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Totally real Einstein submanifolds of CP^n and the spectrum of the Jacobi operator

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Abstract. We consider n-dimensional compact totally real parallel Einstein submanifolds of the complex projective space CP^n and we use invariants determined by the spectrum of the Jacobi operator J to characterize such submanifolds.

1. Introduction

Let M be an n-dimensional compact (connected and smooth) Riemannian manifold without boundary, isometrically immersed in a Riemannian manifold $\bar{M}$. Then a second order elliptic operator J, called the *Jacobi operator*, is associated to the isometric immersion. Such operator is defined on the space of smooth sections of the normal bundle $TM^{\perp}$ by the formula

$$J = D + \tilde{R} - \tilde{A},$$

where D is the rough Laplacian of the normal connection $\nabla^{\perp}$ on $TM^{\perp}$, $\tilde{R}$ and $\tilde{A}$ are linear transformations of $TM^{\perp}$ defined by means of a partial Ricci tensor of $\bar{M}$ and of the second fundamental form A, respectively. J is also called the *second variation operator* because it appears in the formula which gives the second variation for the area function of a compact minimal

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submanifold (see [15]). Its spectrum, denoted by

$$\operatorname{spec}(M,J)=\{\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots+\uparrow\infty\}$$

is discrete, as a consequence of the compactness of M.

H. Donnelly [4] and T. Hasegawa [8], applying Gilkey's results [6] to the asymptotic expansion of the partition function $Z(t)$ associated to $\operatorname{spec}(M,J)$, found spectral invariants and studied spectral geometry for compact minimal submanifolds of the Euclidean sphere and for compact Kaehlerian submanifolds of the complex projective space CP^n. Some results about spectral geometry of Sasakian submanifolds were given in [14]. Moreover, an analogous study was made about spectral geometry determined by the Jacobi operator associated to the energy of a harmonic map in [16] and in [12] for Riemannian foliations. Recently, the inverse spectral problem of the Jacobi operator of a harmonic map has been further investigated in [2], [9], [10], [17].

Besides Kaehlerian submanifolds, another typical class of submanifolds of the complex projective space CP^n is the one of totally real minimal submanifolds. In [1], the author and D. Perrone determined the first three terms of the asymptotic expansion for the partition function associated to the spectrum of the Jacobi operator of an n-dimensional totally real submanifold of CP^n. The corresponding Riemannian spectral invariants have been used to characterize n-dimensional totally real parallel conformally flat submanifolds of CP^n.

In this paper, we use Riemannian invariants determined by $\operatorname{spec}(M,J)$ to characterize n-dimensional totally real parallel Einstein submanifolds of CP^n. In Section 2 we shall make some preliminaries about n-dimensional totally real minimal submanifolds of CP^n. Section 3 is devoted to the description of n-dimensional totally real parallel Einstein submanifolds of CP^n. In Section 4, we shall characterize such submanifolds, for a wide range of dimensions, using some spectral invariants of J.

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2. Preliminaries

Let CP^n denote the complex projective space equipped with the Fubini-Study metric $\bar{g}$ of constant holomorphic sectional curvature $c > 0$. An n-dimensional *totally real submanifold* of CP^n is a Riemannian manifold (M, g) isometrically immersed in CP^n such that IT_xM is orthogonal to T_xM for all $x \in M$, where I denotes the almost complex structure of CP^n. We shall denote by ∇ (respectively, $\bar{\nabla}$) and R (respectively, $\bar{R}$) the Levi–Civita connection and the curvature tensor of M (respectively, CP^n), taken with the sign convention

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

Note that this sign convention is the opposite from that used by Simons in [15].

