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On the composite Pexider equation modulo a subgroup

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Abstract. Let X, Y, Z be arbitrary nonempty sets, E be a subgroup of the group of all bijections of Z (with composition of functions as the group operation), and K be a nonempty set with a binary operation defined on $D(K) \subset K^2$. Conditions are established under which functions F, G, H mapping K into Z^X, Y^X, Z^Y, resp., and satisfying the generalized composite Pexider equation $F(st) = p(s,t) \circ H(s) \circ G(t)$, $(s,t) \in D(K)$, for some function $p : D(K) \to E$, can be represented in terms of solutions of the corresponding generalized Cauchy equation.

1. Introduction

Let K be a nonempty set endowed with a binary operation i.e. a mapping of a subset $D(K)$ of $K \times K$ into K (for convenience we will not use any symbol for the operation). Following [10] we say that K is a *groupoid*. Let X, Y, Z be arbitrary nonempty sets and E be a subgroup of the group of all bijections of Z with composition of functions as the group operation. To simplify our presentation we adopt the following terminology (see [4]).

We say that functions F, G, H mapping K into Z^X, Y^X, Z^Y, resp., (as usual, Z^X stands for the set of all functions mapping X into Z) satisfy the composite Pexider equation

$$F(st) = H(s) \circ G(t), \qquad (s,t) \in D(K) \tag{PE}$$

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modulo E, if there exists a function $p : D(K) \ni (s,t) \mapsto p(s,t) \in E$ such that a generalized composite Pexider equation holds, that is,

$$F(st) = p(s,t) \circ H(s) \circ G(t), \qquad (s,t) \in D(K). \tag{GPE}$$

Analogously, we say that $T : K \to Z^Z$ satisfies the composite Cauchy equation

$$T(st) = T(s) \circ T(t), \qquad (s,t) \in D(K), \tag{CE}$$

modulo E, if

$$T(st) = c(st) \circ T(s) \circ T(t), \qquad (s,t) \in D(K), \tag{GCE}$$

for some function $c : D(K) \to E$.

Note that the equation (PE) ((CE), resp.) can be seen as the generalized composite Pexider (Cauchy, resp.) equation modulo the trivial group $E = \{\mathrm{id}_Z\}$ (id_Z means the identity function on Z). Further, observe that if $F(t)$, $G(t)$, $H(t)$ and $T(t)$ are invertible functions for $t \in K$ then (GPE) and (GCE) can be rewritten as follows

$$F(st) \circ G(t)^{-1} \circ H(s)^{-1} \in E, \qquad (s,t) \in D(K),$$

$$T(st) \circ T(t)^{-1} \circ T(s)^{-1} \in E, \qquad (s,t) \in D(K).$$

Thus, (GPE) and (GCE) correspond to the well-known notion of the Pexider and Cauchy difference (see e.g. [6], [7], [8]), respectively. Actually, our notion of the Pexider and Cauchy difference generalizes the classical one introduced in [6] for functions F, G, H, T taking values in a group, because any group can be considered as a group of transformations of a set (see Remark 3).

The equation (GPE) on a groupoid with unity has been treated in [2], [3] and [4]. It has been shown in [2] that functions satisfying (PE) can be represented in terms of a function satisfying (CE). In [3], (GPE) has been reduced to (CE) under the additional assumption that function p depends only on the product st. The case of an arbitrary function p has been considered in [4], where (GPE) has been reduced to (GCE). In [5] the assumption that the groupoid possess unity has been weakened. Namely, it has been shown that if function p depends only on the product st then (GPE) can be reduced to (GCE), supposing basically that each member of the groupoid has left and right unities. The general case, that is without any additional assumptions on function p, is considered in the present paper which, among others, generalizes some of the results presented in [4] and [5].

2. Preliminaries

In the sequel if we say nothing about the symbols X, Y, Z then we mean that they stand for arbitrary nonempty sets. As usual, $\mathbb{N}$, $\mathbb{R}$ stand for the sets of all positive integers, real numbers, respectively and $:=$ means that the right hand side defines the left hand side. If $f \in Y^X$, $g \in Z^Y$ then the composite $g \circ f \in Z^X$ is the function defined by $g \circ f(x) := g(f(x))$ for each $x \in X$. $\operatorname{Ran} f (\operatorname{Dom} f$, resp.) means the range (the domain, resp.) of the function f.

By $\operatorname{In}(X, Y)$ $(\operatorname{Sur}(X, Y), \operatorname{Bij}(X, Y)$, resp.) we denote the set of all injections (surjections, bijections, resp.) of a set X into (onto, resp.) a set Y. For simplicity of notation, we write $\operatorname{In} X$, $\operatorname{Sur} X$, $\operatorname{Bij} X$ in the case when $X = Y$.

Let K be a groupoid. If $(s, t) \in D(K)$ then we say that the product st is *defined*. An element $e \in K$ will be called a *unity* if for every $t \in K$ the products te and et are defined and $te = et = t$. For $t \in K$ denote by $K_l(t)$ ($K_r(t)$, resp.) the set of all *left* (*right*, resp.) *unities* of t, that is, the set of all elements $e \in K$ such that et (te, resp.) is defined and $et = t$ ($te = e$, resp.). Further, an element $e_l \in K$ such that for every $t \in K$ the product $e_l t$ is defined and $e_l t = t$ will be called a *left unity*.

Let K be a groupoid such that for every $t \in K$ the sets $K_l(t)$ and $K_r(t)$ are nonempty. Fix $t \in K$ and denote by $D_r(t)$ ($D_l(t)$, resp.) the set of all elements $s \in K$ such that the product ts (st, resp.) is defined. Further, let $L_0(t) := tD_r(t)$,

$$L_1(t) := \bigcup_{i_1} \{t_{i_1}\}, \quad \text{where} \quad (t_{i_1} D_r(t_{i_1})) \cap L_0(t) \neq \emptyset$$

and for $k \in \mathbb{N}, \quad k \geq 2$

$$L_k(t) := \bigcup_{i_k} \{t_{i_k}\}, \quad \text{where} \quad (t_{i_k} D_r(t_{i_k})) \cap \Big(\bigcup_{s_i \in L_{k-1}(t)} s_i D_r(s_i) \Big) \neq \emptyset.$$

Finally, we set

$$L(t) := \bigcup_{k=1}^{\infty} L_k(t).$$

Analogously, starting with $R_0(t) := D_l(t)t$ we define

$$R(t) := \bigcup_{k=1}^{\infty} R_k(t).$$

It was shown in [5] (Section 4) that the sets

$$\mathcal{L} := \{L(t),\ t \in K\} \quad \text{and} \quad \mathcal{R} := \{R(t),\ t \in K\}$$

are partitions of K. Throughout the paper $\mu : K \to \mathcal{L}$, $\nu : K \to \mathcal{R}$ stand for the canonical surjections i.e. $\mu(t) = L(t)$ and $\nu(t) = R(t)$ for $t \in K$.

