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A note on spheres in a Euclidean space

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Abstract. For an orientable compact and connected positively curved hypersurface in the Euclidean space R^{n+1}, $n \geq 2$, with scalar curvature S, shape operator A and mean curvature α, it is shown that the inequality

$$\|A\|^2 S \geq \frac{1}{2}\|R\|^2 + \|Q\|^2 + 2n(n-1)\|\nabla\alpha\|^2$$

implies that the hypersurface is a sphere, where $\nabla\alpha$ is the gradient of α, and $\|R\|$, $\|Q\|$ are the lengths of the curvature tensor field R, the Ricci operator Q of the hypersurface respectively.

1. Introduction

The class of positively curved compact hypersurfaces in the Euclidean space R^{n+1} is quite large and therefore it is an interesting question in Geometry to obtain conditions which characterize the spheres in this class. We denote by R and Ric the curvature tensor field and the Ricci curvature tensor field of the hypersurface M of the Euclidean space R^{n+1}. The Ricci operator Q is defined as $\operatorname{Ric}(X,Y) = g(QX,Y)$, $X, Y \in \mathfrak{X}(M)$, where g is the induced metric and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. For a local orthonormal frame $\{e_1, ., e_n\}$ on M the lengths $\|R\|$ and $\|Q\|$ are respectively given by $\|R\|^2 = \sum_{ijk} \|R(e_i, e_j)e_k\|^2$,

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$\|Q\|^2 = \sum_{ij} g(Qe_i, e_j)^2$. Let A be the shape operator of the hypersurface, α its mean curvature and S be its scalar curvature. An interesting question in Geometry is using the invariants α, S, $\|A\|$, $\|R\|$, $\|Q\|$ of the hypersurface, how to characterize the spheres in R^{n+1}? For instance, a sphere $S^n(c)$ in R^{n+1}, satisfies the equality

$$\|A\|^2 S = \frac{1}{2}\|R\|^2 + \|Q\|^2 + 2n(n-1)\|\nabla\alpha\|^2$$

$\nabla\alpha$ being the gradient of the mean curvature α. This raises a question, does a compact hypersurface satisfying above equality necessarily a sphere? In this paper we show that the answer is in affirmative for positively curved hypersurfaces, and indeed we prove the following:

Theorem. *Let M be an orientable compact and connected positively curved hypersurface of the Euclidean space R^{n+1}, $n \geq 2$. If the scalar curvature S, the shape operator A, the mean curvature α, the curvature tensor field R and the Ricci operator Q of M satisfy*

$$\|A\|^2 S \geq \frac{1}{2}\|R\|^2 + \|Q\|^2 + 2n(n-1)\|\nabla\alpha\|^2,$$

then α is a constant and $M = S^n(\alpha^2)$.

2. Preliminaries

Let M be an orientable hypersurface of the Euclidean space R^{n+1}. We denote the induced metric on M by g. Let $\overline{\nabla}$ be the Euclidean connection and ∇ be the Riemannian connection on M with respect to the induced metric g. Let N be the unit normal vector field and A be the shape operator. Then the Gauss and Weingarten formulas for the hypersurface are

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX, \qquad X, Y \in \mathfrak{X}(M) \quad (2.1)$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. We also have the following Gauss and Codazzi equations

$$R(X,Y)Z = g(AY,Z)AX - g(AX,Z)AY \quad (2.2)$$

$$(\nabla A)(X,Y) = (\nabla A)(Y,X), \qquad X,Y,Z \in \mathfrak{X}(M) \tag{2.3}$$

where R is the curvature tensor field of the hypersurface and $(\nabla A)(X,Y) = \nabla_X AY - A\nabla_X Y$. The mean curvature α of the hypersurface is given by $n\alpha = \sum_i g(Ae_i, e_i)$, where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M. If $A = \lambda I$ holds for a constant λ, then the hypersurface is said to be totally umbilical. The square of the length of the shape operator A is given by

