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Square free part of products of consecutive integers

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Dedicated to Professor K. Ramachandra on his 70th birthday

Abstract. Defining $\Delta(n,k) = n(n+1)\dots(n+k-1)$, it is proved that, for $k \geq 10$ and $n > k^2$, there are at least 8 distinct primes exceeding k dividing $\Delta(n,k)$ to odd powers except a few explicitly given values of n and k. We also list all the squares which can be written as a product of $k-2$ distinct terms out of k consecutive positive integers.

1. Introduction

Let n and $k \geq 3$ be positive integers. For an integer $\nu > 1$, we denote by $P(\nu)$ the greatest prime factor of ν and we write $P(1) = 1$. Further we put

$$\Delta(n,k) = n(n+1)\cdots(n+k-1).$$

We write $G = G(n,k)$ for the set of all i with $0 \leq i \leq k-1$ such that $n+i$ is divisible by a prime $> k$ to odd power. Further we denote by $G' = G'(n,k)$ the set of prime divisors of $\Delta(n,k)$ exceeding k. We put $g = g(n,k) = |G|$ and $g' = g'(n,k) = |G'|$. A theorem of SYLVESTER [12]

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dating back to 1892 states that

$$g' > 0 \quad \text{if } n > k.$$

Here the assumption $n > k$ is necessary since $P(1 \times 2 \times \cdots \times k) \leq k$. Further Saradha and Shorey [10] showed that

$$g' \geq [\pi(k)/3] + 2 \quad \text{if } n > k \tag{1}$$

unless

$$\begin{cases} n \in \{4,6,7,8,16\} \quad \text{if} \ \ k = 3; \\ n \in \{6\} \quad \text{if} \ \ k = 4; \\ n \in \{6,7,8,9,12,14,15,16,23,24\} \quad \text{if} \ \ k = 5; \\ n \in \{7,8,15\} \quad \text{if} \ \ k = 6; \\ n \in \{8,9,10,12,14,15,24\} \quad \text{if} \ \ k = 7; \\ n \in \{9,14\} \quad \text{if} \ \ k = 8; \\ n \in \{14,15,16,18,20,21,24\} \quad \text{if} \ \ k = 13; \\ n \in \{15,20\} \quad \text{if} \ \ k = 14; \\ n = \{20\} \quad \text{if} \ \ k = 17. \end{cases} \tag{2}$$

We observe that

$$g' = \pi(2k) - \pi(k) \quad \text{if } n = k + 1.$$

Therefore $\frac{1}{3}$ cannot be replaced by a constant larger than 1. Shanta Laishram and Shorey [6] sharpened (1) for $k \geq 19$ to

$$g' \geq \left[\frac{3}{4}\pi(k)\right] - 1 \quad \text{if } n > k \geq 19 \tag{3}$$

unless (n,k) is given by

$$\begin{cases}
n \in \{20-22, 24\} \quad \text{if} \quad k=19; \quad n \in \{21\} \quad \text{if} \quad k=20;\\
n \in \{48-50, 54\} \quad \text{if} \quad k=47; \quad n \in \{49\} \quad \text{if} \quad k=48;\\
n \in \{74, 75\} \quad \text{if} \quad k=71; \quad n \in \{74\} \quad \text{if}\ k=72;\\
n \in \{74-76, 84\} \quad \text{if} \quad k=73;\\
n \in \{75\} \quad \text{if} \quad k=74; \quad n \in \{84\} \quad \text{if} \quad k=79;\\
n \in \{84, 90, 108, 110\} \quad \text{if} \quad k=83;\\
n \in \{90, 102, 104\} \quad \text{if} \quad k=89;\\
n \in \{108, 110, 111, 114, 115\} \quad \text{if} \quad k=103;\\
n \in \{110, 114\} \quad \text{if} \quad k=104; \quad n \in \{108-119\} \quad \text{if} \quad k=107;\\
n \in \{109-118\} \quad \text{if} \quad k=108; \quad n \in \{110-118\} \quad \text{if} \quad k=109;\\
n \in \{111-117\} \quad \text{if} \quad k=110; \quad n \in \{112-116\} \quad \text{if} \quad k=111;\\
n \in \{113-115\} \quad \text{if} \quad k=112;\\
n \in \{114-120, 138, 140, 141\} \quad \text{if} \quad k=113;\\
n \in \{115-119, 140\} \quad \text{if} \quad k=114;\\
n \in \{116-118\} \quad \text{if} \quad k=115;\\
n \in \{117\} \quad \text{if} \quad k=116; \quad n \in \{174\} \quad \text{if} \quad k=173;\\
n \in \{198, 200, 201\} \quad \text{if} \quad k=181;\\
n \in \{200\} \quad \text{if} \quad k=182; \quad n \in \{200, 201\} \quad \text{if} \quad k=193;\\
n \in \{200\} \quad \text{if} \quad k=194; \quad n \in \{200\} \quad \text{if} \quad k=197;\\
n \in \{200-202\} \quad \text{if} \quad k=199; \quad n \in \{201\} \quad \text{if} \quad k=200;\\
n \in \{282-286\} \quad \text{if} \quad k=271;\\
n \in \{282, 284, 285\} \quad \text{if} \quad k=272;\\
n \in \{284\} \quad \text{if} \quad k=273;\\
n \in \{278-280, 282-286\} \quad \text{if} \quad k=277;\\
n \in \{279, 282-285\} \quad \text{if} \quad k=278;\\
n \in \{282-284\} \quad \text{if} \quad k=279;\\
n \in \{282\} \quad \text{if} \quad k=280; \quad n \in \{282-288\} \quad \text{if} \quad k=281;\\
n \in \{283-287\} \quad \text{if} \quad k=282;\\
n \in \{284-288, 294\} \quad \text{if} \quad k=283;\\
n \in \{285-287\} \quad \text{if} \quad k=284;\\
n \in \{286\} \quad \text{if} \quad k=285; \quad n \in \{294\} \quad \text{if} \quad k=293.
\end{cases} \tag{4}$$