Suppose now, that functions M and N mapping the groupoid K into a nonempty set V satisfy the equations

$$M(st) = M(s), \qquad (s,t) \in D(K), \tag{1}$$

$$N(st) = N(t), \qquad (s,t) \in D(K). \tag{2}$$

The following result from [5] (Proposition 3) will be useful in the sequel.

Lemma 1. *The general solution of ((1) (2), resp.) among functions mapping K into V, is given by $M = m \circ \mu$ ($N = n \circ \nu$, resp.), where $m : \mathcal{L} \to V$ ($n : \mathcal{R} \to V$, resp.) is an arbitrary function.*

Assume the following two hypotheses:

H(l,r): K is a groupoid such that the sets $K_l(t)$, $K_r(t)$ are nonempty for every $t \in K$ and there exist functions $l : K \ni t \mapsto l(t) =: l_t \in K_l(t)$, $r : K \ni t \mapsto r(t) =: r_t \in K_r(t)$ such that

$$l_{st} = l_s, \qquad (s,t) \in D(K), \tag{3}$$

$$r_{st} = r_t, \qquad (s,t) \in D(K). \tag{4}$$

H(r): the same as hypothesis H(l,r) except that function l is not required to fulfill (3).

Before we proceed to our first result, it is worth pointing out that there exist natural and simple examples of structures satisfying the above hypotheses. For instance, one can check that a groupoid with unity (see Corollary 3), a small category (defined in the sense used e.g. in [19] or [20]), aproppriately chosen subsets of $\mathbb{R}^2$ with one of the binary operations "min" or "max" (defined in the usual way), an orthogonality space (see e.g. [9] or [14] for the definition), satisfy H(l,r). For details we refer to [5] (Examples 1, 2 and 3). The case of orthogonality spaces will be considered in the last section of this paper.

3. Main results

Let us begin with the following proposition.

Proposition 1. *Let* H(r) *be valid and E be a subgroup of the group* $(\mathrm{Bij}\, Z, \circ)$. *Suppose that functions F, G, H mapping K into Z^X, Y^X, Z^Y, resp., satisfy* (PE) *modulo E and $H(l_t) \in \mathrm{In}(Y,Z)$, $G(r_t) \in \mathrm{Sur},(X,Y)$ for $t \in K$. Then there exist functions $u, v : K \to E$, $c : D(K) \to E$ and $T : K \ni t \mapsto T(t) \in [\mathrm{Ran}\, H(l_t)]^{\mathrm{Ran}\, H(l_t)}$ such that*

$$T(st) = c(s,t) \circ T(s) \circ H(l_s) \circ H(l_t)^{-1} \circ T(t) \circ H(l_t) \circ H(l_{st})^{-1}, \qquad (s,t) \in D(K) \tag{5}$$

and

$$\begin{cases} F(t) = u(t) \circ T(t) \circ H(l_t) \circ G(r_t), \\ G(t) = H(l_t)^{-1} \circ T(t) \circ H(l_t) \circ G(r_t), \\ H(t) = v(t) \circ T(t) \circ H(l_t), \qquad t \in K. \end{cases} \tag{6}$$

Proof. Assume that functions $F : K \to Z^X$, $G : K \to Y^X$, $H : K \to Z^Y$ such that $H(l_t) \in \mathrm{In}(Y,Z)$, $G(r_t) \in \mathrm{Sur}(X,Y)$ for $t \in K$ satisfy (PE) modulo E i.e. (GPE) holds for some function $p : D(K) \to E$. Setting alternately $s = l_t$ and $t = r_s$ into (GPE), we get respectively

$$F(t) = u(t) \circ H(l_t) \circ G(t), \qquad t \in K, \tag{7}$$

$$F(s) = d(s) \circ H(s) \circ G(r_s), \qquad s \in K, \tag{8}$$

where

$$u(t) := p(l_t, t), \quad d(t) := p(t, r_t) \quad \text{for } t \in K. \tag{9}$$

Note that from (7) we have

$$\operatorname{Ran}(u(t)^{-1} \circ F(t)) \subset \operatorname{Dom} H(l_t)^{-1} \quad \text{for } t \in K. \tag{10}$$

Hence

$$(u(t) \circ H(l_t))^{-1} \circ F(t) = H(l_t)^{-1} \circ (u(t)^{-1} \circ F(t)), \qquad t \in K. \tag{11}$$

Fix arbitrary $t \in K$ and introduce on the set X an equivalence relation "ρ_t" putting

$$x \, \rho_t \, y \quad \text{if and only if} \quad G(r_t)(x) = G(r_t)(y).$$

Denote by X/ρ_t the set of all equivalence classes under ρ_t. Let $g_t : X/\rho_t \to X$ be a mapping such that $g_t([x]_t) \in [x]_t$, where $[x]_t$ stands for the equivalence class containing x. Define

$$\widetilde{G}_r(t) := G(r_t) \circ g_t \quad \text{and} \quad \widetilde{F}(t) := F(t) \circ g_t, \quad t \in K.$$

It is evident that for every $t \in K$ the mapping $\widetilde{G}_r(t)$ is a bijection onto Y.

On account of (8), we have

$$\widetilde{F}(t) = d(t) \circ H(t) \circ \widetilde{G}_r(t), \qquad t \in K.$$

Hence

$$H(t) = d(t)^{-1} \circ \widetilde{F}(t) \circ \widetilde{G}_r(t)^{-1}, \qquad t \in K. \tag{12}$$

By (7) and (11), we get

$$G(t) = H(l_t)^{-1} \circ (u(t)^{-1} \circ F(t)), \qquad t \in K. \tag{13}$$

Substituting (8) into (13), we obtain

$$G(t) = H(l_t)^{-1} \circ (u(t)^{-1} \circ d(t) \circ H(t) \circ G(r_t)), \qquad t \in K. \tag{14}$$

Next, putting (12) and (13) into (GPE), we have for $(s, t) \in D(K)$

$$F(st) = p(s,t) \circ d(s)^{-1} \circ \widetilde{F}(s) \circ \widetilde{G}_r(s)^{-1} \circ H(l_t)^{-1} \circ (u(t)^{-1} \circ F(t)). \tag{15}$$

By (GPE), we can write

$$F(st) = F((st)r_{st}) = p(st, r_{st}) \circ H(st) \circ G(r_{st}), \quad (s,t) \in D(K). \tag{16}$$

Hence, by (4), we get

$$F(st) = p(st, r_{st}) \circ H(st) \circ G(r_t), \qquad (s,t) \in D(K). \tag{17}$$

Formulae (16) and (17) imply respectively

$$\widetilde{F}(st) = d(st) \circ H(st) \circ \widetilde{G}_r(st), \qquad (s,t) \in D(K) \tag{18}$$

and

$$F(st) \circ g_t = d(st) \circ H(st) \circ \widetilde{G}_r(t), \qquad (s,t) \in D(K). \tag{19}$$

Calculating $H(st)$ from (18) and (19), and comparing the obtained formulae, we get

$$\widetilde{F}(st) \circ \widetilde{G}_r(st)^{-1} = F(st) \circ g_t \circ \widetilde{G}_r(t)^{-1}, \qquad (s,t) \in D(K), \tag{20}$$

since $d(st)$ is a bijection for all $(s,t) \in D(K)$.

