Publ. Math. Debrecen
64/1-2 (2004), 15–20

A consequence of the Proper Forcing Axiom in topology

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Abstract. If $\langle L, <\rangle$ is a dense linear order without end-points and if $A_1, A_2 \subset L$ are disjoint dense subsets of L, then $\mathcal{O}_{A_1A_2}$ denotes the topology on L generated by the closed intervals $[a_1, a_2]$, where $a_1 \in A_1$ and $a_2 \in A_2$. It is proved that under the Proper Forcing Axiom each two spaces of the form $\langle \mathbb{R}, \mathcal{O}_{A_1A_2}\rangle$, where A_1 and A_2 are $\aleph_1$-dense subsets of reals, are homeomorphic.

0. Introduction

The standard topology on a linearly ordered set $\langle L, <\rangle$ (generated by the family of all open intervals) is extensively investigated. It is T_5, collectionwise normal, connected iff the order is continuous, compact iff each subset of L has the supremum, there holds $\chi(L) \leq d(L) \leq w(L) \leq |L|$, etc. (see [4]).

If $\langle L, <\rangle$ is a dense linear order without end-points and if $A_1, A_2 \subset L$ are disjoint dense subsets of L, then $\mathcal{O}_{A_1A_2}$ denotes the topology on L generated by the closed intervals $[a_1, a_2]$, where $a_1 \in A_1$, $a_2 \in A_2$ and $a_1 < a_2$.

Modifying the methods used in the theory of linearly ordered spaces it can be proved that in general, the spaces $\langle L, \mathcal{O}_{A_1A_2}\rangle$ are zero-dimensional and collectionwise normal. Also, there holds $\chi(L, \mathcal{O}_{A_1A_2}) \leq d(L, \mathcal{O}_{A_1A_2})$ $\leq \min\{|A_1|, |A_2|\} \leq \max\{|A_1|, |A_2|\} = w(L, \mathcal{O}_{A_1A_2}) \leq |L|$ and all the

Mathematics Subject Classification: 03E35, 03E65, 03E75, 54A35, 54F05.
Key words and phrases: Proper Forcing Axiom, real line, closed intervals, linear orders.

inequalities can be strict. If the sets A_1 and A_2 are countable, the space $\langle L, \mathcal{O}_{A_1 A_2} \rangle$ is metrizable.

Specially, if $\mathbb{R}$ is the real line, then the space $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$ is a hereditarily separable, hereditarily Lindelöf, first countable T_6-space.

The space $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$ is second countable if and only if the sets A_1 and A_2 are countable. Such a space is a universal second countable zero-dimensional space. By the well known theorem of Cantor each two countable dense linear orders without end-points are isomorphic. If A_1, A_2 and B_1, B_2 are pairs of disjoint dense countable subsets of reals, then using Cantor's "back and forth" method it is easy to make an order-isomorphism from $A_1 \cup A_2$ onto $B_1 \cup B_2$ which maps A_1 onto B_1 (and A_2 onto B_2). Extending the isomorphism continuously we obtain an homeomorphism $F : \langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle \to \langle \mathbb{R}, \mathcal{O}_{B_1 B_2} \rangle$. So, each two spaces of the form $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$, where $|A_1| = |A_2| = \aleph_0$, are homeomorphic.

Can this result be extended for uncountable cardinals? In this paper we will show that for $\aleph_1$-dense sets A_1 and A_2 the answer in the affirmative is consistent with ZFC, namely we will prove

Theorem 1. PFA $\Rightarrow$ *All the spaces* $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$, *where* A_1 *and* A_2 *are* $\aleph_1$*-dense subsets of* $\mathbb{R}$, *are homeomorphic.*

In [2], using iterated forcing Baumgartner proved the consistency of MA + (today called) Baumgartner's Axiom, BA: "All $\aleph_1$-dense sets of reals are order-isomorphic". A different proof was obtained by Shelah (see [1]) and, amalgamating these two ideas, Baumgartner showed that the PFA implies BA (see [3]). In Corollary 8.3 of [7] Todorčević presented an elegant proof of PFA $\Rightarrow$ BA using

Theorem 2 (Theorem 8.2 of [7])**.** (PFA) *If* A *and* B *are sets of reals of size* $\aleph_1$, *then there is an injection* $f : A \to B$ *which is the union of countably many increasing subfunctions.*

In fact, Todorčević deduced BA from $\mathrm{MA}_{\aleph_1}$ for σ-centred posets + the consequence of the PFA given in the previous theorem.

For the proof of Theorem 1 we will modify the construction of Todorčević and use the following elementary fact.

Fact 1. If A and B are dense subsets of $\mathbb{R}$ and $f : A \to B$ is an order isomorphism, then the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \sup\{f(a) : a \in A \wedge a < x\}$ is the unique isomorphism which extends f.