Thus the estimate (3) always holds for $k > 293$. Further they derived from their result that

$$g' \geq \min\left(\left[\frac{3}{4}\pi(k)\right] - 1,\ \pi(2k) - \pi(k) - 1\right) \quad \text{if} \ \ n > k. \tag{5}$$

Now we turn to giving lower bounds for g. If $k < n \leq k^2$, we see that $G = G'$ implying $g = g'$ and lower bounds for g' have already been given above. Thus we assume that $n > k^2$. Erdős and Selfridge [4], developing on the method of Erdős [2] and Rigge [7], proved that there exists a prime $p \geq k$ dividing $\Delta(n, k)$ to odd power unless $(n, k) = (48, 3)$. Further Saradha [9] sharpened the assertion $p \geq k$ to $p > k$ in the preceding result. Thus

$$g \geq 1 \quad \text{if } (n, k) \neq (48, 3).$$

Next Saradha and Shorey [10] showed that

$$g \geq 2 \quad \text{if } k \geq 4,\ (n, k) \neq (24, 4), (47, 4), (48, 4). \tag{6}$$

In fact (6) is stated in [10] for the number of distinct prime divisors $> k$ dividing $\Delta(n, k)$ to odd powers but it is clear from the proof that the assertion is valid for g. We sharpen (6) as follows.

Theorem 1. *Let $k \geq 10$ and $n > k^2$. Then*

$$g \geq 8$$

unless

$$\begin{aligned} n \in \{&103 - 105, 112, 116 - 126, 135, 138 - 144, 159 - 162, 166 - 168,\\ &187 - 189, 191, 192, 216, 234 - 245, 247 - 250, 280, 285 - 288, 315,\\ &334 - 336, 354 - 360, 375, 441, 477 - 484, 498 - 500, 503, 504,\\ &667 - 672, 717 - 722, 726, 836 - 841, 959, 960, 1080, 1343, 1344,\\ &1436 - 1440, 1443, 1444, 1673 - 1681, 2016, 2019 - 2023,\\ &2518 - 2520, 2879 - 2883, 3355 - 3360, 4796 - 4800, 5034 - 5041,\\ &6718 - 6724, 10077 - 10080, 13447, 13448, 15116 - 15123,\\ &6375621\} \quad \textit{if} \ \ k = 10; \end{aligned}$$

$$n \in \{122-126, 140, 144, 158-162, 165-168, 188-192, 215, 216, 235-243, 287, 288, 375, 440, 480, 719, 720, 837-840, 1680, 2880, 5036-5040, 6718-6720, 15119, 15120\} \quad \textit{if } k = 11;$$

$$n \in \{158-160, 165, 189, 239-242\} \quad \textit{if } \ k = 12;$$

$$n \in \{188, 189, 240\} \quad \textit{if } \ k = 13.$$

Since $x^2 - 2y^2 = -1$ has infinitely many solutions in integers x and y, it is clear that the assumption $k \geq 10$ is necessary in Theorem 1. We also observe that $g \leq 7$ for every exception stated in Theorem 1. Further we notice that the number of distinct primes $> k$ dividing $\Delta(n,k)$ to odd powers is at least g. Therefore Theorem 1 implies the following results immediately.

Corollary 1. *For $k \geq 10$ and $n > k^2$, there are at least 8 distinct primes exceeding k dividing $\Delta(n,k)$ to odd powers unless*

$$n \in \{103-105, 112, 116-126, 144, 159-162, 166-168, 188, 189, 191, 192, 234-243, 287, 288, 354-360, 482, 483, 672, 717-721, 837-841, 1444, 5039\} \quad \textit{if } \ k = 10;$$

$$n \in \{122-126, 140, 144, 158-162, 165-168, 188-192, 235, 236, 240, 242, 287, 288, 719, 720, 837-840, 1680\} \quad \textit{if } \ k = 11;$$

$$n \in \{158-160, 165, 189\} \quad \textit{if } \ k = 12;$$

$$n \in \{188, 189, 240\} \quad \textit{if } \ k = 13.$$

Corollary 2. *Let $k \geq 10$ if $n \geq 5040$ and $k \geq 14$ otherwise. Assume that $n > k^2$. Then there are at least 8 distinct primes exceeding k each dividing $\Delta(n,k)$ to odd power.*

We observe that the exceptions mentioned in Corollary 1 are necessary. Sharper lower bounds for g have been given when $n > k^2$ and k is sufficiently large. Erdős [3] showed that

$$g \geq C_1 \frac{k}{\log k}$$

where $C_1 > 0$ is an effectively computable absolute constant. This has been improved by Shorey [11] to

$$g \geq C_2 \frac{k \log\log k}{\log k}$$

where $C_2 > 0$ is an effectively computable absolute constant. The improvement depends upon a theorem of Baker [1] that a hyperelliptic equation, under necessary assumptions, has only finitely many solutions. The constants C_1, C_2 turn out to be small and therefore, the above estimates for g are of interest only when k is large. As an immediate consequence of the result of Baker referred above, we have

$$g \geq k - 2$$

whenever $n \geq n_0(k)$ and $n_0(k)$ is a sufficiently large number depending only on k.