Define

$$T(t) := u(t)^{-1} \circ \widetilde{F}(t) \circ \widetilde{G}_r(t)^{-1} \circ H(l_t)^{-1}, \quad t \in K, \tag{21}$$

$$v(t) := d(t)^{-1} \circ u(t), \quad t \in K, \tag{22}$$

$$c(s,t) := u(st)^{-1} \circ p(s,t) \circ v(s), \quad (s,t) \in D(K). \tag{23}$$

In view of (9) and (22) functions u, v map K into E and consequently, by (23), c maps $D(K)$ into E. Further, note that, by (10) and (21), each $T(t)$ maps $\operatorname{Ran} H(l_t)$ into itself.

Using (21), (20), (15), (22), (23) and (21) again, in this order, we can write for $(s,t) \in D(K)$

$$\begin{aligned}
T(st) &= u(st)^{-1} \circ \widetilde{F}(st) \circ \widetilde{G}_r(st)^{-1} \circ H(l_{st})^{-1} \\
&= u(st)^{-1} \circ F(st) \circ g_t \circ \widetilde{G}_r(t)^{-1} \circ H(l_{st})^{-1} \\
&= u(st)^{-1} \circ p(s,t) \circ d(s)^{-1} \circ \widetilde{F}(s) \circ \widetilde{G}_r(s)^{-1} \circ H(l_t)^{-1} \\
&\quad \circ (u(t)^{-1} \circ F(t)) \circ g_t \circ \widetilde{G}_r(t)^{-1} \circ H(l_{st})^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= u(st)^{-1} \circ p(s,t) \circ v(s) \circ u(s)^{-1} \circ \widetilde{F}(s) \circ \widetilde{G}_r(s)^{-1} \circ H(l_t)^{-1} \\
&\quad \circ (u(t)^{-1} \circ F(t)) \circ g_t \circ \widetilde{G}_r(t)^{-1} \circ H(l_{st})^{-1} \\
&= c(s,t) \circ T(s) \circ H(l_s) \circ H(l_t)^{-1} \circ T(t) \circ H(l_t) \circ H(l_{st})^{-1}.
\end{aligned}
$$

That is, (5) holds.

Taking into account (21) and (22), from (12) we get

$$
H(t) = v(t) \circ T(t) \circ H(l_t), \qquad t \in K. \tag{24}
$$

Substituting (24) into (14), we obtain

$$
G(t) = H(l_t)^{-1} \circ (u(t)^{-1} \circ d(t) \circ v(t) \circ T(t) \circ H(l_t) \circ G(r_t)), \qquad t \in K.
$$

Hence, by (22),

$$
G(t) = H(l_t)^{-1} \circ T(t) \circ H(l_t) \circ G(r_t), \qquad t \in K.
$$

Consequently, by (7), we get

$$
F(t) = u(t) \circ T(t) \circ H(l_t) \circ G(r_t), \qquad t \in K.
$$

Thus (6) holds and the proof is complete. □

Remark 1. It is easily seen from the above proof (using (23), (22), (9) in (5) and (9), (22) in (6), resp.) that the following formulae hold

$$
\begin{aligned}
T(st) = p(l_s, st)^{-1} \circ p(s,t) \circ p(s, r_s)^{-1} \circ p(l_s, s) \circ T(s) \circ H(l_s)& \\
\circ H(l_t)^{-1} \circ T(t) \circ H(l_t) \circ H(l_{st})^{-1}, \quad (s,t) \in D(K),&
\end{aligned} \tag{25}
$$

where $T(t) \in \operatorname{Ran} H(l_t)^{\operatorname{Ran} H(l_t)}$, $t \in K$, and

$$
\begin{cases}
F(t) = p(l_t, t) \circ T(t) \circ H(l_t) \circ G(r_t), \\
G(t) = H(l_t)^{-1} \circ T(t) \circ H(l_t) \circ G(r_t), \\
H(t) = p(t, r_t)^{-1} \circ p(l_t, t) \circ T(t) \circ H(l_t), \quad t \in K,
\end{cases} \tag{26}
$$

Moreover, observe that the assumptions imposed on function p can be weakened. In fact, assuming that (GPE) holds for a function $p : D(K) \ni (s,t) \mapsto p(s,t) \in Z^{\operatorname{Ran}(H(s)\circ G(t))}$ such that $p(t, r_t) \in \operatorname{In}(\operatorname{Ran} H(t), Z)$ and $p(l_t, t) \in \operatorname{In}(\operatorname{Ran}(H(l_t) \circ G(t)), Z)$ for $t \in K$, one can easily repeat the above proof and obtain the representations (25) and (26).

Corollary 1 (see [5], Theorem 1)**.** *Let* H(l,r) *be valid and functions* F, G, H *mapping* K *into* Z^X, Y^X, Z^Y, *resp., satisfy*

$$F(st) = k(st) \circ H(s) \circ G(t), \qquad (s,t) \in D(K), \tag{27}$$

where $k : K \to \operatorname{Bij} Z$. *If* $H(l_t) \in \operatorname{In}(Y,Z)$, $G(r_t) \in \operatorname{Sur}(X,Y)$ *for* $t \in K$ *then there exists a function* $W : K \to Y^Y$ *satisfying* (CE) *and functions* $m : \mathcal{L} \to \operatorname{In}(Y,Z)$, $n : \mathcal{R} \to \operatorname{Sur}(X,Y)$ *such that*

$$\begin{cases} F(t) = k(t) \circ (m \circ \mu)(t) \circ W(t) \circ (n \circ \nu)(t), \\ G(t) = W(t) \circ (n \circ \nu)(t), \\ H(t) = (m \circ \mu)(t) \circ W(t), \quad t \in K. \end{cases} \tag{28}$$

Conversely, if $k : K \to Z^Z$, $m : \mathcal{L} \to Z^Y$, $n : \mathcal{R} \to Y^X$ *are arbitrary functions and* $W : K \to Y^Y$ *satisfies* (CE) *then the functions* F, G, H *given by* (28) *satisfy* (27).

