertices of degree 3 such that their convex hull is a regular tetrahedron with circum-radius r_1;
- type 2, consisting of vertices of degree 4 such that their convex hull is a regular octahedron with circum-radius $r_2 > r_1$;
- type 3, consisting of vertices of degree 3 such that their convex hull is a regular tetrahedron with circum-radius $r_3 > r_2$.

It is not a perfect polytope, since its dihedral angles can change without altering its symmetry type. Accordingly, the ratio $r_1 : r_2 : r_3$ can change within certain limits as well.

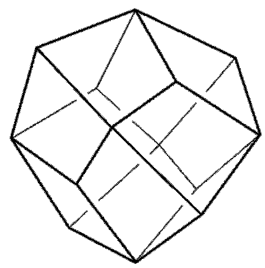


Figure 6

Now take a deltoidal dodecahedron with a suitable ratio $r_1 : r_3$, and include a copy of it in each ATT cell of X so that the following condition holds:

Condition 5.1. Vertices of type 1 coincide with points of type H and vertices of type 3 coincide with points of type T.

It is checked that such a deltoidal dodecahedron exists and its vertices of type 2 will get into the interior of the cell it is included in. Moreover, Condition 5.1 implies the following

Proposition 5.2.

(a) *Vertices of type 2 are lying just on the line segments connecting the midpoints of edges to the centroid of the cell.*

(b) *The shape of the deltoidal dodecahedron is unique, i.e. the ratio $r_1 : r_2 : r_3$ is fixed.*

Finally, take the convex hull of the figure consisting of the 10 deltoidal dodecahedra. We obtain a polytope for which it can be proved that these dodecahedra are facets of the one type. We may regard the facets of this type the "characteristic facets" of our new polytope and therefore we shall call it a *deltoid-dodecahedral perfect polytope.* We shall denote it by DdP_{10}.

A next type of facets can be obtained as follows. First, we note that we are given three distinct types of points that play role in our construction.

These are: type T, type H and a third one, which, on account of property 5.2(a), we shall call again a type E'. Consider 2 points of type T and 4 points of type E' in the vicinity of a vertex V of X. As we have seen above, the local symmetry around V implies that the quadruple of type E' form a tetrahedron, actually, a disphenoid. However, it is checked that this tetrahedron cannot occur as a facet of DdP_{10}. For, some calculation shows that the centroid of this quadruple is at a smaller distance from the centroid of our polytope as compared to the centroid of the pair of type T, and this justifies our statement. Since the two centroids do not coincide, a scalenohedron cannot occur in the structure either. Thus, under the condition of the local symmetry imposed by the stabilizer of V, the only possibility to form facets from these six points is as follows. Take a quadruple consisting of the pair of type T as well as of two points of type E' located in adjacent ATT cells. The convex hull of this quadruple is obviously a tetrahedron, and the stabilizer D_{2d} permutes just 4 such tetrahedra. (Note that it is as if the scalenohedron considered in our former construction were split to 4 congruent tetrahedra so that their arrangement preserved the original symmetry. The exercise of the decomposition of a tetragonal scalenohedron with this conditon has a unique solution.) We observe that this tetrahedron is symmetrical to a half-turn and there is no other symmetry of it. The corresponding axis is just that is indicated in Figure 2 by broken line. By further considerations, based on symmetry arguments, it can be shown that this tetrahedron forms in fact a facet of our new polytope.

Take now a deltoidal face, e.g. $T_{34}E'_{359}H_{35}E'_{357}$ (subscript notation is as above). This is a face of the deltoidal dodecahedron included in the cell III of X. Its vertex H_{35} is shared with a neighbouring dodecahedron in cell V. The vertex of this second dodecahedron closest to the deltoid in question is E'_{345}. The convex hull of the deltoid and this latter point is a pyramid, and it is checked that this pyramid is a representative of the third and last type of facets. It has its triangular face $E'_{345}E'_{357}H_{35}$ with another pyramid in common (observe that the symmetry group of the arrangement of the 6 pyramids having the vertex H_{35} in common is in fact equal to the stabilizer of that vertex, i.e. isomorphic to D_{3d}). Its triangular face of the other type, namely $E'_{345}E'_{357}T_{34}$, is shared by the tetrahedron $E'_{345}E'_{357}T_{34}T_{57}$.

Thus we have altogether 3 types of facets, and these are transitivity classes as well. The following numbers for the facets of various types can directly be obtained: there are 10 deltoidal dodecahedra, 120 four-sided pyramids and 120 tetrahedra.

Representatives of 3 types of 2-faces we already met before. There is a fourth type, namely type $E'TT$. Triangles of this type are common faces of tetrahedra. Here again, the types correspond to transitivity classes. We remark that a peculiar property of the deltoidal faces is that their angle at a vertex of type T is just right angle.