The notation used in the paper is standard. The facts concerning forcing can be found in [5] or [6], while all the facts related to the PFA are contained in [3].

1. A combinatorial consequence of the PFA

A set $A \subset \mathbb{R}$ is called $\aleph_1$-dense iff $|A \cap (x, y)| = \aleph_1$ for each $x, y \in \mathbb{R}$ satisfying $x < y$. In this section we extend Corollary 8.3 of [7].

Theorem 3. (PFA) *Let $\theta \geq 1$ be a countable ordinal, let $\{A_\alpha : \alpha < \theta\}$ and $\{B_\alpha : \alpha < \theta\}$ be two families of pairwise disjoint $\aleph_1$-dense subsets of $\mathbb{R}$ and let $A = \bigcup_{\alpha<\theta} A_\alpha$ and $B = \bigcup_{\alpha<\theta} B_\alpha$. Then there exists an order isomorphism $f : A \to B$ such that $f[A_\alpha] = B_\alpha$, for each $\alpha < \theta$.*

Proof. Let $\mathcal{B}$ denote the family of all open intervals with rational end-points. Using Theorem 2, for each $I, J \in \mathcal{B}$ and each $\alpha < \theta$ we choose injections $f_{I,J,\alpha} : A_\alpha \cap I \to B_\alpha \cap J$ and $g_{I,J,\alpha} : B_\alpha \cap J \to A_\alpha \cap I$ such that $f_{I,J,\alpha} = \bigcup_{k\in\omega} f_{I,J,\alpha,k}$ and $g_{I,J,\alpha} = \bigcup_{k\in\omega} g_{I,J,\alpha,k}$, where $f_{I,J,\alpha,k}$ and $g_{I,J,\alpha,k}$, $k \in \omega$, are increasing functions. Let

$$\mathbb{Q} = \Big\{p \subset \bigcup_{I,J\in\mathcal{B}} \bigcup_{\alpha<\theta} (f_{I,J,\alpha} \cup g^{-1}_{I,J,\alpha}) : p \text{ is a finite increasing function}\Big\}.$$

Clearly the elements $p, q \in \mathbb{Q}$ are compatible if and only if $p \cup q$ is an increasing function. □

Claim 1. The partial ordering $\langle \mathbb{Q}, \supset \rangle$ is σ-centred.

Proof of Claim 1. For the elements $I = (a, b)$ and $J = (c, d)$ of $\mathcal{B}$ we write $I \prec J$, if $b < c$. We will say that a finite sequence of rational open rectangles $\langle I_i \times J_i : i < r \rangle$ is increasing if $i < j < r$ implies $I_i \prec I_j$ and $J_i \prec J_j$. Clearly, the set $\mathcal{R}$ of all finite increasing sequences of rational open rectangles is countable. On the other hand, the set Φ of all finite sequences of elements of the set $\bigcup_{I,J\in\mathcal{B}}\bigcup_{\alpha<\theta}\bigcup_{k\in\omega}\{f_{I,J,\alpha,k}, g^{-1}_{I,J,\alpha,k}\}$ is countable. Firstly we prove that $\mathbb{Q}$ is the union of the sets

$$\mathbb{Q}^{\langle\varphi_i:i<r\rangle}_{\langle I_i\times J_i:i<r\rangle} = \Big\{p\in\mathbb{Q} : p\subset\bigcup_{i<r}(I_i\times J_i)\cap\varphi_i\Big\}, \tag{1}$$

where $r\in\omega$, $\langle I_i\times J_i : i<r\rangle\in\mathcal{R}$ and $\langle\varphi_i : i<r\rangle\in\Phi$. (Clearly, some of these sets are equal to $\{\emptyset\}$.) Let $p=\{\langle x_i,y_i\rangle : i<r\}\in\mathbb{Q}$, where $x_0<x_1<\cdots<x_{r-1}$ (which implies $y_0<y_1<\cdots<y_{r-1}$). Then it is easy to find an increasing sequence of rectangles $\langle I_i\times J_i : i<r\rangle\in\mathcal{R}$ such that $\langle x_i,y_i\rangle\in I_i\times J_i$, for $i<r$. According to the definition of $\mathbb{Q}$, for each $i<r$ there are $I'_i, J'_i\in\mathcal{B}$, $\alpha'_i<\theta$, $k'_i\in\omega$ and $\varphi_i\in\{f_{I'_i,J'_i,\alpha'_i,k'_i}, g^{-1}_{I'_i,J'_i,\alpha'_i,k'_i}\}$ such that $\langle x_i,y_i\rangle\in\varphi_i$. Now $p\subset\bigcup_{i<r}(I_i\times J_i)\cap\varphi_i$ and consequently $p\in\mathbb{Q}^{\langle\varphi_i:i<r\rangle}_{\langle I_i\times J_i:i<r\rangle}$.