$$\|A\|^2 = \sum_{ij} g(Ae_i, e_j)^2 = \mathrm{tr}.A^2.$$

From equation (2.2) we get the following expression for the Ricci tensor field

$$\mathrm{Ric}(X,Y) = n\alpha g(AX,Y) - g(AX,AY). \tag{2.4}$$

The scalar curvature S of the hypersurface is given by

$$S = n^2\alpha^2 - \|A\|^2. \tag{2.5}$$

The Ricci operator Q is the symmetric operator $Q : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $\mathrm{Ric}(X,Y) = g(QX,Y)$, $X,Y \in \mathfrak{X}(M)$. Then from equation (2.4) we get

$$Q = n\alpha A - A^2. \tag{2.6}$$

3. Some lemmas

Let M be a hypersurface of R^{n+1} and $\nabla\alpha$ be the gradient of the mean curvature function α. Then we have

Lemma 3.1. *Let M be an orientable hypersurface of R^{n+1} and $\{e_1, \dots, e_n\}$ be a local orthonormal frame on the hypersurface M. Then*

$$\sum_i (\nabla A)(e_i, e_i) = n\nabla\alpha.$$

The proof is straightforward and follows from the symmetry of A and the equation (2.3).

Lemma 3.2. *Let M be an orientable hypersurface of R^{n+1}. Then the length $\|R\|$ of the curvature tensor field of the hypersurface M is given by*

$$\frac{1}{2}\|R\|^2 = \|A\|^4 - \left\|A^2\right\|^2.$$

The proof follows immediately from equation (2.2).

Lemma 3.3. *Let M be an orientable hypersurface of R^{n+1}. Then the length $\|Q\|$ of the Ricci operator Q of the hypersurface M is given by*

$$\|Q\|^2 = n^2\alpha^2\|A\|^2 + \left\|A^2\right\|^2 - 2n\alpha(\mathrm{tr}.A^3)$$

The proof follows immediately from the equation (2.6).

Lemma 3.4. *Let M be an orientable hypersurface of R^{n+1}, $n \geq 2$. Then*

$$\|\nabla A\|^2 \geq n\|\nabla\alpha\|^2,$$

where $\|\nabla A\|^2 = \sum_{ij}\|(\nabla A)(e_i, e_j)\|^2$ for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, moreover for a positively curved M if the equality holds then M is totally umbilical.

Proof. Define an operator $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $B = A - \alpha I$. Then we have

$$(\nabla B)(X, Y) = (\nabla A)(X, Y) - (X\alpha)Y,$$

which gives

$$\begin{aligned}
\|\nabla B\|^2 &= \|\nabla A\|^2 + n\|\nabla\alpha\|^2 - 2\sum_{ij} g\left((\nabla A)(e_i, e_j), e_j\right) g(\nabla\alpha, e_i) \\
&= \|\nabla A\|^2 + n\|\nabla\alpha\|^2 - 2\sum_{j} g\left(\nabla\alpha, (\nabla A)(e_j, e_j)\right) \\
&= \|\nabla A\|^2 - n\|\nabla\alpha\|^2.
\end{aligned}$$

This proves that $\|\nabla A\|^2 \geq n\|\nabla\alpha\|^2$. The equality holds if and only if $\nabla B = 0$. If M is positively curved, then in this case we shall have $B = \lambda I$ for some constant λ (as M is irreducible being positively curved). However $B = A - \alpha I$ gives that $\mathrm{tr}.B = 0$. Hence $\lambda = 0$ and consequently $B = 0$ that is $A = \alpha I$ and thus M is totally umbilical ($\alpha =$ a constant, follows from equation (2.3) and $\dim M \geq 2$). □

Lemma 3.5. *Let M be an orientable compact hypersurface of the Euclidean space R^{n+1}. Then*

$$\int_M \left(\sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) \right) dV = -n \int_M \|\nabla\alpha\|^2 \, dV$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

