Publ. Math. Debrecen
64/1-2 (2004), 1–13

Horospherical surfaces of curves in Hyperbolic space

By S. IZUMIYA (Sapporo), D. PEI (Changchun) and T. SANO (Sapporo)

Abstract. We consider the contact between curves and horospheres in Hyperbolic 3-space as an application of singularity theory of functions. We define the osculating horosphere of the curve. We also define the horospherical surface of the curve whose singular points correspond to the locus of polar vectors of osculating horospheres of the curve. One of the main results is to give a generic classification of singularities of horospherical surface of curves.

1. Introduction

In [2], [3] we have applied singularity theory to local differential geometry on hypersurfaces in Hyperbolic space. We have constructed some basic tools for the study of these subjects. These tools work very well for hypersurfaces. The next step is to consider the case for submanifolds with higher codimensions. In this paper we stick to hyperbolic space curves because this is the simplest case with higher codimensions. Here, we study the contact between hyperbolic space curves and horospheres as an application of singularity theory of smooth functions. One of the basic tools we have given in [3] is the notion of horospherical height functions on hypersurfaces. We can also define the horospherical height function of a hyperbolic space curve here. By the aid of the technique of singularity theory on such a function, we can give the definition of osculating horospheres along a hyperbolic space curve (cf., §4). We can define the

Mathematics Subject Classification: Primary 53A25; Secondary 58C27.
Key words and phrases: Hyperbolic space, horosphere, horospherical height function.

horospherical surface of a hyperbolic space curve as the discriminat set of the horospherical height function on the curve. Compared with the case for curves in Euclidean space, the situation is rather different because the horospherical surface is defined in the light cone. It might be considered as a kind of the dual surface of the curve. The main results in this paper are Theorems 2.1 and 2.2. These theorems give generic classifications of singularities of horospherical surfaces of hyperbolic space curves. Moreover, we study the geometric meanings of singularities of horospherical surfaces of hyperbolic space curves and introduce a new invariant $\sigma_h(s)$. We can show that $\sigma_h(s) \equiv 0$ if and only if the curve is located on a horosphere under a certain generic assumption (cf., §4).

This is one of the papers of the authors project on "generic differential geometry" of submanifolds in Hyperbolic space (cf., [2], [3]).

All maps considered here are of class C^∞ unless otherwise stated.

2. Basic notions and results

We adopt the Lorentzian model of the hyperbolic 3-space. Let $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) \mid x_i \in \mathbb{R}\ (i = 0, 1, 2, 3)\}$ be a 4-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$, $\boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i.$$

We call $(\mathbb{R}^4, \langle , \rangle)$ *Minkowski space.* We denote $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. For a vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number c, we define *the hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *Hyperbolic 3-space* by

$$H^3_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1,\ x_0 \geq 1\}.$$

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x^1_0 & x^1_1 & x^1_2 & x^1_3 \\ x^2_0 & x^2_1 & x^2_2 & x^2_3 \\ x^3_0 & x^3_1 & x^3_2 & x^3_3 \end{vmatrix},$$

where $\boldsymbol{e}_0,\ \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3$ is the canonical basis of $\mathbb{R}^4_1$. We can easily show that $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}\ \boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3)$, so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, 2, 3)$.

We also define a set $LC_a = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\}$, which is called *a closed lightcone* with the vertex $\boldsymbol{a}$. We denote that

$$LC^*_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in LC_0 \mid x_0 > 0\}$$

and we call it *the future lightcone* at the origin. We can also define the notion of *the past lightcone*. We have three kinds of surfaces in $H^3_+(-1)$ which are given by intersections of $H^3_+(-1)$ and hyperplanes in $\mathbb{R}^4_1$. A surface $H^3_+(-1) \cap HP(\boldsymbol{v}, c)$ is called a *sphere*, an *equidistant plane* or a *horosphere* if $H(\boldsymbol{v}, c)$ is spacelike, timelike or lightlike respectively. Especially we write a horosphere as $HS^2(\boldsymbol{v}, c) = H^3_+(-1) \cap H(\boldsymbol{v}, c)$. If we consider a lightlike vector $\boldsymbol{v}_0 = -1/c\boldsymbol{v}$, we have $HS^2(\boldsymbol{v}, c) = HS^2(\boldsymbol{v}_0, -1)$. We call $\boldsymbol{v}_0$ the *polar vector* of $HS^2(\boldsymbol{v}_0, -1)$.