The normal connection is defined by

$$\begin{aligned} \nabla^\perp : TM \times TM^\perp &\longrightarrow TM^\perp \\ (X, \xi) &\longmapsto \nabla^\perp_X \xi, \end{aligned}$$

where $\nabla^\perp_X \xi$ denotes the normal component of $\bar{\nabla}_X \xi$. The second fundamental form σ and the Weingarten operator A are respectively defined by

$$\sigma(X, Y) = \bar{\nabla}_X Y - \nabla_X Y, \qquad A_\xi X = -\bar{\nabla}_X \xi + \nabla^\perp_X \xi,$$

for all $X, Y \in TM$ and $\xi \in TM^\perp$. Moreover, $\bar{g}(\sigma(X, Y), \xi) = g(A_\xi X, Y)$ and, since M is totally real, $A_{IX} Y = A_{IY} X$, for all $X, Y \in TM$ and $\xi \in TM^\perp$ (see [3]).

Let $R^\perp$ denote the curvature tensor associated to the normal connection $\nabla^\perp$. The curvature tensors R, $\bar{R}$ and $R^\perp$ satisfy the Gauss and the Ricci equations:

$$\begin{aligned} R(X, Y, Z, W) = g(R(X, Y)Z, W) &= \bar{R}(X, Y, Z, W) \\ &+ \bar{g}(\sigma(X, Z), \sigma(Y, W)) - \bar{g}(\sigma(Y, Z), \sigma(X, W)), \end{aligned}$$

$$R^\perp(X, Y, \xi, \eta) = \bar{g}(R^\perp(X, Y)\xi, \eta) = \bar{R}(X, Y, \xi, \eta) - g([A_\xi, A_\eta]X, Y),$$

where $[A_\xi, A_\eta] = A_\xi \circ A_\eta - A_\eta \circ A_\xi$, for all $X, Y, Z, W \in TM$ and $\xi, \eta \in TM^\perp$.

Let $\{e_1, \dots, e_n, e_1^* = Ie_1, \dots, e_n^* = Ie_n\}$ be a local orthonormal frame on CP^n such that, restricted to M, the vector fields $e_1, \dots, e_n$ are tangent to M. We put $A_{i^*} = A_{e_i^*}$, $R_{ijkh} = R(e_i, e_j, e_k, e_h)$ and $R^\perp_{ijk^*h^*} = R^\perp(e_i, e_j, e_k^*, e_h^*)$. Since

$$\begin{aligned}\bar{R}(X,Y,Z,W) = \frac{c}{4}\{&\bar{g}(X,Z)\bar{g}(Y,W) - \bar{g}(Y,Z)\bar{g}(X,W)\\ &+ 2\bar{g}(X,IY)\bar{g}(Z,IW) + \bar{g}(X,IZ)\bar{g}(Y,IW)\\ &- \bar{g}(Y,IZ)\bar{g}(X,IW)\},\end{aligned}$$

the Gauss and Ricci equations become

$$\begin{aligned}R_{ijkh} = &\frac{c}{4}(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih})\\ &+ \bar{g}(\sigma(e_i,e_k),\sigma(e_j,e_h)) - \bar{g}(\sigma(e_j,e_k),\sigma(e_i,e_h)),\end{aligned} \tag{2.1}$$

and

$$R^\perp_{ijk^*h^*} = \frac{c}{4}(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih}) - g([A_{k^*}, A_{h^*}]e_i, e_j). \tag{2.2}$$

The mean curvature vector is defined by

$$H = \text{trace}(\sigma) = \sum_i \sigma(e_i, e_i) = \sum_i (\text{tr}\, A_{i^*})e_i^*.$$

M is said to be *minimal* if $H = 0$, *totally geodesic* if $\sigma = 0$, *parallel* (or *with parallel second fundamental form*) if $\nabla'\sigma = 0$, where

$$(\nabla'_X\sigma)(Y,Z) = \nabla^\perp_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

The scalar curvature of M is given by

$$\tau = n(n-1)\frac{c}{4} + \|H\|^2 - \|\sigma\|^2, \tag{2.3}$$

where $\|\sigma\|^2 = \sum \text{tr}\, A_{i^*}^2$ and $\|H\|^2 = \sum(\text{tr}\, A_{i^*})^2$.