Proof. Assume that the functions F, G, H mapping K into Z^X, Y^X, Z^Y, resp., satisfy (27) and $H(l_t) \in \operatorname{In}(Y,Z)$, $G(r_t) \in \operatorname{Sur}(X,Y)$. Applying Proposition 1 for $E = (\operatorname{Bij} Z, \circ)$ and $p(s,t) = k(st)$, $(s,t) \in D(K)$, we get $T : K \ni t \mapsto T(t) \in [\operatorname{Ran} H(l_t)]^{\operatorname{Ran} H(l_t)}$ such that (25) and (26) hold. Since $p(l_s, st) = p(s,t) = k(st)$ for $(s,t) \in D(K)$ and $p(s, r_s) = p(l_s, s) = k(s)$ for $s \in K$, equation (25) takes the form

$$T(st) = T(s) \circ H(l_s) \circ H(l_t)^{-1} \circ T(t) \circ H(l_t) \circ H(l_{st})^{-1}, \quad (s,t) \in D(K).$$

Further, the injectivity of $H(l_{st})$ for $(s,t) \in D(K)$ allows to rewrite the above equation in the following equivalent form

$$\begin{aligned} H(l_{st})^{-1} \circ T(st) \circ H(l_{st}) = H(l_{st})^{-1} \circ H(l_s) \circ H(l_s)^{-1} \circ T(s) \circ H(l_s) \\ \circ H(l_t)^{-1} \circ T(t) \circ H(l_t), \quad (s,t) \in D(K). \end{aligned}$$

Hence, function $W : K \to Y^Y$ defined by $W(t) := H(l_t)^{-1} \circ T(t) \circ H(l_t)$, $t \in K$, satisfies (CE) since, by (3), $H(l_{st}) = H(l_s)$ for $(s,t) \in D(K)$. Now, (26) can be rewritten as

$$\begin{cases} F(t) = k(t) \circ M(t) \circ W(t) \circ N(t), \\ G(t) = W(t) \circ N(t), \\ H(t) = M(t) \circ W(t), \quad t \in K, \end{cases}$$

where

$$M(t) := H(l_t) \quad \text{and} \quad N(t) := G(r_t) \quad \text{for } t \in K. \tag{29}$$

It is clear that functions M, N map K into $\operatorname{In}(Y,Z)$, $\operatorname{Sur}(X,Y)$, respectively. Furthermore, observe that by (3) and (4), M and N satisfy (1) and (2), respectively. Thus, by Lemma 1, we have the following decompositions:

$$M = m \circ \mu \quad \text{and} \quad N = n \circ \nu, \tag{30}$$

where $\mu : K \to \mathcal{L}$, $\nu : K \to \mathcal{R}$ are the canonical surjections and $m : \mathcal{L} \to \operatorname{In}(Y,Z)$, $n : \mathcal{R} \to \operatorname{Sur}(X,Y)$. The converse is easy and the proof is complete. □

Theorem 1. *Let* H(l,r) *be valid, group E and functions F, G, H satisfying* (PE) *modulo E be defined as in Proposition 1. Assume that $H(l_t) \in \operatorname{In}(Y,Z)$, $G(r_t) \in \operatorname{Sur}(X,Y)$ for $t \in K$ and*

$$H(l_s) = H(l_t) \quad \textit{for} \quad (s,t) \in D(K). \tag{31}$$

Then there exist functions $u, v : K \to E$, $m : \mathcal{L} \to \operatorname{In}(Y,Z)$, $n : \mathcal{R} \to \operatorname{Sur}(X,Y)$ and $T : K \ni t \mapsto T(t) \in [\operatorname{Ran}(m \circ \mu)(t)]^{\operatorname{Ran}(m\circ\mu)(t)}$ such that T satisfies (CE) *modulo E,*

$$(m \circ \mu)(t) = (m \circ \mu)(s) \quad \textit{for} \quad (s,t) \in D(K) \tag{32}$$

and

$$\begin{cases} F(t) = u(t) \circ T(t) \circ (m \circ \mu)(t) \circ (n \circ \nu)(t), \\ G(t) = [(m \circ \mu)(t)]^{-1} \circ T(t) \circ (m \circ \mu)(t) \circ (n \circ \nu)(t), \\ H(t) = v(t) \circ T(t) \circ (m \circ \mu)(t), \qquad t \in K. \end{cases} \tag{33}$$

Conversely, if $u, v : K \to E$, $n : \mathcal{R} \to Y^X$ are arbitrary functions, $m : \mathcal{L} \to \operatorname{In}(Y,Z)$ satisfies (32) *and $T : K \ni t \mapsto T(t) \in [\operatorname{Ran}(m \circ \mu)(t)]^{\operatorname{Ran}(m\circ\mu)(t)}$ satisfies* (CE) *modulo E then functions F, G, H given by* (33) *satisfy* (PE) *modulo E.*

Proof. Assume that (GPE) holds for some $p : D(K) \to E$ and functions F, G, H mapping K into Z^X, Y^X, Z^Y, resp., satisfying (31) and such that $H(l_t) \in \operatorname{In}(Y,Z)$, $G(r_t) \in \operatorname{Sur}(X,Y)$ for $t \in K$. By Proposition 1, there exist $u, v : K \to E$, $c : D(K) \to E$ and $T \ni t \mapsto T(t) \in$

$[\operatorname{Ran} H(l_t)]^{\operatorname{Ran} H(l_t)}$ satisfying (5) such that (6) holds. Combining (3) and (31) we get

$$H(l_{st}) = H(l_t) \qquad (s,t) \in D(K). \tag{34}$$

Using (3) and (34) in (5), we obtain

$$T(st) = c(s,t) \circ T(s) \circ T(t), \qquad (s,t) \in D(K),$$

that is, T satisfies (CE) modulo E. Further, define functions M and N by (29). Then M, N map K into $\operatorname{In}(Y,Z)$, $\operatorname{Sur}(X,Y)$, respectively. Furthermore, (3) and (4) imply that M and N satisfy (1) and (2), respectively. Thus, by Lemma 1, we get the decompositions (30). Now this with (6) yields (33). Finally, by the definition of M and (31) it is easily seen that (32) holds.

To prove the converse observe that, by Lemma 1, functions $m \circ \mu$ and $n \circ \nu$ satisfy equations (1) and (2), resp., i.e.

$$(m \circ \mu)(st) = (m \circ \mu)(s), \quad (s,t) \in D(K), \tag{35}$$

$$(n \circ \nu)(st) = (n \circ \nu)(t), \quad (s,t) \in D(K). \tag{36}$$

By (32), we get from (35)

$$(m \circ \mu)(st) = (m \circ \mu)(t), \quad (s,t) \in D(K). \tag{37}$$

Now, assuming that $T : K \ni t \mapsto T(t) \in [\operatorname{Ran}(m\circ\mu)(t)]^{\operatorname{Ran}(m\circ\mu)(t)}$ satisfies (GCE) for some function $c : D(K) \to E$ and using (32), (36) and (37) it is easy to verify that functions F, G, H given by (33) satisfy (GPE) with $p(s,t) := u(st) \circ c(s,t) \circ v(s)^{-1}$ for $(s,t) \in D(K)$. This ends the proof of the theorem. $\square$

From the above proof one can easily derive the following corollary.