We now construct the explicit differential geometry on curves in $H^3_+(-1)$. Let $\boldsymbol{\gamma} : I \longrightarrow H^3(-1)$ be a regular curve. Since $H^3_+(-1)$ is a Riemannian manifold, we can reparametrise $\boldsymbol{\gamma}$ by the arc-length. Hence, we may assume that $\boldsymbol{\gamma}(s)$ is a unit speed curve. So we have the tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ with $\|\boldsymbol{t}(s)\| = 1$, where $\|\boldsymbol{v}\| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v} \rangle|}$. In the case when $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, then we have a unit vector $\boldsymbol{n}(s) = \frac{\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)}{\|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|}$. Moreover, define $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4_1$ along $\boldsymbol{\gamma}$. By standard arguments, under the assumption that $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq -1$, we have the following *Frenet–Serret type formula*:

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = \kappa_h(s)\boldsymbol{n}(s) + \boldsymbol{\gamma}(s) \\ \boldsymbol{n}'(s) = -\kappa_h(s)\boldsymbol{t}(s) + \tau_h(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) = -\tau_h(s)\boldsymbol{n}(s) \end{cases},$$

where $\kappa_h(s) = \|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|$ and $\tau_h(s) = -\frac{\det(\boldsymbol{\gamma}(s),\boldsymbol{\gamma}'(s),\boldsymbol{\gamma}''(s),\boldsymbol{\gamma}'''(s))}{(\kappa_h(s))^2}$.

Since $\langle \boldsymbol{t}'(s) - \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\rangle = \langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle + 1$, the condition $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq -1$ is equivalent to the condition $\kappa_h(s) \neq 0$. Moreover, we can show that the curve $\boldsymbol{\gamma}(s)$ satisfies the condition $\kappa_h(s) \equiv 0$ if and only if there exists a lightlike vector $\boldsymbol{c}$ such that $\boldsymbol{\gamma}(s) - \boldsymbol{c}$ is a geodesic. Such a curve is called *an equidistant line*. We can study many properties of hyperbolic space curves by using this fundamental equation. Here, we consider the contact between hyperbolic space curves and horospheres. This is the special subject in hyperbolic differential geometry.

Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve. We now define a map

$$HS_\gamma : I \times J \longrightarrow LC^*_+$$

by $HS_\gamma(s,\theta) = \boldsymbol{\gamma}(s) + \cos\theta\boldsymbol{n}(s) + \sin\theta\boldsymbol{e}(s)$, where J is an open interval or the unit circle in Euclidean plane. We call HS_γ the *horospherical surface* of $\boldsymbol{\gamma}$. We also introduce a hyperbolic invariant $\sigma_h(s) = ((\kappa_h')^2 - (\kappa_h)^2(\tau_h)^2((\kappa_h)^2 - 1))(s)$. The geometric meaning of these objects will be discussed in §4. Our main result is given as follows:

Theorem 2.1. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. Then we have the following:*

(1) *The horospherical surface HS_γ of $\boldsymbol{\gamma}$ is singular at (s_0,θ_0) if and only if $\cos\theta_0 = 1/\kappa_h(s_0)$.*

(2) *The horospherical surface HS_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0,θ_0) if $\cos\theta_0 = 1/\kappa_h(s_0)$ and $\sigma_h(s_0) \neq 0$.*

(3) *The horospherical surface HS_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallow tail SW at (s_0,θ_0) if $\cos\theta_0 = 1/\kappa_h(s_0)$, $\sigma_h(s_0) = 0$ and $\sigma_h'(s_0) \neq 0$.*

Here, $C \times \mathbb{R} = \{(x_1, x_2, x_3) \mid {x_1}^2 = {x_2}^3\}$ *is the cuspidal edge (c.f., Figure 1) and* $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *is the swallow tail (c.f., Figure 2).*