We shall make use of the following

Lemma 2.1 ([13]). *Let M be an n-dimensional totally real submanifold of CP^n. Then*

$$\|R\|^2 = c\tau - 2n(n-1)\frac{c^2}{16} - \sum_{i,j} \text{tr}[A_{i^*}, A_{j^*}]^2. \tag{2.4}$$

If in addition M is minimal, then

$$\|\varrho\|^2 = 2(n-1)\frac{c}{4}\tau - n(n-1)^2\frac{c^2}{16} + \sum(\operatorname{tr} A_{i^*}A_{j^*})^2, \tag{2.5}$$

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \|R\|^2 - \|\varrho\|^2 + (n+1)\frac{c}{4}\tau \tag{2.6}$$

where ϱ is the Ricci tensor of M.

3. Totally real parallel Einstein submanifolds of CP^n

H. Naitoh [11] classified n-dimensional totally real parallel submanifolds of CP^n. We now recall some basic ideas of [11], in order to determine all n-dimensional totally real parallel Einstein submanifolds of CP^n.

Fix an n-dimensional simply connected Riemannian symmetric space M. By $\bar{\mathcal{T}}_M$ (respectively, $\bar{\mathcal{S}}_M$) we denote the set of equivalence classes of totally real parallel isometric immersions of M into CP^n (respectively, of complete totally real parallel submanifolds in CP^n, having M as universal covering).

Since M is symmetric, there exists a Lie group G acting isometrically and transitively on M. M is isometric to a quotient M/K and the Lie algebra $\boldsymbol{g}$ of G splits as $\boldsymbol{g} = \boldsymbol{k} + \boldsymbol{p}$, with $\boldsymbol{p}$ isometric to the tangent space T_oM at a point o of M. By $\mathcal{M}_M$ it is denoted the set of all $\boldsymbol{p}$-valued bilinear forms $\tilde{\sigma}$ on $\boldsymbol{p}$, satisfying

(1) $\tilde{\sigma}$ is a symmetric trilinear form on $\boldsymbol{p}$, under the canonical identification of $\boldsymbol{p}^* \otimes \boldsymbol{p}^* \otimes \boldsymbol{p}$ with $\boldsymbol{p}^* \otimes \boldsymbol{p}^* \otimes \boldsymbol{p}^*$ through the Riemannian metric $\langle\, , \rangle$ on $\boldsymbol{p}$,

(2) $\boldsymbol{t} \cdot \tilde{\sigma} = 0$, and

(3) $\frac{c}{4}(\langle Y, Z\rangle X - \langle X, Z\rangle Y) = R(X,Y)Z - [\tilde{\sigma}(X), \tilde{\sigma}(Y)](Z)$, for all vectors $X, Y, Z \in \boldsymbol{p}$.

When $f : M \to CP^n$ is a totally real parallel isometric immersion, then $(\tilde{\sigma}_f)_o$ belongs to $\mathcal{M}_M$, where σ_f is the second fundamental form associated to f and $\tilde{\sigma}_f$ is defined by

$$\tilde{\sigma}_f(X,Y) = J\sigma_f(X,Y),$$

J being the complex structure of CP^n. A suitable equivalence is introduced in $\mathcal{M}_M$, so that the quotient set $\bar{\mathcal{M}}_M$ of $\mathcal{M}_M$ has a natural one-to-one correspondance with $\bar{\mathcal{T}}_M$ (and so, also with $\bar{\mathcal{S}}_M$). Therefore, the problem of classifying totally real parallel submanifolds of CP^n reduces to the problem of studying $\bar{\mathcal{M}}_M$, for any given simply connected symmetric space M. We refer to [11] for more details.

The following results were proved in [11, Section 4], where M is supposed to be without Euclidean factor.

Lemma 3.1 ([11]). *Assume that $\mathcal{M}_M$ is not empty. Then the simply connected symmetric space M without Euclidean factor is irreducible and of compact type.*

Note that, as it is well-known, an irreducible symmetric Riemannian manifold is Einsteinian.

Theorem 3.2 ([11]). *Let M be a simply connected symmetric space without Euclidean factor. Then the set $\bar{\mathcal{M}}_M$ is not empty if and only if M is one of the followings:*

$$\begin{gathered} SO(n+1)/SO(n)\ (n \geq 2), \quad SU(k),\ (k \geq 3), \\ SU(k)/SO(k),\ (k \geq 3), \quad SU(2k)/Sp(k),\ (k \geq 3), \quad E_6/F_4. \end{gathered} \tag{3.1}$$

In this case, the metric on M is determined uniquely by the constant c (the holomorphic sectional curvature of CP^n) and the set $\bar{\mathcal{M}}_M$ consists of one point.