Corollary 2. *Under the assumptions of Theorem 1 functions F, G, H fulfill* (26) *with $T : K \ni t \mapsto T(t) \in \operatorname{Ran} H(l_t)^{\operatorname{Ran} H(l_t)}$ satisfying* (CE) *modulo E.*

Corollary 3 (see [4], Theorem)**.** *Let K be a groupoid with unity $e \in K$ and E be a subgroup of the group* $(\operatorname{Bij} Z, \circ)$*. Assume that functions F, G, H mapping K into Z^X, Y^X, Z^Y, resp., satisfy* (PE) *modulo E,*

$H(e) \in \mathrm{In}(Y,Z)$ *and* $G(e) \in \mathrm{Sur}(X,Y)$. *Then there exist functions* $a \in \mathrm{In}(Y,Z)$, $b \in Y^X$, $u,v : K \to E$, $T : K \ni t \to T(t) \in \operatorname{Ran} a^{\operatorname{Ran} a}$ *such that* T *satisfies* (CE) *modulo* E *and*

$$\begin{cases} F(t) = u(t) \circ T(t) \circ b, \\ G(t) = a^{-1} \circ T(t) \circ b, \\ H(t) = v(t) \circ T(t) \circ a, \qquad t \in K. \end{cases} \tag{38}$$

Conversely, if $a \in \mathrm{In}(Y,Z)$, $b \in Y^X$, $u,v : K \to E$ *and* $T : K \to \operatorname{Ran} a^{\operatorname{Ran} a}$ *satisfies* (CE) *modulo* E *then functions* F, G, H *given by* (38) *satisfy* (PE) *modulo* E.

Proof. Assume that the functions F, G, H mapping K into Z^X, Y^X, Z^Y, resp., such that $H(e) \in \mathrm{In}(Y,Z)$ and $G(e) \in \mathrm{Sur}(X,Y)$ satisfy (PE) modulo E. Define functions $l : K \ni t \mapsto l_t \in K_l(t)$ and $r : K \ni t \mapsto r_t \in K_r(t)$ setting $l_t = r_t = e$. It is obvious that l, r satisfy equations (3), (4), respectively. Therefore, hypothesis H(l,r) holds true. Moreover, note that condition (31) is trivially fulfilled. Thus, by Theorem 1, there exist functions $u,v : K \to E$, $m : \mathcal{L} \to \mathrm{In}(Y,Z)$, $n : \mathcal{R} \to \mathrm{Sur}(X,Y)$, $T : K \ni t \mapsto T(t) \in [\operatorname{Ran}(m \circ \mu)(t)]^{\operatorname{Ran}(m\circ\mu)(t)}$ such that T satisfies (CE) modulo E and (32), (33) hold. Further, observe that we have $card\,\mathcal{L} = card\,\mathcal{R} = 1$ ($card\,A$ stands for the cardinality of the set A), since $L(e) = R(e) = K$. Consequently, functions m and n are constant. Putting $a := [(m \circ \mu)(t)]$ and $b := (m \circ \mu)(t) \circ (n \circ \nu)(t)$ for $t \in K$, we obtain from (33) formulae (38), as claimed.

An easy computation shows that the converse holds true. □

Taking into account the above proof and Corollary 2 one can easily obtain the subsequent result which we will use in the last section of the paper.

Corollary 4. *Under the assumptions of Corollary 3, there exists function* $T : K \to \operatorname{Ran} H(e)^{\operatorname{Ran} H(e)}$ *satisfying* (CE) *modulo* E *and such that*

$$\begin{cases} F(t) = p(e,t) \circ T(t) \circ H(e) \circ G(e), \\ G(t) = H(e)^{-1} \circ T(t) \circ H(e) \circ G(e), \\ H(t) = p(t,e)^{-1} \circ p(e,t) \circ T(t) \circ H(e), \quad t \in K. \end{cases} \tag{39}$$

Now, we are able to solve (GPE) on a groupoid with a left unity.

Corollary 5. *Let K be a groupoid with left unity $e_l \in K$ such that $K_r(t) \neq \emptyset$ for $t \in K$ and there exists a function $r : K \ni t \mapsto r_t \in K_r(t)$ satisfying* (4)*. Let E, F, G and H be defined as in Corollary 3. If the functions F, G, H satisfy* (PE) *modulo E and $H(e_l) \in \mathrm{In}(Y,Z)$, $G(r_t) \in \mathrm{Sur}(X,Y)$ for $t \in K$ then there exist a, u, v, T as in Corollary 3 and $n : \mathcal{R} \to Z^X$ such that*

$$\begin{cases} F(t) = u(t) \circ T(t) \circ (n \circ \nu)(t), \\ G(t) = a^{-1} \circ T(t) \circ (n \circ \nu)(t), \\ H(t) = v(t) \circ T(t) \circ a, \quad t \in K. \end{cases} \tag{40}$$

Conversely, if $n : \mathcal{R} \to Z^X$, $u, v : K \to E$ are arbitrary functions, $a \in \mathrm{In}(Y,Z)$ and $T : K \to \mathrm{Ran}\, a^{\mathrm{Ran}\, a}$ satisfies (CE) *modulo E then the functions F, G, H given by* (40) *satisfy* (PE) *modulo E.*

Proof. Assume that functions F, G, H mapping K into Z^X, Y^X, Z^Y, resp., such that $H(e_l) \in \mathrm{In}(Y,Z)$, $G(r_t) \in \mathrm{Sur}(X,Y)$ satisfy (GPE) with a function $p : D(K) \to E$. Define function $l : K \ni t \mapsto l_t \in K_l(t)$ putting $l_t = e_l$. It is seen at once that function l satisfies (3). Moreover, (31) is trivially fulfilled. Thus, by Corollary 2, there exists a function $T : K \to \mathrm{Ran}\, H(e_l)^{\mathrm{Ran}\, H(e_l)}$ satisfying (CE) modulo E and such that

$$\begin{cases} F(t) = p(e_l,t) \circ T(t) \circ H(e_l) \circ G(r_t), \\ G(t) = H(e_l)^{-1} \circ T(t) \circ H(e_l) \circ G(r_t), \\ H(t) = p(t,r_t)^{-1} \circ p(e_l,t) \circ T(t) \circ H(e_l), \quad t \in K. \end{cases} \tag{41}$$

Put $a := H(e_l) \circ G(r_t)$ for $t \in K$ and $u(t) := p(e_l,t)$, $v(t) := p(t,r_t)^{-1} \circ p(e_l,t)$, $N(t) := H(e_l) \circ G(r_t)$ for $t \in K$. It is clear that function N satisfies (2). Hence, by Lemma 1, we get $N = n \circ \nu$, where $\nu : K \to \mathcal{R}$ is the canonical surjection and $n : \mathcal{R} \to Z^X$. Now, formulae (41) yield (40). A trivial verification shows that the converse holds true. □