TYPE	DEGREE	INCIDENCES	NUMBER OF OCCURRENCE	LENGTH
$E'H$	4	1 deltoidal dodecahedron 3 four-sided pyramids	120	$\frac{\sqrt{17}}{4} \approx 1.0308$
$E'T$	5	1 deltoidal dodecahedron 2 four-sided pyramids 2 tetrahedra	120	$\frac{5}{4}$
$E'T$	6	2 four-sided pyramids 4 tetrahedra	60	$\frac{\sqrt{5}}{4} \approx 0.5590$
$E'E'$	3	2 four-sided pyramids 1 tetrahedron	120	$\frac{\sqrt{70}}{8} \approx 1.0458$
TT	4	4 tetrahedra	30	$\frac{2\sqrt{5}}{3} \approx 1.4907$

Table 3

The data of the transitivity classes of edges are listed in Table 3.

The orbit vector and f-vector is found to be $\theta(DdP_{10}) = (3,5,4,3)$ and $f(DdP_{10}) = (100,450,600,250)$, respectively. Here, as in the former cases, the original symmetry is preserved through the construction, thus the symmetry group of DdP_{10} is $[[3,3,3]]$.

Proposition 5.3. *DdP_{10} is perfect and semi-nodal.*

Proof. Consider the set of vertices of type T and the set of vertices of type H. Both sets consist of nodal vertices. Hence these sets cannot be displaced with respect to each other in the spherical image without altering the symmetry of the union of them. Thus in Euclidean space the only possibility of changing is to take the homothetic copy of the one set with respect to the origin. But in this case the centroid of nodes of type T and the centroid of nodes of type H in an ATT cell would not coincide, i.e. the deltoidal dodecahedron facet could not exist, contradicting the starting point of our construction. On the other hand, having fixed the position of the vertices of these two types of a deltoidal dodecahedron, the locaton of its vertices of type E' is uniquely determined. Hence our polytope is perfect.

Now here again, the vertices of type E' are non-nodal. Furthermore, the existence of non-nodal vertices of the polar of DdP_{10} is implied by the location of the four-sided pyramid facets. Thus DdP_{10} is semi-nodal. □

We remark that in contrast to our former two cases, here we have no analogous construction from the perfect 48-cell. Although an analogue of the deltoidal dodecahedron exists, namely, the deltoidal icositetrahedron, it can be included in an analogous way in an Archimedean truncated octahedron, and not in an Archimedean truncated cube.