We shall derive Theorem 1 from the following more general result which also covers smaller values $k < 10$.

Theorem 2. *Let $2 \leq g_1 \leq 7$, $k \geq 3 + g_1$ and $n > k^2$. Then all values of n and k for which $g = g_1$ are given in Table 1.*

g_1	k	n	g_1	k	n	g_1	k	n	g_1	k	n
2	5	45-48			78-80			15119-15120			287-288
		96			94-96			15123			336
		239-242			119	4	8	119-121			356-360
		359-360			121-125			238-240			479-480
	6	45			144			840			483-484
		240			238			5039-5040			500
3	6	44			241-242	4	9	120			669-672
		46-49			250	5	8	68-70			719-720
		95-96			288			74-75			722
		120			357-360			77-80			838-839
		238-239			480			93-96			841
		241-242			484			98			1438-1440
		358-360			670-672			105			1675-1680
		1440			720			118			2021-2023
		4800			839-841			122-125			2520
		5041			1439-1440			140			2883

3	7	120			1676-1680			143-144			3357-3360
		239-240			2022-2023			162			4798-4800
		5040			3358-3360			168			5036-5038
4	7	50			4799-4800			189			5041
		54			5037-5039			236-237			6720-6724
		58-60			5041			241-243			10079-10080
		69-70			6722-6724			245			15118-15120
		75			10080			249-250			15122-15123
5	9	118-119			841			112			1443-1444
		121			960			116			1673-1681
		237-240			1344			122-126			2016
		242			1437-1440			135			2019-2023
		839-840			1444			138-144			2518-2520
		5038-5040			1674-1680			159			2879-2883
		6720			2020-2023			161-162			3355-3360
5	10	119-120			2519-2520			166-168			4796-4800
6	9	90			2880			187-189			5034-5036
		92-98			2882-2883			191-192			5041
		100			3356-3360			216			6718
		104-105			4797-4800			234-235			6721-6724
		117			5035-5037			244-245			10077-10080
		122-125			5041			247-250			13447-13448
		139-140			6719			280			15116-15119
		142-144			6721-6724			285-288			15121-15123
		160-162			10078-10080			315			6375621
		167-168			13448			334-336	7	11	122-124
		188-189			15117-15123			354-360			126
		192	6	10	117-118			375			140
		235-236			121			441			144
		241			160			477-484			158
		243-245			236-237			498-500			161-162
		248-250			238-243			503-504			165-168
		286-288			720			667-672			188-192
		335-336			838-840			717-719			215-216
		355-360			5037-5040			721-722			235-239
		478-480			6719-6720			726			241
		482-484			15120			836-837			243
		499-500	6	11	125			841			287-288
		504			159-160			959-960			375
		668-672			240			1080			440
		718-722			242			1343-1344			480
		837-838	7	10	103-105			1436-1440			719-720

Table 1

g_1	k	n	g_1	k	n	g_1	k	n	g_1	k	n
7	11	837-840			5036-5040	7	12	158-160			239-242
		1680			6718-6720			165	7	13	188-189
		2880			15119-15120			189			240

Table 1 (contiued)

The assumption $2 \leq g_1 \leq 7$ in Theorem 2 can be relaxed and the assertion $g \geq 8$ in Theorem 1 can be strengthened but this will increase the computations and the number of exceptions. We prove Theorem 2 by induction and the first step of induction is given by (6). We write $G = \{i_1, \cdots, i_g\}$ with $i_1 < i_2 < \cdots < i_g$. Then

$$\frac{\Delta(n,k)}{\prod_{i \in G}(n+i)} = by^2 \tag{7}$$

where b and y are positive integers such that b is square free and $P(b) \leq k$. We derive from (7) that

$$n + i = a_i x_i^2 \quad \text{for } 0 \leq i \leq k-1,\ i \notin G \tag{8}$$

where a_i's are square free positive integers with $P(a_i) \leq k$. Further we see that a_i's are distinct whenever $n > k^2$. We observe that the assumptions $k \geq 3 + g_1$ and $n > k^2$ in Theorem 2 are necessary otherwise (7) has infinitely many solutions.

The proof of Theorem 2 depends on elementary and combinatorial arguments of Erdős [2] and Rigge [7] as developed by Erdős and Selfridge [4]. We shall also use SIMATH for solving elliptic curves

$$X(X + bp)(X + bq) = Y^2, \quad 1 \leq p < q \leq 12,\ P(b) \leq 7$$

in positive integers X and Y. We shall apply some combinatorial arguments to keep a check on the number of elliptic curves and securing the ones that can be solved by SIMATH.