Theorem 2. *Let* H(l,r) *be satisfied, E be a subgroup of* $(\mathrm{Bij}\, X, \circ)$ *and functions F, G, H mapping K into X^X satisfy* (PE) *modulo E. Assume that $H(l_t) \in \mathrm{In}\, X$, $G(r_t) \in \mathrm{Sur}\, X$ for $t \in K$ and*

$$H(l_s) \circ E = E \circ H(l_s), \quad s \in K \tag{42}$$

(i.e. for every $f \in E$ there is $g \in E$ such that $H(l_s) \circ f = g \circ H(l_s)$, $s \in K$). Then there exist functions $u, v : K \to E$, $m : \mathcal{L} \to \operatorname{In} X$, $n : \mathcal{R} \to \operatorname{Sur} X$ and $T : K \to X^X$ satisfying (CE) *modulo E such that*

$$(m \circ \mu)(s) \circ E = E \circ (m \circ \mu)(s), \quad s \in K \tag{43}$$

and

$$\begin{cases} F(t) = u(t) \circ (m \circ \mu)(t) \circ T(t) \circ (n \circ \nu)(t), \\ G(t) = T(t) \circ (n \circ \nu)(t), \\ H(t) = v(t) \circ (m \circ \mu)(t) \circ T(t), \quad t \in K. \end{cases} \tag{44}$$

Conversely, if $u, v : K \to E$, $n : \mathcal{R} \to X^X$ are arbitrary functions, $m : \mathcal{L} \to X^X$ satisfies (43) *and $T : K \to X^X$ satisfies* (CE) *modulo E then functions F, G, H given by* (44) *satisfy* (PE) *modulo E.*

Proof. Assume that functions $F, G, H : K \to X^X$ such that $H(l_t) \in \operatorname{In} X$, $G(r_t) \in \operatorname{Sur} X$ and (42) holds satisfy (GPE), with a function $p : D(K) \to E$. By Proposition 1 (for $X = Y = Z$), there exist $u, v : K \to E$ and $W : K \ni t \mapsto W(t) \in [\operatorname{Ran} H(l_t)]^{\operatorname{Ran} H(l_t)}$ such that

$$\begin{gathered} W(st) = c(s,t) \circ W(s) \circ H(l_s) \circ H(l_t)^{-1} \circ W(t) \circ H(l_t) \circ H(l_{st})^{-1}, \\ (s,t) \in D(K) \end{gathered} \tag{45}$$

and

$$\begin{cases} F(t) = u(t) \circ W(t) \circ H(l_t) \circ G(r_t), \\ G(t) = H(l_t)^{-1} \circ W(t) \circ H(l_t) \circ G(r_t), \\ H(t) = v(t) \circ W(t) \circ H(l_t), \quad t \in K. \end{cases} \tag{46}$$

Since $H(l_s) \in \operatorname{In} X$ for $s \in K$, (45) is equivalent to

$$\begin{aligned} H(l_{st})^{-1} \circ W(st) \circ H(l_{st}) = H(l_{st})^{-1} \circ c(s,t) \circ W(s) \circ H(l_s) \\ \circ H(l_t)^{-1} \circ W(t) \circ H(l_t) \end{aligned} \tag{47}$$

for $(s,t) \in D(K)$. Put $T(t) := H(l_t) \circ W(t) \circ H(l_t)$ for $t \in K$. Then $T : K \to X^X$ and from (47), by (3), we obtain

$$T(st) = H(l_s)^{-1} \circ c(s,t) \circ H(l_s) \circ T(s) \circ T(t), \quad (s,t) \in D(K). \tag{48}$$

Hence, by (42) it is easily seen that T satisfies (CE) modulo E. Define functions M, N by (29). Then M, N map K into $\operatorname{In} X$, $\operatorname{Sur} X$, respectively and (3), (4) imply that M, N satisfy (1) and (2), respectively. Thus, by Lemma 1, we get the decompositions (30), where μ, ν are the canonical surjections and $m : \mathcal{L} \to \operatorname{In} X$, $n : \mathcal{R} \to \operatorname{Sur} X$. It is clear that (43) holds. Now, it is immediate that (46) implies (44).

An easy verification shows that the converse holds true. $\square$

Remark 2. Observe that if E is a normal subgroup of the group $(\operatorname{Bij} X, \circ)$ and $H(l_t) \in \operatorname{Bij} X$ for $t \in K$, then condition (42) is trivially fulfilled.

Theorem 3. *Let* H(r) *be valid, E be a subgroup of* $(\operatorname{Bij} X, \circ)$ *and functions F, G, H mapping K into X^X satisfy* (PE) *modulo E. Suppose that $G(r_t) \in \operatorname{Sur} X$ for $t \in K$ and*

$$H(l_s) \in E, \quad s \in K. \tag{49}$$

Then there exist functions $r, q : K \to E$, $n : \mathcal{R} \to \operatorname{Sur} X$ and $T : K \to X^X$ such that T satisfies (CE) *modulo E and*

$$\begin{cases} F(t) = r(t) \circ T(t) \circ (n \circ \nu)(t), \\ G(t) = T(t) \circ (n \circ \nu)(t), \\ H(t) = q(t) \circ T(t), \quad t \in K. \end{cases} \tag{50}$$

Conversely, if $r, q : K \to E$, $n : \mathcal{R} \to X^X$ are arbitrary functions and $T : K \to X^X$ satisfies (CE) *modulo E then functions F, G, H given by* (50) *satisfy* (PE) *modulo E.*

Proof. By Proposition 1 there exist $u, v : K \to E$ and $W : K \to X^X$ such that (45) and (46) hold. In the same way as in the proof of Theorem 2 we obtain $T : K \to X^X$ satisfying (48), which means, by (49), that T satisfies (CE) modulo E. Further, (46) takes the form

$$\begin{cases} F(t) = u(t) \circ H(l_t) \circ T(t) \circ N(t), \\ G(t) = T(t) \circ N(t), \\ H(t) = v(t) \circ H(l_t) \circ T(t), \quad t \in K, \end{cases} \tag{51}$$

where $N(t) := G(r_t)$, $t \in K$. Define $r, q : K \to E$ setting $r(t) := u(t) \circ H(l_t)$ and $q(t) := v(t) \circ H(l_t)$. Since function N satisfies (2), by Lemma 1, we get $N = n \circ \nu$, where ν is the canonical surjection and $n : \mathcal{R} \to \operatorname{Sur} X$. Now, from (51), we easily get (50) as claimed.

The converse is easy to check. □

As a consequence of Theorems 1 and 2 we have the following result concerning the classical notion of the Pexider difference.

Theorem 4. *Let* H(l,r) *be satisfied,* $(P, +)$ *be a group (not necessarily abelian) and* E *be a subgroup of* P. *A triple* (F, G, H) *of functions mapping* K *into* P *such that* (31) *((*42) *(in additive notation), resp.) holds satisfies the condition*

$$F(st) - G(t) - H(s) \in E \quad \text{for} \quad (s, t) \in D(K) \tag{52}$$

if and only if there exist functions $u, v : K \to E$, $m : \mathcal{L} \to P$, $n : \mathcal{R} \to P$, $T : K \to P$ *such that* (32) *((*43) *(in additive notation), resp.) holds,*

$$T(st) - T(t) - T(s) \in E, \qquad (s, t) \in D(K) \tag{53}$$

and (33) *((*44)*, resp.), switched to additive notation, holds.*

Proof. By the well-known Cayley's Theorem (see e.g. [12]) any group is isomorphic to a group of bijections with composition of functions as the group operation. Thus, without loss of generality we can assume that P is a subgroup of the group $(\operatorname{Bij} P, \circ)$. Now, condition (52) means that $F(st) = p(s,t) \circ H(s) \circ G(t)$ for $(s, t) \in D(K)$ and a function $p : D(K) \to E$, i.e. functions F, G, H satisfy (PE) modulo E. Thus, by Theorem 1, if (31) holds, or by Theorem 2, in the case when (42) is satisfied, we get the statement. □

By analysis similar to that in the proof of Corollary 3, from Theorem 4 one may derive easily the subsequent corollary.