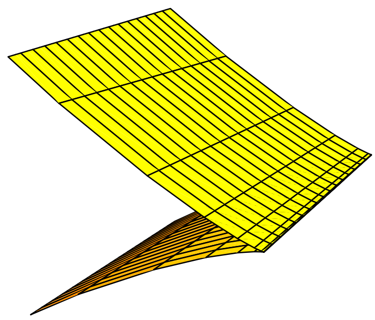

Figure 1: cuspidal edge

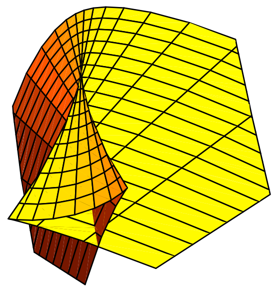

Figure 2: swallowtail

Moreover, we can assert that the above theorem gives a generic classification of singularities of horospherical surfaces of hyperbolic space curves. Let $\mathrm{Emb}(I, H^3_+(-1))$ be the space of proper embeddings $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ equipped with Whitney C^∞-topology. The generic classification theorem is given as follows:

Theorem 2.2. *There exists an open and dense subset* $\mathcal{O} \subset \mathrm{Emb}(I, H^3_+(-1))$ *such that for any* $\boldsymbol{\gamma} \in \mathcal{O}$*, the horospherical surface* HS_γ *of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the cuspidal edge or the swallowtail at any singular point.*

3. Horospherical height functions

In this section we introduce a family of functions on a curve which is useful for the study of invariants of hyperbolic space curves. For a hyperbolic space curve $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$, we define a function $H : I \times LC^*_+ \longrightarrow \mathbb{R}$ by $H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle + 1$. We call H a *horospherical height* *function* on $\boldsymbol{\gamma}$. We denote that $h(s) = H_{v_0}(s) = H(s, \boldsymbol{v}_0)$ for any $\boldsymbol{v}_0 \in LC^*_+$.

Proposition 3.1. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. Then we have the followings:*

(1) *$h(s_0) = 0$ if and only if there exist real numbers λ, μ, η with $\lambda^2 + \mu^2 + \eta^2 = 1$ such that $\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \lambda \boldsymbol{t}(s_0) + \mu \boldsymbol{n}(s_0) + \eta \boldsymbol{e}(s_0)$.*

(2) *$h(s_0) = h'(s_0) = 0$ if and only if there exists $\theta_0 \in [0, 2\pi]$ such that $\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$.*

(3) *$h(s_0) = h'(s_0) = h''(s_0) = 0$ if and only if $\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$ and $\cos\theta_0 = 1/\kappa_h(s_0)$.*

(4) *$h(s_0) = h'(s_0) = h''(s_0) = h^{(3)}(s_0) = 0$ if and only if $\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$, $\cos\theta_0 = 1/\kappa_h(s_0)$ and $\sigma_h(s_0) = ((\kappa'_h)^2 - (\kappa_h)^2(\tau_h)^2((\kappa_h)^2 - 1))(s_0) = 0$.*

(5) *$h(s_0) = h'(s_0) = h''(s_0) = h^{(3)}(s_0) = h^{(4)}(s_0) = 0$ if and only if $\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$, $\cos\theta_0 = 1/\kappa_h(s_0)$ and $\sigma_h(s_0) = \sigma'_h(s_0) = 0$.*

Proof. Since $h(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle + 1$, we have

(a) $h'(s) = \langle \boldsymbol{t}(s), \boldsymbol{v} \rangle$,

(b) $h''(s) = \langle \kappa_h(s)\boldsymbol{n}(s) + \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle$,

(c) $h^{(3)}(s) = \langle (1 - \kappa_h^2(s))\boldsymbol{t}(s) + \kappa'_h(s)\boldsymbol{n}(s) + \kappa_h(s)\tau_h(s)\boldsymbol{e}(s), \boldsymbol{v} \rangle$,

(d) $h^{(4)}(s) = \langle (1 - \kappa_h^2(s))\boldsymbol{\gamma}(s) - 3\kappa_h(s)\kappa'_h(s)\boldsymbol{t}(s) + (\kappa_h(s) - \kappa_h^3(s)$
$-\kappa_h(s)\tau_h^2(s) + \kappa''_h(s))\boldsymbol{n}(s) + (2\kappa'_h(s)\tau_h(s) + \kappa_h(s)\tau'_h(s))\boldsymbol{e}(s), \boldsymbol{v} \rangle$.