We conclude from Theorem 2 that $g \geq 8$ unless (n, k) with $k \geq 10$ is included in Table 1. This is the assertion of Theorem 1. By omitting all exceptions (n, k) in Theorem 1 for which the number of distinct primes $> k$ dividing $\Delta(n, k)$ to odd power is at least 8, we conclude Corollary 1. For Corollary 2, we observe that there are no exceptions in Corollary 1 whenever $n \geq 5040$ or $k \geq 14$.

Erdős [4] and Rigge [7], independently, proved that a product of two or more consecutive positive integers is never a square. Further Saradha and Shorey [10] showed that any product of distinct $k-1$ terms out of k consecutive positive integers is a square only if

$$\frac{6!}{5} = 12^2, \quad \frac{10!}{7} = 720^2. \tag{9}$$

This confirms a conjecture of Erdős and Selfridge [4, p. 300]. We re-write (9) as

$$\frac{6!}{1\cdot 5} = \frac{7!}{5\cdot 7} = 12^2, \quad \frac{10!}{1\cdot 7} = \frac{11!}{7\cdot 11} = 720^2. \tag{10}$$

These may be viewed as examples of squares which are products of $k-2$ distinct terms out of k consecutive positive integers. There are more examples:

$$\begin{cases} \frac{4!}{2\cdot 3} = 2^2, \ \frac{6!}{4\cdot 5} = 6^2, \ \frac{8!}{2\cdot 5\cdot 7} = 24^2, \ \frac{10!}{2\cdot 3\cdot 4\cdot 6\cdot 7} = 60^2, \ \frac{9!}{2\cdot 5\cdot 7} = 72^2, \\ \frac{10!}{2\cdot 3\cdot 6\cdot 7} = 120^2, \ \frac{10!}{2\cdot 7\cdot 8} = 180^2, \ \frac{10!}{7\cdot 9} = 240^2, \ \frac{10!}{4\cdot 7} = 360^2, \\ \frac{21!}{13!\cdot 17\cdot 19} = 5040^2, \ \frac{14!}{2\cdot 3\cdot 4\cdot 11\cdot 13} = 5040^2, \ \frac{14!}{2\cdot 3\cdot 11\cdot 13} = 10080^2. \end{cases} \tag{11}$$

We derive from Theorem 3 that there are no more.

Corollary 3. *Let $k \geq 4$. A product of $k-2$ distinct terms out of k consecutive positive integers is a square only if it is given by* (10) *and* (11).

It is clear that the assumption $k \geq 4$ is necessary in Corollary 3 otherwise there are infinitely many solutions. Let $\kappa(t)$ and $\kappa'(t)$ be given by

$$\kappa(2) = 8, \ \kappa(3) = 9, \ \kappa(4) = 11, \ \kappa(5) = 15, \ \kappa(6) = 16, \ \kappa(7) = 24 \tag{12}$$

and

$$\begin{aligned} &\kappa'(2) = 11, \quad \kappa'(3) = 25, \quad \kappa'(4) = 28, \\ &\kappa'(5) = 30, \quad \kappa'(6) = 46, \quad \kappa'(7) = 50. \end{aligned} \tag{13}$$

We prove

Theorem 3. *Let* $2 \le t \le 7$ *and* $k \ge 2 + t$. *Let* $d_1, d_2, \ldots, d_{k-t}$ *be distinct integers in* $[0, k-1]$. *Assume that*

$$(n + d_1)(n + d_2) \cdots (n + d_{k-t}) = z^2 \tag{14}$$

where $z > 0$ *is an integer. If* $n > k^2$, *then the solution of* (14) *are given by*

$$240.243.245 = 3780^2, \quad 242.245.250 = 3850^2, \qquad 240.242.243.250 = 59400^2. \tag{15}$$

Further

$$k \le \kappa(t) \quad \textit{if } k < n \le k^2 \tag{16}$$

and

$$k \le \kappa'(t) \quad \textit{if } n \le k \tag{17}$$

The proof of Theorem 3 depends on Theorem 2 and inequalities (1), (3), (5). The values $\kappa(t)$ and $\kappa'(t)$ given in (12) and (13) are optimal. For $3 \le t \le 7$, we can compute all squares which are products of $k - t$ distinct terms out of k consecutive positive integers such that t is minimal. But the number of these squares turn out to be much larger than given by (11) in the case $t = 2$. We shall follow the notation introduced in Section 1 throughout the paper. We shall use Mathematica for computations in this paper.

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2. Lemmas

This section consists of lemmas for the proof of Theorem 2. We shall assume that $n > k^2$ throughout this section so that a_i with $0 \le i \le k-1$ and $i \notin G$ are distinct. We begin with the following result which will be applied inductively on g to assume without loss of generality that k is prime if $k \ge 7$, $g = 2$; $k \ge 11$, $g = 3, 4, 5$; $k \ge 13$, $g = 6$ and $k \ge 17$, $g = 7$. This decreases the computational load considerably.