Note that $SO(n+1)/SO(n)$ is the Euclidean sphere $S^n(\beta)$, for some $\beta > 0$.

Suppose now that M is an n-dimensional totally real parallel Einstein submanifold of $CP^n(c)$.

a) If M has no Euclidean factor, then M is one of the spaces listed in Theorem 3.2. Note that all these spaces are compact and their immersions in $CP^n(c)$ are minimal [11, Remark 5.3].

We can determine explicitly the metric of the Einstein submanifolds of $CP^n(c)$ listed in (3.1). As claimed in Theorem 3.2, their metrics are determined by c. In fact, since M is parallel and compact, integrating

(2.6) we get

$$\int_M \|R\|^2 dv + \int_M \|\varrho\|^2 dv - (\dim M + 1)\frac{c}{4}\int_M \tau dv = 0,$$

from which, if $\|R\|^2$ is constant (τ and $\|\varrho\|^2 = \tau^2/\dim M$ are constant, M being an Einstein manifold), it follows

$$\|R\|^2 + \frac{1}{\dim M}\tau^2 - (\dim M + 1)\frac{c}{4}\tau = 0. \tag{3.2}$$

Curvature invariants of symmetric spaces of rank 1 and of classical symmetric spaces were calculated in [7] (some corrections were successively needed and they have been made in [5]). In particular, for the spaces listed in (3.1), we have

M	dim	τ	$\|R\|^2$
$S^n(\beta)$	n	$n(n-1)\beta$	$2n(n-1)\beta^2$
$SU(k)$	k^2-1	$4k(k^2-1)\beta$	$16k^2(k^2-1)\beta^2$
$SU(k)/SO(k)$	$\frac{1}{2}(k-1)(k+2)$	$k(k-1)(k+2)\beta$	$2k(k-1)(k+2)^2\beta^2$
$SU(2k)/Sp(k)$	$(k-1)(2k+1)$	$4k(k-1)(2k+1)\beta$	$16k(k-1)^2(2k+1)\beta^2$

Table I

The metric on M is defined up to a homothetic transformation and so, curvature invariants of M depend on $\beta > 0$. We did not report the value of $\|\varrho\|^2$ since M is an Einstein space and so, $\|\varrho\|^2 = \tau^2/\dim M$. Using (3.2), we can determine β for such spaces, in function of c. We get

M	$S^n(\beta)$	$SU(k)$	$SU(k)/SO(k)$	$SU(2k)/Sp(k)$
β	$\frac{c}{4}$	$\frac{c}{16k}$	$\frac{kc}{32}$	$\frac{kc}{16}$

Table II

The Riemannian curvature invariants τ and $\|R\|^2$ of M can now be calculated from the above Table I, using the values of β listed in Table II.

For what concerns E_6/F_4, it was noted in [11, Remark 5.4] that its immersion f in CP^n is $\frac{\sqrt{c}}{2\sqrt{2}}$-*isotropic*, that is, $\sigma_f(X.X) = \frac{\sqrt{c}}{2\sqrt{2}}$ for any unit tangent vector X of M. In particular, this implies that $\|\sigma_f\|^2 = \dim M \frac{1}{8}c = \frac{13}{4}c$ and so, by (2.3), we get $\tau = \frac{637}{4}c$. For E_6/F_4, being an Einstein space, we have $\|\varrho\|^2 = \tau^2/\dim M = \frac{31213}{32}c^2$. Finally, using (3.2) we can also compute $\|R\|^2$ and we get $\|R\|^2 = \frac{3185}{32}c^2$. In this way, we determined all n-dimensional totally real parallel Einstein submanifolds of $CP^n(c)$ without Euclidean factor, and calculated explicitly their Riemannian curvature invariants τ, $\|\varrho\|^2$ and $\|R\|^2$.

b) Suppose now that M is an n-dimensional totally real parallel Einstein submanifold of $CP^n(c)$, having a Euclidean factor. Therefore, we have

$$M = \mathbb{R}^{n_0} \times M_1^{n_1} \times \cdots \times M_r^{n_r},$$

with $n = \sum_{j=0}^{r} n_j$, $n_0 > 0$, and $M_i^{n_i}$ is an n_i-dimensional irreducible simply connected symmetric space for each i [11].