The assertion (1) is trivial by definition. By the relation (a), $h(s_0) = h'(s_0) = 0$ if and only if $\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \mu \boldsymbol{n}(s_0) + \eta \boldsymbol{e}(s_0)$ with $\mu^2 + \eta^2 = 1$. Therefore, we might write $\mu = \cos\theta$ and $\eta = \sin\theta$. By the relation (b), $h(s_0) = h'(s_0) = h''(s_0) = 0$ if and only if $\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$ and $0 = \langle \kappa_h(s)\boldsymbol{n}(s) + \boldsymbol{\gamma}(s), \boldsymbol{v}_0 \rangle = -1 + \cos\theta \kappa_h(s_0)$. This means that the assertion (3) holds. The other assertions also follow from the above relations exactly the same way as the above, we need, however, rather long calculations, so that we omit the detail. $\square$

4. Invariants of hyperbolic space curves

In §2 we found that the function

$$\sigma_h(s) = ((\kappa_h')^2 - (\kappa_h)^2(\tau_h)^2((\kappa_h)^2 - 1))(s)$$

on $\boldsymbol{\gamma}$ has a special meaning. Here, we try to understand the geometric meaning of this invariant. Let $\boldsymbol{v}$ be a lightlike vector and $\boldsymbol{w}$ be a spacelike vector. A hyperbolic space curve given by $HS^2(\boldsymbol{v}, -1) \cap HP(\boldsymbol{w}, 0)$ is called a *horocycle.* We have the following proposition.

Proposition 4.1. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \geq 1$. We consider the vector field along $\boldsymbol{\gamma}$ given by $\boldsymbol{v}(s) = \boldsymbol{\gamma}(s) + \cos\theta \boldsymbol{n}(s) + \sin\theta \boldsymbol{e}(s)$ with $\cos\theta = 1/\kappa_h(s)$.*

(1) *Suppose that $\kappa_h(s) \equiv 1$. Then the following conditions are equivalent:*
  (a) *$\boldsymbol{v}(s)$ is a constant vector.*
  (b) *$\tau_h(s) \equiv 0$.*
  (c) *$\boldsymbol{\gamma}$ is a part of horocycle.*

(2) *Suppose that the set $\{s \in I \mid \kappa_h(s) = 1\}$ consists of isolated points. Then the following conditions are equivalent:*
  (a) *$\boldsymbol{v}(s)$ is a constant vector.*
  (b) *$\sigma_h(s) \equiv 0$.*
  (c) *$\boldsymbol{\gamma}$ is located on a horosphere.*

Proof. Suppose that $\kappa_h(s) \equiv 1$. Then $\boldsymbol{v}(s) = \boldsymbol{\gamma}(s) + \boldsymbol{n}(s)$, so that we have $\boldsymbol{v}'(s) = \tau_h(s)\boldsymbol{e}(s)$. Therefore $\boldsymbol{v}(s)$ is constant if and only if $\tau_h(s) \equiv 0$. For any $s \in I$, we consider the horocycle given by $HS^2(\boldsymbol{v}(s), -1) \cap \langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s)\rangle_{\mathbb{R}}$. If $\boldsymbol{v}(s)$ is constant, then $\tau_h(s) \equiv 0$. This means that $\boldsymbol{e}(s)$ is constant, so that the hyperplane $\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s)\rangle_{\mathbb{R}}$ is also constant. In this case the horosphere $HS^2(\boldsymbol{v}(s), -1)$ is also constant. Thus the image of $\boldsymbol{\gamma}$ is a part of horocycle given by $HS^2(\boldsymbol{v}(s), -1) \cap \langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s)\rangle_{\mathbb{R}}$. If $\boldsymbol{\gamma}$ is a part of a horocycle, then it is a hyperbolic plane curve. Therefore we have $\tau_h(s) \equiv 0$. This completes the proof the assertion (1).