Lemma 1. *Let $n > 0$ and $t \geq 1$ be integers. Assume that $k_1 < k_2$ be consecutive primes. Suppose that*

$$g(n,k) \neq t \quad \textit{for } k \geq k_1 \tag{18}$$

and

$$g(n,k) \neq t+1 \quad \textit{for } k = k_1. \tag{19}$$

Then

$$g(n,k) \neq t+1 \quad \textit{for } k_1 < k < k_2.$$

Proof. Let k with $k_1 < k < k_2$ be given and assume that $g(n,k) = t+1$. We put $i_{t+1} = k - r$ and $k = k_1 + j$ with j, r positive integers. If $r \geq j+1$, we observe that $t+1 = g(n,k) = g(n,k_1) \neq t+1$ by (19). Thus $r \leq j$. Then $k_1 \leq k - r$ and $g(n, k-r) = t$. This contradicts (18). □

Let $m \geq 1$ be an integer. We denote by $f(k,m,G)$ the number of a_i's with $0 \leq i \leq k-1$ and $i \notin G$ composed of the first m primes $2 = p_1 < p_2 < \cdots < p_m$. Then

$$f(k,m,G) \geq f_0(k,m,g) := k - g - \sum_{j \geq m+1} \left(\left[\frac{k}{p_j} \right] + \epsilon_j \right)$$

where $\epsilon_j = 0$ if either $p_j > k$ or if $p_j \mid k$ and $\epsilon_j = 1$ otherwise. Since a_i's are square free, we see that $f(k,m,G) \leq 2^m$ and hence

$$f_0(k,m,g) \leq 2^m. \tag{20}$$

This function with $G = \Phi$ was introduced by Erdős and Selfridge [4]. We check the values of this function given in the next two lemmas.

Lemma 2. *We have*

$$\begin{aligned}
&f_0(k,3,2) \geq 9 && \textit{for } 29 \leq k \leq 73 \quad \textit{and}\\
&f_0(k,4,2) \geq 17 && \textit{for } 74 \leq k \leq 216,\\
&f_0(k,4,3) \geq 17 && \textit{for } 53 \leq k \leq 263 \quad \textit{and}\\
&f_0(k,5,3) \geq 33 && \textit{for } 264 \leq k \leq 276,\\
&f_0(k,4,4) \geq 17 && \textit{for } 59 \leq k \leq 233 \quad \textit{and}
\end{aligned}$$

$$
\begin{array}{lll}
f_0(k,5,4) \geq 33 & \textit{for } 234 \leq k \leq 338, & \\
f_0(k,4,5) \geq 17 & \textit{for } 67 \leq k \leq 229 & \textit{and} \\
f_0(k,5,5) \geq 33 & \textit{for } 230 \leq k \leq 401, & \\
f_0(k,4,6) \geq 17 & \textit{for } 83 \leq k \leq 211, & \\
f_0(k,5,6) \geq 33 & \textit{for } 212 \leq k \leq 433 & \textit{and} \\
f_0(k,6,6) \geq 65 & \textit{for } 434 \leq k \leq 466, & \\
f_0(k,4,7) \geq 17 & \textit{for } 97 \leq k \leq 197, & \\
f_0(k,5,7) \geq 33 & \textit{for } 198 \leq k \leq 433 & \textit{and} \\
f_0(k,6,7) \geq 65 & \textit{for } 434 \leq k \leq 533. &
\end{array}
$$

Lemma 3. *We have*

(i) $f_0(5,3,2)=3,\ f_0(6,3,2)=f_0(7,3,2)=4,\ f_0(11,3,2)=f_0(13,3,2)=6,$
$f_0(17,3,2)=f_0(19,3,2)=f_0(23,3,2)=7.$

(ii) $f_0(6,3,3)=3,\ f_0(7,3,3)=3,\ f_0(11,3,3)=f_0(13,3,3)=5,$
$f_0(17,3,2)=f_0(19,3,3)=f_0(23,3,3)=6,$
$f_0(29,3,3)=f_0(31,3,3)=8.$

(iii) $f_0(7,4,4)=3,\ f_0(11,4,4)=f_0(13,4,4)=6,\ f_0(17,4,4)=f_0(19,4,4)=8,$
$f_0(23,4,4)=9,\ f_0(29,4,4)=f_0(31,4,4)=12,$
$f_0(37,4,4)=f_0(41,4,4)=f_0(43,4,4)=f_0(47,4,4)=14.$

(iv) $f_0(8,4,5)=3,\ f_0(9,4,5)=4,\ f_0(10,4,5)=5,$
$f_0(11,4,5)=f_0(13,4,5)=5,\ \ f_0(17,4,5)=f_0(19,4,5)=7,$
$f_0(23,4,5)=8,\ f_0(29,4,5)=f_0(31,4,5)=11,$
$f_0(37,4,5)=f_0(41,4,5)=f_0(43,4,5)=f_0(47,4,5)=13,$
$f_0(53,4,5)=15.$

(v) $f_0(9,4,6)=3,\ f_0(10,4,6)=4,\ f_0(11,4,6)=f_0(13,4,6)=4,$
$f_0(17,4,6)=f_0(19,4,6)=6,\ f_0(23,4,6)=7,$
$f_0(29,4,6)=f_0(31,4,6)=10,$
$f_0(37,4,6)=f_0(41,4,6)=f_0(43,4,6)=f_0(47,4,6)=12,$
$f_0(53,4,6)=14,\ f_0(59,4,6)=f_0(61,4,6)=15.$

(vi) $f_0(10,4,7)=f_0(11,4,7)=f_0(13,4,7)=3,$
$f_0(17,4,7)=f_0(19,4,7)=5,\ f_0(23,4,7)=6,$
$f_0(29,4,7)=f_0(31,4,7)=9,$

$f_0(37,4,7) = f_0(41,4,7) = f_0(43,4,7) = f_0(47,4,7) = 11,$
$f_0(53,4,7) = 13,\ f_0(59,4,7) = f_0(61,4,7) = 14,$
$f_0(67,4,7) = f_0(71,4,7) = f_0(73,4,7) = f_0(79,4,7) = 15.$

The following result is due to ROSSER and SCHOENFELD [8, p. 69, 71].