Proof. Choosing a point wise covariant constant local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, we compute

$$\begin{aligned} \operatorname{div}(A(\nabla\alpha)) &= \sum_i e_i g(\nabla\alpha, Ae_i) \\ &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + \sum_i g(\nabla\alpha, (\nabla A)(e_i, e_i)) \\ &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + n \|\nabla\alpha\|^2 . \end{aligned}$$

Integrating this equation we get the lemma. □

We define the second covariant derivative $(\nabla^2 A)(X, Y, Z)$ as

$$(\nabla^2 A)(X, Y, Z) = \nabla_X (\nabla A)(Y, Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

then using the Ricci identity we get

$$(\nabla^2 A)(X, Y, Z) - (\nabla^2 A)(Y, X, Z) = R(X, Y)AZ - AR(X, Y)Z. \quad (3.1)$$

4. Proof of the theorem

Let M be an orientable compact and connected hypersurface of the Euclidean space R^{n+1}. Define a function $f : M \to R$ by $f = \frac{1}{2} \|A\|^2$. Then by a straightforward computation we get the Laplacian Δf of the smooth function f as

$$\Delta f = \|\nabla A\|^2 + \sum_{ij} g\left((\nabla^2 A)(e_j, e_j, e_i), Ae_i\right) \quad (4.1)$$

where $\{e_1, \ldots, e_n\}$ is local orthonormal frame on M.

Using the equation (2.3), we arrive at

$$g\big(\left(\nabla^2 A\right)(e_j, e_j, e_i), Ae_i\big) = g\big(\left(\nabla^2 A\right)(e_j, e_i, e_j), Ae_i\big).$$

Now using the Ricci identity (3.1) in above equation we get

$$\begin{aligned} g\left(\left(\nabla^2 A\right)(e_j, e_j, e_i), Ae_i\right) &= g\left(\left(\nabla^2 A\right)(e_i, e_j, e_j), Ae_i\right) \\ &\quad + g\left(R(e_j, e_i)Ae_j, Ae_i\right) - g(R(e_j, e_i)e_j, A^2 e_i\right). \end{aligned}$$

Thus in light of this equation the equation (4.1) takes the form

$$\begin{aligned} \Delta f = \|\nabla A\|^2 &+ \sum_{ij} g\left(\left(\nabla^2 A\right)(e_i, e_j, e_j), Ae_i\right) \\ &+ \sum_{ij}\left[g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2 e_i)\right]. \end{aligned} \tag{4.2}$$

Using Lemma 3.1, we get

$$\sum_j \left(\nabla^2 A\right)(e_i, e_j, e_j) = n\nabla_{e_i}(\nabla\alpha). \tag{4.3}$$

Now we use equations (2.2) and (2.4) to compute

$$\begin{aligned} &\sum_{ij}\left[g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2 e_i)\right] \\ &\quad = \left\|A^2\right\|^2 - \|A\|^4 - \left[\left\|A^2\right\|^2 - n\alpha(\mathrm{tr}.A^3)\right] = n\alpha(\mathrm{tr}.A^3) - \|A\|^4. \end{aligned}$$

Using this last equation together with (4.3) in (4.2), we arrive at

$$\Delta f = \|\nabla A\|^2 + n\sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + n\alpha(\mathrm{tr}.A^3) - \|A\|^4.$$

Integrating this equation and using Lemmas 3.2, 3.3 and 3.5 and equation (2.5) we arrive at

$$\begin{aligned} \int_M \Bigg\{ &\left[\|\nabla A\|^2 - n\|\nabla\alpha\|^2\right] + \frac{1}{2}\|A\|^2 S \\ &- \frac{1}{2}\left(\|Q\|^2 + \frac{1}{2}\|R\|^2 + 2n(n+1)\|\nabla\alpha\|^2\right)\Bigg\} dV = 0. \end{aligned}$$

The condition in the statement of the theorem together with Lemma 3.4 and above equation yields

$$\|\nabla A\|^2 = n\,\|\nabla \alpha\|^2 .$$

Since M is positively curved, the above equality again in view of Lemma 3.4 gives that M is totally umbilical hypersurface of R^{n+1} and thus the theorem is proved.

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