In our case, since M is an Einstein space given by a Riemannian product of Einstein spaces, we must have

$$0 = \frac{\tau_0}{n_0} = \frac{\tau_1}{n_1} = \cdots = \frac{\tau_r}{n_r},$$

that is, $\tau_i = 0$ for all i. But none of the spaces listed in a) has zero scalar curvature. Therefore, if M has a Euclidean factor, then M itself is Euclidean. In particular, if M is compact, then M is the n-dimensional flat torus, T^n.

Therefore, we proved the following

Theorem 3.3. *Let M be an n-dimensional totally real parallel Einstein submanifold of the complex projective space $CP^n(c)$. If M has no Euclidean factor, then M is one of the spaces listed in* (3.1)*, equipped with a Riemannian metric uniquely determined by c. In particular, M is compact and its immersion in $CP^n(c)$ is minimal. If M has an Euclidean factor, then M is flat (in particular, if M is compact, then $M = T^n$).*

The following Table III describes all n-dimensional compact totally real parallel Einstein submanifolds of $CP^n(c)$.

M	dim	τ	$\|R\|^2$
$S^n(\frac{c}{4})$	n	$\frac{n(n-1)}{4}c$	$\frac{n(n-1)}{8}c^2$
$SU(k)$	k^2-1	$\frac{(k^2-1)}{4}c$	$\frac{(k^2-1)^2}{16}c^2$
$SU(k)/SO(k)$	$\frac{1}{2}(k-1)(k+2)$	$\frac{k^2(k-1)(k+2)}{32}c$	$\frac{k^3(k-1)(k+2)^2}{512}c^2$
$SU(2k)/Sp(k)$	$(k-1)(2k+1)$	$\frac{k^2(k-1)(2k+1)}{4}c$	$\frac{k^3(k-1)^2(2k+1)}{16}c^2$
E_6/F_4	26	$\frac{637}{4}c$	$\frac{3185}{32}c^2$
T^n	n	0	0

Table III

It is easy to check that for two of such manifolds, having the same dimension, it never occurs that the pair of Riemannian curvature invariants $(\tau, \|R\|^2)$ attains the same value. Therefore, we proved the following

Theorem 3.4. *Each compact n-dimensional totally real parallel Einstein submanifold of $CP^n(c)$ is uniquely determined by the pair of Riemannian curvature invariants $(\tau, \|R\|^2)$.*

4. Spectral geometry of J and totally real Einstein submanifolds of $CP^n(c)$

Let M be an n-dimensional Riemannian manifold immersed in a Riemannian manifold $\bar{M}$ of dimension $\bar{n} = n + r$. The normal bundle $TM^\perp$ is a real r-dimensional vector bundle on M, with inner product induced by the metric $\bar{g}$ of $\bar{M}$. Let D denote the so-called *rough Laplacian* associated to the normal connection $\nabla^\perp$ of $TM^\perp$, that is,

$$D\xi = -\nabla^\perp_{e_i}\nabla^\perp_{e_i}\xi + \nabla^\perp_{\nabla_{e_i}e_i}\xi,$$

where ξ is a section of $TM^\perp$. Next, let $\tilde{A}$ be the *Simons operator* defined in [15] by

$$\bar{g}(\tilde{A}\xi, \eta) = \mathrm{tr}(A_\xi \circ A_\eta),$$

for $\xi, \eta \in TM^{\perp}$. Moreover, we consider the operator $\tilde{R}$ defined by

$$\tilde{R}(\xi) = -\sum_{i=1}^{n} (\bar{R}(e_i, \xi)e_i)^{\perp},$$

where $(\bar{R}(e_i, \xi)e_i)^{\perp}$ denotes the normal component of $\bar{R}(e_i, \xi)e_i$.