Lemma 4. *We have*

(i) $\pi(2x) - \pi(x) > \dfrac{3x}{5\log x}$ *for* $x \geq 20.5$

(ii) $\prod_{p \leq x} p < (2.78)^x$.

We apply Lemmas 2 and 4 in the next result.

Lemma 5. *Let* $2 \leq g \leq 7$, $n > k^2$ *and* k *prime. Then* $k \leq k_0(g)$ *where* $k_0(2) = 23$, $k_0(3) = 31$, $k_0(4) = 47$, $k_0(5) = 53$, $k_0(6) = 61$ *and* $k_0(7) = 79$.

PROOF. Suppose that the assumptions of Lemma 5 are satisfied. We recall that (7) holds and a_i's are square free and they are distinct since $n > k^2$. Let R be the set of integers in $[0, k-1]$ which do not belong to G. We give an upper bound and a lower bound for $\prod_{i \in R} a_i$. For a prime $p_0 \leq k$, we write

$$\gamma_{p_0} = \mathrm{ord}_{p_0}\Big(\prod_{i \in \mathbb{R}} a_i\Big).$$

Then

$$\gamma_{p_0} \leq \left[\frac{k-1}{p_0}\right] + 1.$$

Since

$$\prod_{i \in \mathbb{R}} a_i = \prod_{p_0 \leq k} p_0^{\gamma_{p_0}},$$

it follows that

$$\prod_{i \in \mathbb{R}} a_i \,\Big|\, \prod_{p_0 \leq k} p_0^{[\frac{k-1}{p_0}]+1}.$$

Thus

$$\prod_{i \in \mathbb{R}} a_i \,\Big|\, (k-1)! \prod_{p_0 \leq k} p_0.$$

Let
$$\gamma'_{p_0} = \mathrm{ord}_{p_0}\Big((k-1)!\prod_{p_0\le k} p_0\Big).$$

Let $p_0^h \le k-1 < p_0^{h+1}$. Then
$$\gamma'_{p_0} = \left[\frac{k-1}{p_0}\right] + \cdots + \left[\frac{k-1}{p_0^h}\right] + 1.$$

We observe that γ_{p_0} is equal to the number of terms $n+i$ with $i \in \mathbb{R}$ divisible by p_0 to an odd power. Let $n+J$ for $J \in \mathbb{R}$ be a term divisible by the maximal power of p_0. We consider the set $S = \{n+i : i \in \mathbb{R}, i \ne J\}$ and let μ be a positive integer. Then the number of elements of S divisible by p_0^μ is at most $[(k-1)/p_0^\mu]$ and at least $[(k-1)/p_0^\mu] - g - 1$. Thus

$$\begin{aligned}\gamma_{p_0} \le{}& \left[\frac{k-1}{p_0}\right] - \left(\left[\frac{k-1}{p_0^2}\right] - g - 1\right)\\ &+ \left[\frac{k-1}{p_0^3}\right] - \left(\left[\frac{k-1}{p_0^4}\right] - g - 1\right) + \cdots + (-1)^\epsilon\left(\left[\frac{k-1}{p_0^h}\right] + \epsilon_1\right) + 1\end{aligned}$$

where $\epsilon = 1$ or 0 and $\epsilon_1 = -(g+1)$ or 0 according as h is even or odd, respectively. Thus we have

$$\begin{aligned}\gamma_{p_0} - \gamma'_{p_0} &\le (g+1)\frac{(h+\epsilon-1)}{2}\\ &\qquad - 2\left(\left[\frac{k-1}{p_0^2}\right] + \left[\frac{k-1}{p_0^4}\right] + \cdots + \left[\frac{k-1}{p_0^{h+\epsilon-1}}\right]\right)\\ &\le (g+1)\frac{(h+\epsilon-1)}{2} - 2\left(\frac{k-1}{p_0^2} + \cdots + \frac{k-1}{p_0^{h+\epsilon-1}} - \frac{h+\epsilon-1}{2}\right)\\ &\le (g+3)\frac{(h+\epsilon-1)}{2} - \frac{2(k-1)}{p_0^2-1}\left(1 - \frac{1}{p_0^{h+\epsilon-1}}\right).\end{aligned}$$

Since $p_0^{h+1} > k-1$ and $h < \log k/\log p_0$, we get
$$\gamma_{p_0} - \gamma'_{p_0} < \frac{\log k}{2\log p_0}(g+3) - \frac{2k}{p_0^2-1} + \delta_{p_0}$$
where
$$\delta_{p_0} = \frac{2+2p_0^2}{p_0^2-1}.$$

We observe that

$$\prod_{i\in\mathbb{R}} a_i \,\Big|\, (k-1)! \prod_{p_0\le k} p_0 \prod_{p_0\le 7} p_0^{\gamma_{p_0}-\gamma'_{p_0}}.$$

We compute that

$$\prod_{p_0\le 7} p_0^{\gamma_{p_0}-\gamma'_{p_0}} \le 296001 k^{2g+6}(2.5907)^{-k}.$$

Thus

$$\prod_{i\in\mathbb{R}} a_i \le 296001(k-1)!k^{2g+6}(1.07307)^k \tag{21}$$

by Lemma 4. On the other hand, we see that

$$\prod_{i\in\mathbb{R}} a_i \ge \prod_{i=1}^{k-g} s_i \tag{22}$$

where s_i denotes the i-th square free integer. Further

$$\prod_{i=1}^{k-g} s_i \ge (k-g)!(1.5)^{k-g} \quad \text{for } k\ge 79. \tag{23}$$

We check (23) for $k=79$. Then (23) follows immediately by induction on k from an inequality of Erdős [2] that $s_i \ge (1.5)i$ for $i\ge 39$.