Corollary 6 (see [4], Corollary 2)**.** *Let* K *be a groupoid with unity,* $(P, +)$ *be a group and* E *be a subgroup of* P. *A triple* (F, G, H) *of functions mapping* K *into* P *satisfies* (52) *if and only if there exist functions* $u, v : K \to E$, $T : K \to P$ *and constants* $a, b \in P$ *such that* T *satisfies* (53) *and* (38) *(switched to additive notation) holds.*

Remark 3. It is easily seen that Theorem 9 from [1] (Section 4.3), Theorem 2 from [6] and Theorem 1 from [16] (Chapter 13, §3) are particular cases of Corollary 6.

Analogously as Theorem 4, combining the Cayley's Theorem and our Theorem 3, we get the following

Theorem 5. *Let* H(r) *be satisfied,* $(P,+)$ *be a group and* E *be a subgroup of* P. *A triple* (F,G,H) *of functions mapping* K *into* P *such that* (49) *holds satisfies* (52) *if and only if there exist functions* $r,q : K \to E$, $n : \mathcal{R} \to P$, *and* $T : K \to P$ *satisfying* (53) *such that* (50) *(switched to additive notation) holds.*

4. Applications

In this section we present some results which can be derived from Proposition 1, Corollaries 4 and 6. The first proposition is a generalization of Corollary 3 from [4] and gives a connection between stability in the Hyers–Ulam sense (see e.g. [13] or [15]) of the translation equation (see e.g. [17]) and stability of the pexidered form of the equation. The proposition gains in interest if we realize that up to now, very few results are known about the stability of the translation equation itself.

Proposition 2. *Let hypothesis* H(l,r) *be fulfilled,* $(V,\|\cdot\|)$ *be a normed space and let* $\varepsilon > 0$. *Suppose that a triple* (F,G,H) *of functions mapping* K *into* V^V *satisfies for all* $x,y \in V$, *and* $(s,t) \in D(K)$ *the following two conditions:*

$$\|F(st)(x) - H(s) \circ G(t)(x)\| < \varepsilon, \tag{54}$$

$$H(s) \circ G(t)(x) = H(s) \circ G(t)(y) \quad \text{iff} \quad F(st)(x) = F(st)(y). \tag{55}$$

If $H(l_t) \in \operatorname{In} V$, $G(r_t) \in \operatorname{Sur} V$, $t \in K$, *and* (31) *holds then there exists a function* $T : K \ni t \mapsto T(t) \in \operatorname{Ran} H(l_t)^{\operatorname{Ran} H(l_t)}$ *such that*

$$\|T(st)(x) - T(s) \circ T(t)(x)\| < 4\varepsilon, \quad (s,t) \in D(K),\ x \in \operatorname{Ran} H(l_t) \tag{56}$$

and

$$\begin{cases} \|F(t)(x) - T(t) \circ H(l_t) \circ G(r_t)(x)\| < \varepsilon, \\ G(t)(x) = H(l_t)^{-1} \circ T(t) \circ H(l_t) \circ G(r_t)(x), \\ \|H(t)(x) - T(t) \circ H(l_t)(x)\| < 2\varepsilon, \quad t \in K, \quad x \in V. \end{cases} \tag{57}$$

Proof. First observe that, by (3) and (31), we have

$$H(l_{st}) = H(l_t), \qquad (s,t) \in D(K) \tag{58}$$

and consequently $\operatorname{Ran} H(l_{st}) = \operatorname{Ran} H(l_t)$ for $(s,t) \in D(K)$. Thus, (56) makes sense. The rest of the proof is similar to the proof of Corollary 3 from [4]. However, for the sake of completeness we present it.

In view of (54) and (55) there exists a function $p : D(K) \ni (s,t) \mapsto p(s,t) \in \operatorname{In}(\operatorname{Ran}(H(s) \circ G(t)), V)$ such that for all $(s,t) \in D(K)$ and $x \in V$, we have

$$F(st)(x) = p(s,t) \circ H(s) \circ G(t)(x) = H(s) \circ G(t)(x) + \varepsilon_{(s,t,x)}, \tag{59}$$

where $\|\varepsilon_{(s,t,x)}\| < \varepsilon$ for $(s,t) \in D(K)$ and $x \in V$. Hence, by (55), we infer that if $H(s) \circ G(t)(x) = H(s) \circ G(t)(y)$ for some $(s,t) \in D(K)$ and $x, y \in V$, then $\varepsilon_{(s,t,x)} = \varepsilon_{(s,t,y)}$. Consequently, by (59), for every $y \in \operatorname{Dom} p(s,t) = \operatorname{Ran}(H(s) \circ G(t))$, we can write $p(s,t)(y) = y + \varepsilon_{(s,t,x)}$, where x is such that $H(s) \circ G(t)(x) = y$. In view of Proposition 1 (see also Remark 1), we get the formulae (25) and (26), where $T : K \ni t \mapsto [\operatorname{Ran} H(l_t)]^{\operatorname{Ran} H(l_t)}$. Using (58) and (31) in (25), we get

$$T(st) = p(l_s, st)^{-1} \circ p(s,t) \circ p(s, r_s)^{-1} \circ p(l_s, s) \circ T(s) \circ T(t), \quad (s,t) \in D(K).$$

Now, it is easily seen that (56) and (57) hold. □

Following [9] (see also e.g. [14]) we say that a pair $(K, \perp)$ is an *orthogonality space*, provided K is a real linear space with $\dim K \geq 2$ and $\perp \subset K^2$ is a relation such that

(1) $s \perp 0$ and $0 \perp s$ for every $s \in K$;

(2) if $s, t \in K$ and $s \perp t$, then s and t are linearly independent;

(3) if $s, t \in K$ and $s \perp t$, then $as \perp bt$ for every $a, b \in \mathbb{R}$;

(4) if W is a 2-dimensional subspace of K, $s \in W$ and $t \in \mathbb{R}$, $a > 0$, then there is $t \in W$ with $s \perp t$ and $s + t \perp as - t$.

Function f mapping a vector space K (an orthogonality space K, resp.) into a group $(P,+)$ is called *additive* (*orthogonally additive,* resp.) if $f(s+t) = f(s)+f(t)$ for all $s,t \in K$ (for $s \perp t$, resp.). Further, recall that function $g : K \to P$ is said to be *quadratic* if $g(s+t)+g(s-t) = 2g(s)+2g(t)$ for $s,t \in K$ (see e.g. [9] or [18]). We will keep the same terminology if f, g take their values in a group of functions with composition as the group operation.