The *Jacobi operator* (or *second variation operator*), acting on cross-sections of $TM^{\perp}$, is the second order elliptic differential operator J defined by (see [15] or [4])

$$J : TM^{\perp} \longrightarrow TM^{\perp}$$

$$\xi \longmapsto (D - \tilde{A} + \tilde{R})\xi.$$

Let $f : M \to \bar{M}$ be an isometric minimal immersion. A *variation* of f is a one parameter family $\{f_t\}$ of immersions $M \to \bar{M}$, such that $f_0 = f$ and $F : M \times [0,1] \to \bar{M}$, with $F(m,t) = f_t(m)$, is C^{∞}. If $\mathcal{A}(t)$ denotes the area associated to f_t, then the Jacobi operator expresses the second variation for $\mathcal{A}$, since

$$\mathcal{A}''(0) = \int_M \langle JV, V \rangle dv$$

(see [15]). Similarly, if $\phi : (M,g) \to (N,h)$ is a harmonic map and $\{\phi_t\}$ a variation of ϕ, then the Jacobi operator J_{ϕ} expresses the second variation of the energy $\mathcal{E}(t) = \mathcal{E}(\phi_t)$ associated to ϕ, by

$$\mathcal{E}''(0) = \int_M h(V, J_{\phi}V) dv$$

(see for example [16]).

When M is compact, we can define an inner product for cross-sections on $TM^{\perp}$, by

$$\langle \xi, \eta \rangle = \int_M \bar{g}(\xi, \eta) dv$$

and J is self-adjoint with respect to this product. Moreover, J is strongly elliptic and it has an infinite sequence of eigenvalues, with finite multiplicities, denoted by

$$\mathrm{spec}(M, J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \uparrow\infty\}.$$

The partition function $Z(t) = \sum_{i=1}^{\infty} \exp(-\lambda_i t)$ has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-n/2}\{a_0(J) + a_1(J)t + a_2(J)t^2 + \dots\}.$$

By GILKEY's results [6] (see also [4] and [8]), the coefficients a_0, a_1 and a_2 are given by the following

Theorem 4.1 ([6])**.**

$$a_0 = r\,\mathrm{vol}(M),$$

$$a_1 = \frac{r}{6}\int_M \tau dv + \int_M \mathrm{tr}\,\tilde{E} dv,$$

$$a_2 = \frac{r}{360}\int_M \{2\|R\|^2 - 2\|\varrho\|^2 + 5\tau^2\} dv + \frac{1}{360}\int_M \{-30\|R^\perp\|^2 + \mathrm{tr}(60\tau\tilde{E} + 180\tilde{E}^2)\} dv,$$

where $\tilde{E} = \tilde{A} - \tilde{R}$.

In the case of an n-dimensional totally real submanifold of CP^n, the coefficients a_0, a_1 and a_2 were computed explicitly in [1], in terms of curvature invariants of M. In particular, the following result has been obtained.

Theorem 4.2 ([1])**.** *On an n-dimensional totally real minimal submanifold M of $CP^n(c)$, the first coefficients of the asymptotic expansion of the partition function of the Jacobi operator are given by*

$$a_0 = n\,\mathrm{vol}(M), \tag{4.1}$$

$$\begin{aligned} a_1 &= \frac{n-6}{6}\int_M \tau dv + 2n(n+1)\frac{c}{4}\,\mathrm{vol}(M) \\ &= \frac{6-n}{6}\int_M \|\sigma\|^2 dv + \frac{n}{6}(n^2+5n+18)\frac{c}{4}\,\mathrm{vol}(M), \end{aligned} \tag{4.2}$$

$$a_2 = \frac{1}{360}\int_M \{2(n-15)\|R\|^2 - 2(n-90)\|\varrho\|^2 + 5(n-12)\tau^2\} dv + \frac{(n+1)(n-6)}{3}\frac{c}{4}\int_M \tau dv$$

$$+ 2n(n+1)^2 \frac{c^2}{16} \operatorname{vol}(M). \tag{4.3}$$

In the sequel, we shall denote by M_0 one of the compact totally real submanifolds of $CP^n(c)$ listed in Table III. Our purpose is to characterize M_0 by its spec(J) in the class of all compact totally real minimal submanifolds of $CP^n(c)$. We first remark that, as an easy consequence of Theorem 3.4, we get the following

Theorem 4.3. *Each compact n-dimensional totally real parallel Einstein submanifold M_0 of $CP^n(c)$ is uniquely determined by its* spec(J).