By combining (21), (22) and (23), we get

$$(1.3978)^k \le 296001(1.5)^g k^{3g+5}$$

which implies that $k\le 216$ if $g=2$; $k\le 276$ if $g=3$; $k\le 338$ if $g=4$; $k\le 401$ if $g=5$; $k\le 466$ if $g=6$ and $k\le 533$ if $g=7$.

Now we apply Lemma 2 and (20). We conclude that $k\le 23$ if $g=2$; $k\le 47$ if $g=3$; $k\le 53$ if $g=4$; $k\le 61$ if $g=5$; $k\le 79$ if $g=6$ and $k\le 89$ if $g=7$. Thus it remains to exclude the cases $k=37,41,43,47$ if $g=3$; $k=53$ if $g=4$; $k=59,61$ if $g=5$; $k=67,71,73,79$ if $g=6$ and $k=83,89$ if $g=7$.

Let $g=3$. We observe that $f_0(37,3,3)=f_0(41,3,3)=9$ which imply that $k\ne 37,41$ by (20). Let $k=43$. Then the primes 43, 41, 37, 31, 29, 23, 19, 17, 13, 11, 7 divide 1, 2, 2, 2, 2, 2, 3, 3, 4, 4, 7 distinct a_i's, respectively, and none of these a_i's is divisible by more than one of these primes. So 41 divides a_0, a_{41} or a_1, a_{42} and 7 divides a_0, a_7, a_{14}, a_{21}, a_{28}, a_{35}, a_{42}. This is not possible. Let $k = 47$. Then exactly 1, 2, 2, 2, 2, 2, 3, 3, 3, 4, 5, 7 distinct a_i's are divisible by 47, 43, 41, 37, 31, 29, 23, 19, 17, 13, 11, 7, respectively, and none of these a_i's is divisible by more than one of these primes. Hence 23 divides a_0, a_{23}, a_{46}. Then 11 divides either a_1, a_{12}, a_{23}, a_{34}, a_{45} or a_2, a_{13}, a_{24}, a_{35}, a_{46} leading to a contradiction in either of the cases.

The proofs for the other cases are similar and we suppress some details. Let $g = 4$. We have $f_0(53, 3, 4) = 8$. Hence 13 divides a_0, a_{13}, a_{26}, a_{39}, a_{52} and 17 divides a_1, a_{18}, a_{35}, a_{52}, a contradiction.

Let $g = 5$. Then $f_0(61, 4, 5) = f_0(59, 4, 5) = 16$. If $k = 61$, then 59 divides a_0, a_{59} or a_1, a_{60}. If 59 divides a_0, a_{59}, then 29 divides a_2, a_{31}, a_{60}, 19 divides a_1, a_{20}, a_{39}, a_{58} and 11 divides a_3, a_{14}, a_{25}, a_{36}, a_{47}, a_{58} which is not possible. If 59 divides a_1, a_{60}, then 29 divides a_0, a_{29}, a_{58}, 19 divides a_2, a_{21}, a_{40}, a_{59} and 11 divides a_3, a_{14}, a_{25}, a_{36}, a_{47}, a_{58} which is impossible. When $k = 59$, then 29 divides a_0, a_{29}, a_{58} and 19 divides a_1, a_{20}, a_{39}, a_{58}. This is a contradiction.