Let us also recall two results concerning the Cauchy difference, which we will need in the sequel.

Lemma 2 (see [8], Lemma 3)**.** *Let K be a real topological vector space, $(P,+)$ be a topological group (possibly non-abelian) and E be a normal discrete subgroup of P. Let $T : K \to P$ be a continuous at the origin function satisfying $T(s+t) - T(s) - T(t) \in E$ for $s,t \in K$. Then, there exists a continuous additive function $A : K \to P$ such that $T(s) - A(s) \in E$ for $s \in K$.*

Lemma 3 (see [9], Theorem 2.9)**.** *Let K be an orthogonality space endowed with a linear topology (i.e. one which makes K into a real linear topological space), $(P,+)$ be a commutative topological group without elements of order 2, and E be a discrete subgroup of P. Then a continuous at the origin function $T : K \to P$ satisfies*

$$T(s+t) - T(s) - T(t) \in E \quad \textit{whenever} \quad s \perp t$$

if and only if there exist a unique continuous additive function $A : K \to P$ and a unique continuous at the origin quadratic and orthogonally additive function $Q : K \to P$ with $T(s) - A(s) - Q(s) \in E$ for $s \in K$.

Now, we are in a position to prove the following

Theorem 6. *Let K be a real topological vector space, $(P,\circ) \subset \mathrm{Bij}\, X$ be a topological group and E be a normal discrete subgroup of P. Suppose that functions F, G, H mapping K into X^X, P, X^X, resp., satisfy* (PE) *modulo E i.e. $F(s+t) = p(s,t) \circ H(s) \circ G(t)$, $(s,t) \in D(K)$, for some function $p : K^2 \to E$, $H(0) \in P$ and the map G is continuous at the origin. Then there exist $a, b \in P$, $q, w, z : K \to E$ and a continuous* *additive function $A : K \to P$ such that*

$$\begin{cases} F(t) = w(t) \circ A(t) \circ b, \\ G(t) = a^{-1} \circ q(t) \circ A(t) \circ b, \\ H(t) = z(t) \circ A(t) \circ a, \quad t \in K. \end{cases} \tag{60}$$

Conversely, if $a, b \in P$, $q, w, z : K \to E$ and $A : K \to P$ is an additive function then functions F, G, H given by (60) *satisfy* (PE) *modulo E.*

Proof. By Corollary 4, there exists $T : K \to X^X$ satisfying (CE) modulo E and such that (see (39))

$$\begin{cases} F(t) = p(0,t) \circ T(t) \circ H(0) \circ G(0), \\ G(t) = H(0)^{-1} \circ T(t) \circ H(0) \circ G(0), \\ H(t) = p(t,0)^{-1} \circ p(0,t) \circ T(t) \circ H(0), \quad t \in K. \end{cases} \tag{61}$$

From the second formula in (61), we get $T(t) = H(0) \circ G(t) \circ G(0)^{-1} \circ H(0)^{-1}$ for $t \in K$. Hence $T(t) \in P$ for $t \in K$ and function $T : K \to P$ is continuous at the origin. By Lemma 2, it follows that there exists a continuous additive function $A : K \to P$ such that

$$T(s) = q(s) \circ A(s), \qquad s \in K \tag{62}$$

for some function $q : K \to E$. Set $a := H(0)$ and $b := H(0) \circ G(0)$. It is obvious that $a, b \in P$. Further, substituting (62) into (61) and putting

$$w(t) := p(0,t) \circ q(t), \quad z(t) := p(t,0)^{-1} \circ p(0,t) \circ q(t) \quad \text{for } t \in K,$$

we get formulae (60), as claimed.

Using the normality of E one can easily check the converse. □

Remark 4. If X is a topological space which is compact Hausdorff or locally compact locally connected Hausdorff space then the group $(P, \circ) \subset \operatorname{Bij} X$ of all homeomorphisms of X equipped with the compact-open topology, is a topological group (see e.g. [11], Propositions 1.22 and 1.24).

Theorem 7. *Let K be an orthogonality space endowed with a linear topology, $(P, \circ) \subset \operatorname{Bij} X$ be an abelian topological group without elements* *of order 2, and E be a discrete subgroup of P. Let functions F, G, H defined on K and taking values in X^X, P, X^X, resp., satisfy*

$$F(s+t) = p(s,t) \circ H(s) \circ G(t) \quad \text{whenever} \quad s \perp t, \tag{63}$$

for some function $p : K^2 \to E$. If $H(0) \in P$ and function G is continuous at the origin then there exist $a, b \in P$, $q, w, z : K \to E$, a continuous additive function $A : K \to P$ and a continuous at the origin quadratic orthogonally additive function $Q : K \to P$ such that

$$\begin{cases} F(t) = w(t) \circ Q(t) \circ A(t) \circ b, \\ G(t) = a^{-1} \circ q(t) \circ Q(t) \circ A(t) \circ b, \\ H(t) = z(t) \circ Q(t) \circ A(t) \circ a, \quad t \in K. \end{cases} \tag{64}$$

Conversely, if $a, b \in P$, $q, w, z : K \to E$, $A, Q : K \to P$ are orthogonally additive maps then the functions F, G, H given by (64) satisfy (63) for some function $p : K^2 \to E$.

Proof. Define on K the binary operation $*$ as follows:

$$s * t := s + t \quad \text{for} \quad s \perp t$$

($+$ means the addition on the linear space K). It is easily seen that $(K, *)$ is a groupoid with the unity 0. Now, by Corollary 4, we infer that there exists function $T : K \to X^X$ such that

$$T(s+t) = c(s,t) \circ T(s) \circ T(t) \quad \text{whenever} \quad s \perp t,$$

for some function $c : K^2 \to E$. Moreover, (61) holds which yields, in an analogous way as in the proof of Theorem 6, that $T(t) \in P$ for $t \in K$ and function $T : K \to P$ is continuous at the origin. Thus, by Lemma 3, there exist a unique continuous additive function $A : K \to P$ and a unique continuous at the origin quadratic and orthogonally additive function $Q : K \to P$ such that

$$T(s) = q(s) \circ Q(s) \circ A(s), \qquad s \in K, \tag{65}$$

where $q : K \to E$. Defining a, b, w and z as in the proof of Theorem 6, by (65), we get from (61) the required formulae (64). Since the converse is easily seen this ends the proof. $\square$

Using Corollary 6 (instead of Corollary 4) and Lemma 3, similarly as Theorem 7, one can also prove the following result concerning the Pexider difference.

Theorem 8. *Let K, P and E be the same as in Lemma 3. If functions F, G, H mapping K into P such that G is continuous at the origin satisfy*

$$F(s+t)-G(t)-H(s)\in E \quad \text{whenever} \quad s\perp t,$$

then there exist constants $a,b\in P$, functions $q,w,z:K\to E$, a continuous additive function $A:K\to P$ and a continuous at the origin quadratic and orthogonally additive function $Q:K\to P$ such that formulae (64) *(switched to additive notation) hold.*

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