Proof. We treat separately the cases $n \neq 6, 15$, $n = 6$ and $n = 15$.

a) If $n \neq 6, 15$, by Theorem 3.4, it is enough to prove that spec(J) determines the pair of Riemannian invariants $(\tau, \|R\|^2)$ of M. In fact, suppose that $\operatorname{spec}(M_0, J) = \operatorname{spec}(M_0', J)$, where M_0, M_0' are n-dimensional compact totally real Einstein submanifolds of $CP^n(c)$. Then, since $n \neq 6$, (4.1) and (4.2) imply that $\tau_0 = \tau_0'$. M_0, M_0' being Einstein manifolds having the same dimension, it follows that $\|\varrho_0\|^2 = \|\varrho_0'\|^2$. Thus, since $n \neq 15$, taking into account that $\|R_0\|^2$ and $\|R_0'\|^2$ are constant, from (4.3) we get $\|R_0\|^2 = \|R_0'\|^2$.

b) If $n = 6$, from Table III we see that $M_0 = S^6(\frac{c}{4})$ or $M_0 = T^6$. Suppose that $\operatorname{spec}(S^6(\frac{c}{4}), J) = \operatorname{spec}(T^6, J)$. Then, in particular, $a_0(S^6(\frac{c}{4})) = a_0(T^6)$ and $a_2(S^6(\frac{c}{4})) = a_2(T^6)$, from which it follows easily that c vanishes, which can not occur.

c) If $n = 15$, from Table III we see that $M_0 = S^{15}(\frac{c}{4})$, T^{15} or $SU(4)$. Suppose that $\operatorname{spec}(M_0, J) = \operatorname{spec}(M_0', J)$. Then, in particular, $a_0(M_0) = a_0(M_0')$ and $a_1(M_0) = a_1(M_0')$, from which it follows easily $\tau_0 = \tau_0'$, which can not occur, because, as it follows from Table III, for $S^{15}(\frac{c}{4})$, T^{15} and $SU(4)$ we respectively have $\tau = \frac{105}{2}c$, 0 and $\frac{15}{4}c$, with $c \neq 0$ □

We now prove the following

Theorem 4.4. *Let M be an n-dimensional compact totally real minimal submanifold of $CP^n(c)$. If $\operatorname{spec}(M, J) = \operatorname{spec}(M_0, J)$, $16 \leq \dim M_0 \leq 52$, then M is isometric to M_0.*

PROOF. Since $\operatorname{spec}(M,J)=\operatorname{spec}(M_0,J)$, we have $\dim M_0=\dim M=n$ and, from Theorem 4.2,

$$\operatorname{vol}(M,g)=\operatorname{vol}(M_0,g_0), \tag{4.4}$$

$$\int_M \tau dv=\int_{M_0}\tau_0 dv, \quad \int_M \|\sigma\|^2 dv=\int_{M_0}\|\sigma_0\|^2 dv, \tag{4.5}$$

$$\begin{aligned}&\int_M \{2(n-15)\|R\|^2+2(90-n)\|\varrho\|^2+5(n-12)\tau^2\}dv\\ &\quad=\int_{M_0}\{2(n-15)\|R_0\|^2+2(90-n)\|\varrho_0\|^2+5(n-12)\tau_0^2\}dv\end{aligned} \tag{4.6}$$

Since τ_0 is constant and vol(M)=vol(M_0), we have

$$\begin{aligned}\int_M \tau^2 dv-\int_{M_0}\tau_0^2 dv&=\int_M \tau^2 dv-2\tau_0\int_{M_0}\tau_0 dv+\int_{M_0}\tau_0^2 dv\\ &=\int_M(\tau-\tau_0)^2 dv\geq 0\end{aligned} \tag{4.7}$$

where the equality holds if and only if $\tau=\tau_0$.