We consider the case $\kappa_h(s) \neq 1$. Since $\cos\theta(s) = 1/\kappa_h(s)$, we have

$$\boldsymbol{v}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\kappa_h(s)}\boldsymbol{n}(s) \pm \frac{\sqrt{\kappa_h^2(s) - 1}}{\kappa_h(s)}\boldsymbol{e}(s).$$

Then we have

$$\boldsymbol{v}'(s) = -\frac{\kappa_h' \pm \kappa_h\tau_h\sqrt{\kappa_h^2 - 1}}{\kappa_h^2}(s)\boldsymbol{n}(s) + \frac{\tau_h\kappa_h\sqrt{\kappa_h^2 - 1} \pm \kappa_h'}{\kappa_h^2\sqrt{\kappa_h^2 - 1}}(s)\boldsymbol{e}(s).$$

Therefore, $\boldsymbol{v}'(s) \equiv 0$ if and only if $\sigma_h(s) \equiv 0$. The conditions (a) and (b) of (2) are equivalent. By the assumption of (2), the set of points with $\kappa_h(s) \neq 1$ is open and dense subset of I. Therefore, the conditions (a) and (b) of (2) are equivalent at any point of I.

We now consider the horospherical height function $H(s, \boldsymbol{v})$ on $\boldsymbol{\gamma}$. If $\boldsymbol{\gamma}$ is located on a horosphere $HS^2(\boldsymbol{v}_0, c)$, we can choose $c = -1$. This means that $H(s, \boldsymbol{v}_0) \equiv 0$. By the assertion (4) of Proposition 3.1, we have $\left(\kappa_h' \pm \kappa_h\tau_h\sqrt{\kappa_h^2 - 1}\right)(s) \equiv 0$. This means that the condition (c) implies the condition (b). If $\boldsymbol{v}(s)$ is a constant vector $\boldsymbol{v}_0$, then $\boldsymbol{\gamma}$ is located on $HS^2(\boldsymbol{v}_0, -1)$. $\square$

Let $F : H^3_+(-1) \longrightarrow \mathbb{R}$ be a submersion and $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a regular curve. We say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have *at least k-point contact* for $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$. If $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have at least k-point contact for $t = t_0$ and satisfies the condition that $g^{(k)}(t_0) \neq 0$, then we say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have *k-point contact* for $t = t_0$. If a horosphere $HS^2(\boldsymbol{v}_0, -1)$ and a hyperbolic space curve $\boldsymbol{\gamma}$ have at least 3-point contact for a point t_0, we call $HS^2(\boldsymbol{v}_0, -1)$ the *osculating horosphere of* $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$. Then we have the following proposition.

Proposition 4.2. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve. Then we have the following:*

(1) *There exists the osculating horosphere of $\boldsymbol{\gamma}$ at a point $\boldsymbol{\gamma}(s_0)$ if and only if $\kappa_h(s_0) \geq 1$.*

(2) *Suppose that $\kappa_h(s_0) \geq 1$. Then the osculating horosphere and $\boldsymbol{\gamma}$ have 4-point contact for $s = s_0$ if and only if $\sigma_h(s_0) = 0$ and $\sigma_h'(s_0) \neq 0$.*

Proof. Let $\mathfrak{H} : H^3_+(-1) \times LC^*_+ \longrightarrow \mathbb{R}$ be a function defined by $\mathfrak{H}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle + 1$. For any $\boldsymbol{v}_0 \in LC^*_+$, $\mathfrak{h}_{v_0}(\boldsymbol{x}) = \mathfrak{H}(\boldsymbol{x}, \boldsymbol{v}_0)$ is a submersion and $\mathfrak{h}_{v_0}^{-1}(0)$ is a horosphere. Moreover, any horosphere can be realized as the zero level set of $\mathfrak{h}_{v_0}$ for some $\boldsymbol{v}_0 \in LC^*_+$. For any $\boldsymbol{\gamma}$, we have