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Appendix

$\mathbf{v}_1(1+\sqrt{2},1,1,-1+\sqrt{2}\,)$	$\mathbf{v}_2(1+\sqrt{2},1,-1+\sqrt{2},1)$
$\mathbf{v}_3(1+\sqrt{2},1,1-\sqrt{2},1)$	$\mathbf{v}_4(1+\sqrt{2},1,-1,-1+\sqrt{2}\,)$
$\mathbf{v}_5(1+\sqrt{2},1,-1,1-\sqrt{2}\,)$	$\mathbf{v}_6(1+\sqrt{2},1,1-\sqrt{2},-1)$
$\mathbf{v}_7(1+\sqrt{2},1,-1+\sqrt{2},-1)$	$\mathbf{v}_8(1+\sqrt{2},1,1,1-\sqrt{2}\,)$
$\mathbf{v}_9(1+\sqrt{2},-1,1,-1+\sqrt{2}\,)$	$\mathbf{v}_{10}(1+\sqrt{2},-1,-1+\sqrt{2},1)$
$\mathbf{v}_{11}(1+\sqrt{2},-1,1-\sqrt{2},1)$	$\mathbf{v}_{12}(1+\sqrt{2},-1,-1,-1+\sqrt{2}\,)$
$\mathbf{v}_{13}(1+\sqrt{2},-1,-1,1-\sqrt{2}\,)$	$\mathbf{v}_{14}(1+\sqrt{2},-1,1-\sqrt{2},-1)$
$\mathbf{v}_{15}(1+\sqrt{2},-1,-1+\sqrt{2},-1)$	$\mathbf{v}_{16}(1+\sqrt{2},-1,1,1-\sqrt{2}\,)$
$\mathbf{v}_{17}(1+\sqrt{2},-1+\sqrt{2},1,1,)$	$\mathbf{v}_{18}(1+\sqrt{2},-1+\sqrt{2},-1,1,)$
$\mathbf{v}_{19}(1+\sqrt{2},-1+\sqrt{2},-1,-1)$	$\mathbf{v}_{20}(1+\sqrt{2},-1+\sqrt{2},1,-1)$
$\mathbf{v}_{21}(1+\sqrt{2},1-\sqrt{2},1,1)$	$\mathbf{v}_{22}(1+\sqrt{2},1-\sqrt{2},-1,1)$
$\mathbf{v}_{23}(1+\sqrt{2},1-\sqrt{2},-1,-1)$	$\mathbf{v}_{24}(1+\sqrt{2},1-\sqrt{2},1,-1)$
$\mathbf{v}_{25}(1,1+\sqrt{2},1,-1+\sqrt{2}\,)$	$\mathbf{v}_{26}(1,1+\sqrt{2},-1+\sqrt{2},1)$
$\mathbf{v}_{27}(1,1+\sqrt{2},1-\sqrt{2},1)$	$\mathbf{v}_{28}(1,1+\sqrt{2},-1,-1+\sqrt{2}\,)$
$\mathbf{v}_{29}(1,1+\sqrt{2},-1,1-\sqrt{2}\,)$	$\mathbf{v}_{30}(1,1+\sqrt{2},1-\sqrt{2},-1)$
$\mathbf{v}_{31}(1,1+\sqrt{2},-1+\sqrt{2},-1)$	$\mathbf{v}_{32}(1,1+\sqrt{2},1,1-\sqrt{2}\,)$
$\mathbf{v}_{33}(2,\sqrt{2},\sqrt{2},0)$	$\mathbf{v}_{34}(2,\sqrt{2},0,\sqrt{2}\,)$
$\mathbf{v}_{35}(2,\sqrt{2},-\sqrt{2},0)$	$\mathbf{v}_{36}(2,\sqrt{2},0,-\sqrt{2}\,)$
$\mathbf{v}_{37}(\sqrt{2},2,\sqrt{2},0)$	$\mathbf{v}_{38}(\sqrt{2},2,0,\sqrt{2}\,)$
$\mathbf{v}_{39}(\sqrt{2},2,-\sqrt{2},0)$	$\mathbf{v}_{40}(\sqrt{2},2,0,-\sqrt{2}\,)$
$\mathbf{v}_{41}(\sqrt{2},0,2,\sqrt{2})$	$\mathbf{v}_{42}(\sqrt{2},0,\sqrt{2},2)$
$\mathbf{v}_{43}(1,-1+\sqrt{2},1,1+\sqrt{2}\,)$	$\mathbf{v}_{44}(-1+\sqrt{2},1,1,1+\sqrt{2}\,)$
$\mathbf{v}_{45}(0,\sqrt{2},\sqrt{2},2)$	$\mathbf{v}_{46}(0,\sqrt{2},2,\sqrt{2}\,)$
$\mathbf{v}_{47}(-1+\sqrt{2},1,1+\sqrt{2},1)$	$\mathbf{v}_{48}(1,-1+\sqrt{2},1+\sqrt{2},1)$
$\mathbf{v}_{49}(2,0,\sqrt{2},\sqrt{2}\,)$	$\mathbf{v}_{50}(1,1,-1+\sqrt{2},1+\sqrt{2}\,)$
$\mathbf{v}_{51}(0,2,\sqrt{2},\sqrt{2}\,)$	$\mathbf{v}_{52}(1,1,1+\sqrt{2},-1+\sqrt{2}\,)$

$\mathbf{v}_{53}(\sqrt{2}, \sqrt{2}, 0, 2)$	$\mathbf{v}_{54}(-1+\sqrt{2}, 1+\sqrt{2}, 1, 1)$
$\mathbf{v}_{55}(\sqrt{2}, \sqrt{2}, 2, 0)$	$\mathbf{v}_{56}(1, 1, 1+\sqrt{2}, 1-\sqrt{2})$
$\mathbf{v}_{57}(1, -1+\sqrt{2}, 1+\sqrt{2}, -1)$	$\mathbf{v}_{58}(\sqrt{2}, 0, 2, -\sqrt{2})$
$\mathbf{v}_{59}(2, 0, \sqrt{2}, -\sqrt{2})$	$\mathbf{v}_{60}(1, 1-\sqrt{2}, 1+\sqrt{2}, 1)$
$\mathbf{v}_{61}(1, -1, 1+\sqrt{2}, -1+\sqrt{2})$	$\mathbf{v}_{62}(\sqrt{2}, -\sqrt{2}, 2, 0)$
$\mathbf{v}_{63}(2, -\sqrt{2}, \sqrt{2}, 0)$	$\mathbf{v}_{64}(1, -1, 1+\sqrt{2}, 1-\sqrt{2})$
$\mathbf{v}_{65}(1, 1-\sqrt{2}, 1+\sqrt{2}, -1)$	

Table 4

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