Next, let $E=\varrho-\frac{\tau}{n}g$ denote the *Einstein curvature tensor* of (M,g). Since $\|E\|^2=\|\varrho\|^2-\frac{\tau^2}{n}$ and $E_0=0$ because M_0 is an Einstein space, (4.6) becomes

$$\begin{aligned}2(n-15)&\left(\int_M\|R\|^2dv-\int_{M_0}\|R_0\|^2dv\right)-2(n-90)\int_M\|E\|^2dv\\ &+\frac{5n^2-62n+180}{n}\left(\int_M\tau^2dv-\int_{M_0}\tau_0^2dv\right)=0.\end{aligned} \tag{4.8}$$

Moreover, from (2.6) we also get

$$\frac{1}{2}\Delta\|\sigma\|^2=\|\nabla'\sigma\|^2-\|R\|^2-\|E\|^2+\frac{1}{n}\tau^2+(n+1)\frac{c}{4}\tau.$$

Integrating over M, we obtain

$$\begin{aligned}\int_M\|\nabla'\sigma\|^2dv&=\int_M\|R\|^2dv+\int_M\|E\|^2dv\\ &\quad+\frac{1}{n}\int_M\tau^2dv-(n+1)\frac{c}{4}\int_M\tau dv.\end{aligned} \tag{4.9}$$

An analogous formula holds for M_0, with $\nabla'\sigma_0 = E_0 = 0$. We use (4.9) to calculate $\int_M \|R\|^2 dv$. Therefore, (4.8) becomes

$$(n-15)\int_M \|\nabla'\sigma\|^2 dv = \alpha(n)\int_M \|E\|^2 dv + \beta(n)\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right), \tag{4.10}$$

where

$$\alpha(n) = 2n - 105,$$
$$\beta(n) = -\frac{5n^2 - 64n + 210}{2n}.$$

It is easy to check that if $16 \le n \le 52$, then $n-15 > 0$ while $\alpha(n), \beta(n) < 0$. Therefore, we get $\nabla'\sigma = 0$, $E = 0$ and $\tau = \tau_0$. Thus, M is an Einstein totally real parallel submanifold of $CP^n(c)$ with the same spec(J) of M_0. So, Theorem 4.3 implies that M is isometric to M_0. □

Remark 4.1. Note that formula (4.10) holds for all n-dimensional compact totally real minimal submanifolds M of CP^n such that spec$(M, J) =$ spec(M_0, J).

In particular, if M is also Einsteinian, then (4.10) becomes

$$(n-15)\int_M \|\nabla'\sigma\|^2 dv = \beta(n)\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right). \tag{4.11}$$

Since $\beta(n) < 0$ for all $n \ge 3$, proceeding as in the proof of Theorem 4.4, we obtain the following

Theorem 4.5. *In the class of all n-dimensional compact totally real Einstein minimal submanifolds of $CP^n(c)$, the parallel ones are characterized by their* spec(J) *for all $n \ge 16$.*

Remark 4.2. If $M_0 = S^n(\frac{c}{4})$, then $\sigma_0 = 0$ and (4.5) gives at once $\sigma = 0$. Therefore:

In the class of compact totally real minimal submanifolds of $CP^n(c)$, $S^n(\frac{c}{4})$ is characterized by its spec(J) *for all $n \ne 6$.*

Remark 4.3. In [1], it was proved that in the class of all n-dimensional compact totally real minimal submanifolds of $CP^n(c)$, the parallel conformally flat ones are characterized by their spec(J) when $53 \leq n \leq 93$. Since the flat torus T^n is at the same time Einstein and conformally flat, combining this result with Theorem 4.4, we obtain the following

Theorem 4.6. *In the class of all n-dimensional compact totally real minimal submanifolds of $CP^n(c)$, the flat torus T^n is characterized by its* spec(J) *when* $16 \leq n \leq 93$.

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