$\mathfrak{h}_{v_0} \circ \boldsymbol{\gamma}(s) = h(s)$, here $h(s) = H(s, \boldsymbol{v}_0)$. Therefore, $h_{v_0}^{-1}(0)$ is an osculating horosphere of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$ if and only if $h(s_0) = h'(s_0) = h''(s_0) = 0$. By Proposition 3.1, this condition is equivalent to the condition that $\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \cos\theta_0 \boldsymbol{n}(s_0) + \sin\theta_0 \boldsymbol{e}(s_0)$ with $\cos\theta_0 = 1/\kappa_h(s_0)$. Of course, $\kappa_h^2(s_0) \geq 1$. The assertion (2) follows from the assertions (3) and (4) of Proposition 2.1. $\square$

Theorem 2.1 asserts that the set of singular points of the horospherical surface of $\boldsymbol{\gamma}$ is the locus the polar vectors of osculating horospheres of $\boldsymbol{\gamma}$. Moreover, the swallow tail point of the horospherical surface of $\boldsymbol{\gamma}$ corresponds to the point $\boldsymbol{\gamma}(s_0)$ at where the osculating horosphere and $\boldsymbol{\gamma}$ have 4-point contact.

On the other hand, we consider the horocycle $HS^2(\boldsymbol{v}(s_0), -1) \cap \langle \boldsymbol{\gamma}(s_0), \boldsymbol{t}(s_0), \boldsymbol{n}(s_0) \rangle_{\mathbb{R}}$ at a point $s_0 \in I$ with $\kappa_h(s_0) \geq 1$. We call it the *osculating horocycle* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$. The assertion (1) of Proposition 4.1, suggests that two invariants $\kappa_h(s_0)$ and $\tau_h(s_0)$ describe the contact between curves and horocycle. We do not, however, proceed to study about this topics here.

5. Unfoldings of functions of one-variable

In this section we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [1]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F(s, x_0)$. We say that f has an *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has an *$A_{\geq k}$-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ has an A_k-singularity $(k \geq 1)$ at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s - s_0)^j$ for $i = 1, \ldots, r$. Then F is called a *versal unfolding* if the $k \times r$ matrix of coefficients (α_{ji}) has rank k $(k \leq r)$.

We now introduce an important set concerning the unfoldings relative to the above notions. The *discriminant set* of F is the set

$$\mathcal{D}_F = \left\{ x \in \mathbb{R}^r \mid \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x) \right\}.$$

Then we have the following well-known result (cf., [1]).

Theorem 5.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has an A_k singularity at s_0. Suppose that F is a versal unfolding.*

(1) *If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

(2) *If $k = 3$, then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

For the proof of Theorem 2.1, we have the following key proposition.

Proposition 5.2. *Let $H : I \times LC^*_+ \longrightarrow \mathbb{R}$ be the horospherical height function on a unit speed hyperbolic space curve $\boldsymbol{\gamma}(s)$. If h_{v_0} has an A_k-singularity ($k = 2, 3$) at s_0, then H is a versal unfolding of h_{v_0}.*

Proof. Let us consider the pseudo orthonormal basis $\boldsymbol{e}_0 = \boldsymbol{\gamma}(s_0)$, $\boldsymbol{e}_1 = \boldsymbol{t}(s_0)$, $\boldsymbol{e}_2 = \boldsymbol{n}(s_0)$ and $\boldsymbol{e}_3 = \boldsymbol{e}(s_0)$ instead of the canonical basis of $\mathbb{R}^4_1$. Then

$$H(s, \boldsymbol{v}) = -v_0 x_0(s) + v_1 x_1(s) + v_2 x_2(s) + v_3 x_3(s) + 1,$$

where v_i and $x_i(s)$ denote respectively the coordinates of $\boldsymbol{v}$ and $\boldsymbol{\gamma}(s)$ with respect to this basis. Since $\boldsymbol{\gamma}(s_0) = \boldsymbol{e}_0$, $\boldsymbol{\gamma}'(s_0) = \boldsymbol{e}_1$, $\boldsymbol{\gamma}''(s_0) = \kappa_h(s_0)\boldsymbol{e}_2 + \boldsymbol{e}_0$ and

$$\boldsymbol{v}_0 = \boldsymbol{e}_0 + \frac{1}{\kappa_h(s_0)}\boldsymbol{e}_2 \pm \left(\frac{\sqrt{\kappa_h(s_0)^2 - 1}}{\kappa_h(s_0)} \right) \boldsymbol{e}_3,$$

we have the matrix

$$A = \begin{pmatrix} 0 & -\frac{1}{\kappa_h(s_0)} & \mp\dfrac{\sqrt{\kappa_h(s_0)^2 - 1}}{\kappa_h(s_0)} \\ 1 & 0 & 0 \\ 0 & \frac{\kappa_h(s_0)^2 - 1}{\kappa_h(s_0)} & \mp\dfrac{\sqrt{\kappa_h(s_0)^2 - 1}}{\kappa_h(s_0)} \end{pmatrix}.$$