Since there is countably many sets defined by (1) it remains to be proved they are centred. The functions $g_{I,J,\alpha,k}$ are increasing, hence the functions $g^{-1}_{I,J,\alpha,k}$ are increasing too, so $\bigcup_{i<r}(I_i\times J_i)\cap\varphi_i$ is an increasing function. Consequently, the union of a finite subset $\{p_k : k<n\}$ of $\mathbb{Q}^{\langle\varphi_i:i<r\rangle}_{\langle I_i\times J_i:i<r\rangle}$ is a finite increasing function, hence belongs to $\mathbb{Q}$ and clearly extends each p_k. Claim 1 is proved. □

Claim 2. The sets $D_a=\{p\in\mathbb{Q} : a\in\mathrm{dom}(p)\}$, $a\in A$, and the sets $D_b=\{p\in\mathbb{Q} : b\in\mathrm{ran}(p)\}$, $b\in B$, are dense subsets of $\mathbb{Q}$.

Proof of Claim 2. Let $a\in A_\alpha$ for some $\alpha<\theta$ and let $q=\{\langle a_i,b_i\rangle : i<r\}\in\mathbb{Q}\setminus D_a$, that is $a\notin\mathrm{dom}(q)$. W.l.o.g. we suppose $a_0<a_1<\cdots<a_{r-1}$. If $a_i<a<a_{i+1}$ for some $i<r-1$, let $I,J\in\mathcal{B}$ where $a\in I\subset(a_i,a_{i+1})$ and $J\subset(b_i,b_{i+1})$. Then $a\in\mathrm{dom}(f_{I,J,\alpha})$, so $\langle a, f_{I,J,\alpha}(a)\rangle\in I\times J$ and, since $p=q\cup\{\langle a,f_{I,J,\alpha}(a)\rangle\}$ is an increasing function, we have $p\in\mathbb{Q}$. Clearly $p\in D_a$ and $p\leq q$. If $a<a_0$ or $a>a_{r-1}$, we proceed similarly. For the sets D_b the proof is analogous. Claim 2 is proved.

Since the partial order $\mathbb{Q}$ is σ-centred, it is ccc and consequently proper, so, by the PFA there exists a filter $G\subset\mathbb{Q}$ intersecting all the sets mentioned in Claim 2. Thus, for $f=\bigcup G\subset A\times B$ we have $\mathrm{dom}(f)=A$ and $\mathrm{ran}(f)=B$. Clearly f is an increasing function thus it is an order isomorphism from A onto B. The fact that $f[A_\alpha]=B_\alpha$, for each $\alpha<\theta$ follows from the construction. □

2. The proof of Theorem 1

Let the PFA hold and let A_1, A_2 and B_1, B_2 be pairs of disjoint $\aleph_1$-dense subsets of $\mathbb{R}$. By Theorem 3, if $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, then there exists an order isomorphism $f : A \to B$ such that $f[A_i] = B_i$, for each $i \in \{1, 2\}$. Now, by Fact 1, there exists an order isomorphism $F : \mathbb{R} \to \mathbb{R}$ extending f. Clearly, if $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$ and $b_2 \in B_2$ where $a_1 < a_2$ and if $F(a_1) = b_1$ and $F(a_2) = b_2$, then

$$F\big[[a_1, a_2]\big] = [b_1, b_2] \quad \text{and} \quad F^{-1}\big[[b_1, b_2]\big] = [a_1, a_2].$$

By the first equality the mapping $F : \langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle \to \langle \mathbb{R}, \mathcal{O}_{B_1 B_2} \rangle$ is open, and by the second it is continuous. So, F is a homeomorphism.

Remark 1. Theorem 3 (and Theorem 1 as its consequence) can be proved without the PFA. Namely, in [3] a ccc partial ordering which adds an isomorphism between two $\aleph_1$ dense sets A and B generically is constructed under the CH. This construction can be modified for the case when $A = \bigcup_{n\in\omega} A_n$ and $B = \bigcup_{n\in\omega} B_n$, where A_n and B_n are $\aleph_1$-dense subsets of $\mathbb{R}$. Then, as in [2], an iteration of length $\aleph_2$ gives a model in which the conclusion of Theorem 3 holds.

Also we note that Theorem 3 is proved under $\mathrm{MA}_{\aleph_1}$ for σ-centred posets + the consequence of the PFA given in Theorem 2.

Acknowledgement. The authors would like to express their gratitude to the referees for constructive suggestions which improved the contents of the paper.

Research supported by the MNTRS (Project 1768: Forcing, Model Theory and Set-theoretic Topology).

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(Received August 27, 2002; revised May 26, 2003)