We give the proof for $k = 3$ at first. The determinant of A is $\mp\sqrt{\kappa_h^2(s_0) - 1}$. If $\kappa_h(s_0) = 1$, then $\kappa_h'(s_0) = 0$ because $\sigma_h(s_0) = 0$. In this case, however, we have $\sigma_h'(s_0) = 0$. This contradicts to the assumption that h_{v_0} has the A_3-type singularity at $s = s_0$. Therefore, H is a versal unfolding of h_{v_0} at $s = s_0$.

We now give the proof for the case $k = 2$. In this case we expect to show that the first and second column of A is non-singular. This fact is trivial. This completes the proof. □

Proof of Theorem 2.1. The assertion (1) follows from the direct calculation and the Frenet–Serret type formula for hyperbolic space curves. By Proposition 3.1, the discriminant set $\mathcal{D}_H$ of the horospherical height function H of $\boldsymbol{\gamma}$ is the image of the horospherical surface of $\boldsymbol{\gamma}$. It also follows from the assertions (4) and (5) that h_{v_0} has the A_2-type singularity (respectively, the A_3-type singularity) at $s = s_0$ if and only if $\cos\theta_0 = 1/\kappa_h(s_0)$ and $\sigma_h(s_0) \neq 0$ (respectively, $\cos\theta_0 = 1/\kappa_h(s_0)$, $\sigma_h(s_0) = 0$ and $\sigma_h'(s_0) \neq 0$). By Theorem 5.1 and Proposition 5.2, we have the assertions (2) and (3). □

6. Generic properties

In this section we consider generic properties of curves in $H^3_+(-1)$. The main tool is a kind of transversality theorems. Let $\mathrm{Emb}(I, H^3_+(-1))$ be the space of proper embeddings $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ with Whitney C^∞-topology. We also consider the function $\mathcal{H} : H^3_+(-1) \times LC^*_+ \longrightarrow \mathbb{R}$ which is given by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v}\rangle + 1$. We claim that $\mathcal{H}_u$ is a submersion for any $\boldsymbol{u} \in LC^*_+$, where $\mathcal{H}_u(\boldsymbol{v}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v})$. For any $\boldsymbol{\gamma} \in \mathrm{Emb}(I, H^3_+(-1))$, we have $H = \mathcal{H} \circ (\boldsymbol{\gamma} \times \mathrm{id}_{LC^*_+})$. We also have the ℓ-jet extension

$$j_1^\ell H : I \times LC^*_+ \longrightarrow J^\ell(I, \mathbb{R})$$

defined by $j_1^\ell H(s, \boldsymbol{v}) = j^\ell h_v(s)$. We consider the trivialisation $J^\ell(I, \mathbb{R}) \equiv I \times \mathbb{R} \times J^\ell(1, 1)$. For any submanifold $Q \subset J^\ell(1, 1)$, we denote that $\widetilde{Q} = I \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [5]. (See also Montaldi [4].)

Proposition 6.1. *Let Q be a submanifold of $J^\ell(1,1)$. Then the set*

$$T_Q = \{\gamma \in \mathrm{Emb}(I, H^3_+(-1)) \mid j^\ell_1 H \text{ is transversal to } \widetilde{Q}\}$$

is a residual subset of $\mathrm{Emb}(I, H^3_+(-1))$. *If Q is a closed subset, then T_Q is open.*

Let $f : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be a function germ which has an A_k-singularity at 0. It is well-known that there exists a diffeomorphism germ $\phi : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ such that $f \circ \phi(s) = \pm s^{k+1}$. This is the classification of A_k-singularities. For any $z = j^\ell f(0) \in J^\ell(1,1)$, we have the orbit $L^\ell(z)$ given by the action of the Lie group of ℓ-jets of diffeomorphism germs. If f has an A_k-singularity, then the codimension of the orbit is k. There is another characterisation of versal unfoldings as follows:

Proposition 6.2. *Let $F : (\mathbb{R} \times \mathbb{R}^r, 0) \longrightarrow (\mathbb{R}, 0)$ be an r-parameter unfolding of $f : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ which has an A_k-singularity at 0. Then F is a versal unfolding if and only if $j^\ell_1 F$ is transversal to the orbit $\widetilde{L^\ell(j^\ell f(0))}$ for $\ell \geq k+1$.*

Here, $j^\ell_1 F : (\mathbb{R} \times \mathbb{R}^r, 0) \longrightarrow J^\ell(\mathbb{R}, \mathbb{R})$ is the ℓ-jet extension of F given by $j^\ell_1 F(s, x) = j^\ell F_x(s)$.

We can prove Theorem 2.2 as a corollary of Proposition 6.1 as follows:

Proof of Theorem 2.2. For $\ell \geq 4$, we consider the decomposition of the jet space $J^\ell(1,1)$ into $L^\ell(1)$ orbits. We now define a semi-algebraic set by

$$\Sigma^\ell = \{z = j^\ell f(0) \in J^\ell(1,1) \mid f \text{ has an } A_{\geq 4}\text{-singularity}\}.$$

Then the codimension of Σ^ℓ is 4. Therefore, the codimension of $\widetilde{\Sigma_0} = I \times \{0\} \times \Sigma^\ell$ is 5. We have the orbit decomposition of $J^\ell(1,1) - \Sigma^\ell$ into

$$J^\ell(1,1) - \Sigma^\ell = L^\ell_0 \cup L^\ell_1 \cup L^\ell_2 \cup L^\ell_3,$$

where L^ℓ_k is the orbit through an A_k-singularity. Thus, the codimension of $\widetilde{L^\ell_k}$ is $k+1$. We consider the ℓ-jet extension $j^\ell_1 H$ of the horospherical height function H. By Proposition 6.1, there exists an open and dense subset $\mathcal{O} \subset \mathrm{Emb}(I, H^3_+(-1))$ such that $j^\ell_1 H$ is transversal to $\widetilde{L^\ell_k}$ $(k =$

$0, 1, 2, 3)$ and the orbit decomposition of $\widetilde{\Sigma^\ell}$. This means that $j_1^\ell H(I \times LC_+^*) \cap \widetilde{\Sigma^\ell} = \emptyset$ and H is a versal unfolding of h at any point (s_0, v_0). By Theorem 5.1, the discriminant set of H (i.e., the horospherical surface of $\boldsymbol{\gamma}$) is locally diffeomorphic to the cuspidal edge or the swallow tail if the point is singular. $\square$

References

[1] J. W. Bruce and P. J. Giblin, Curves and singularities (2nd edn), *Cambridge University Press*, 1992.

[2] S. Izumiya, D-H. Pei, T. Sano and E. Torii, Evolutes of hyperbolic plane curves (*to appear in* Acta Mathematica Sinica).

[3] S. Izumiya, D-H. Pei and T. Sano, Singularities of hyperbolic Gauss maps, *Proceedings of the London Mathematical Society* **86** (2003), 485–512.

[4] J. A. Montaldi, On generic composites of maps, *Bull. London Math. Soc.* **23** (1991), 81–85.

[5] G. Wassermann, Stability of Caustics, *Math. Ann.* **210** (1975), 43–50.

SHYUICHI IZUMIYA
DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY
SAPPORO 060-0810
JAPAN

E-mail: izumiya@math.sci.hokudai.ac.jp

DONGHE PEI
DEPARTMENT OF MATHEMATICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN 130024
P.R. CHINA

E-mail: northlcd@public.cc.jl.cn, pei@math.sci.hokudai.ac.jp

TAKASHI SANO
FACULTY OF ENGINEERING
HOKKAIGAKUEN UNIVERSITY
SAPPORO 062-8605
JAPAN

E-mail: t-sano@cvl.hokkai-s-u.ac.jp

(Received March 28, 2002; revised May 26, 2003)