Let $g = 6$. We see that $f_0(79, 4, 6) = f_0(73, 4, 6) = f_0(71, 4, 6) = f_0(67, 4, 6) = 16$. Let $k = 79$. Then 13 divides a_0, a_{13}, a_{26}, a_{39}, a_{52}, a_{65}, a_{78} and 11 divide a_1, a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{67}, a_{78}. This is impossible. Let $k = 73$. Then 71 divides a_0, a_{71} or a_1, a_{72}. Let 71 divide a_0, a_{71}. Then 23 divides a_1, a_{24}, a_{47}, a_{70} or a_3, a_{26}, a_{49}, a_{72}. Suppose that the first possibility holds. Then 17 divides a_4, a_{21}, a_{38}, a_{55}, a_{72} and 13 divides a_2, a_{15}, a_{28}, a_{41}, a_{54}, a_{67}. Therefore 11 divides either a_3, a_{14}, a_{25}, a_{36}, a_{47}, a_{58}, a_{69} or a_4, a_{15}, a_{26}, a_{37}, a_{48}, a_{59}, a_{70} but neither is possible. Let 23 divide a_3, a_{26}, a_{49}, a_{72}. Then 17 divides either a_1, a_{18}, a_{35}, a_{52}, a_{69} or a_2, a_{19}, a_{36}, a_{53}, a_{70}. In case of the former possibility, we see that 13 divides a_2, a_{15}, a_{28}, a_{41}, a_{54}, a_{67}, 11 divides a_4, a_{15}, a_{26}, a_{37}, a_{48}, a_{59}, a_{70} which is not possible. When the latter possibility holds, we observe that 13 divides a_4, a_{17}, a_{30}, a_{43}, a_{56}, a_{69}, 11 divides a_1, a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{67}, a contradiction. The case 71 dividing a_1, a_{72} is excluded similarly by considering divisibility of a_i's by primes 23, 17, 13, 11 and 67. Let $k = 71$. We observe that 23 divides a_0, a_{23}, a_{46}, a_{69} or a_1, a_{24}, a_{47}, a_{70}. In the first case, 17 divides a_2, a_{19}, a_{36}, a_{53}, a_{70}, 67 divides a_1, a_{68} and hence 11 can not divide 7 distinct a_i's. This is also the case whenever the latter possibility holds. This is a contradiction. Let $k = 67$. We observe that 11 divides a_0, a_{11}, a_{22}, a_{33}, a_{44}, a_{55}, a_{66} and 13 divides a_1, a_{14}, a_{27}, a_{40}, a_{53}, a_{66} which is not possible.

Let $g = 7$. Then we have $f_0(83, 4, 7) = f_0(89, 4, 7) = 16$. Let $k = 89$. Then 11 divides a_0, a_{11}, a_{22}, a_{33}, a_{44}, a_{55}, a_{66}, a_{77}, a_{88} and 29 divides a_1, a_{30}, a_{59}, a_{88}, a contradiction. Let $k = 83$. Then 41 divides a_0, a_{41}, a_{82}. If 79 divides a_1, a_{80}, then 13 divides a_3, a_{16}, a_{29}, a_{42}, a_{55}, a_{68}, a_{81} and 11 divides a_2, a_{13}, a_{24}, a_{35}, a_{46}, a_{57}, a_{68}, a_{79} which is impossible. The case 79 dividing a_2, a_{81} is excluded similarly. $\square$

3. Proof of Theorem 2

Let $2 \leq g_1 \leq 7$, $k \geq 3 + g_1$ and $n > k^2$. Assume that $g = g_1$. We recall that (7) is valid. We first give a proof of Theorem 2 under the assumption that k is prime if $k \geq 7$, $g = 2$; $k \geq 11$, $g = 3, 4, 5$; $k \geq 13$ $g = 6$ and $k \geq 17$, $g = 7$. We conclude from Lemma 5 that $k \leq k_0(g)$.

Let $g = 2$. We first consider $k = 23$. We have at least 7 a_i's composed only of 2, 3, 5 by Lemma 3 (i). Hence there are at least 3 a_i's such that the corresponding i's belong to exactly one of the intervals $[0, 7]$, $[8, 15]$, $[16, 22]$. Therefore we see from (8) that

$$(n + i_0)(n + i_0 + p)(n + i_0 + q) = by^2, \quad 1 \leq p < q \leq 7, \ P(b) \leq 5.$$

We shall always denote by i_0 a non-negative integer and $X = b(n + i_0)$ in the proof of Theorem 2. Putting $b^2y = Y$, we get the following set of elliptic curves

$$X(X + bp)(X + bq) = Y^2, \quad 1 \leq p < q \leq 7, \ P(b) \leq 5. \tag{24}$$

For $k = 19, 17, 13, 11, 7, 6, 5$, we divide $0 \leq i \leq k - 1$ into 3, 3, 2, 2, 1, 1, 1 parts, respectively, and apply Lemma 3(i) as above to obtain elliptic curves (24). Thus we need to solve (24) in integers.

We apply Lemma 3(ii), (iii), (iv), (v), (vi) as above according as $g = 3, 4, 5, 6, 7$, respectively. Then we obtain the following set of elliptic equations

$$X(X + bp)(X + bq) = Y^2, \qquad 1 \leq p < q \leq 11, \ P(b) \leq 5, \tag{25}$$

$$X(X+bp)(X+bq) = Y^2, \qquad 1 \le p < q \le 7,\ P(b) \le 7, \tag{26}$$

$$X(X+bp)(X+bq) = Y^2, \qquad 1 \le p < q \le 9,\ P(b) \le 7 \tag{27}$$

and

$$X(X+bp)(X+bq) = Y^2, \qquad 1 \le p < q \le 12,\ P(b) \le 7 \tag{28}$$

according as $g = 3$, $g = 4, 5$, $g = 6$ and $g = 7$, respectively. For the preceding assertion, we need to make few additional observations in the cases $k = 9$, $g = 5$ and $k = 11, 13$, $g = 6$. Let $k = 9$, $g = 5$. Then $f_0(9, 4, 5) = 4$ and there are at least 3 a_i's with $1 \le i \le 8$ composed only of 2, 3, 5, 7. Let $k = 11$, $g = 6$. Then $f_0(11, 4, 6) = 4$ and there are at least 3 a_i's with $1 \le i \le 10$ composed only of 2, 3, 5, 7. Finally let $k = 13$, $g = 6$. Then $f_0(13, 4, 6) = 4$. We may assume that 11 divides a_0, a_{11} or a_1, a_{12}. Thus we find at least three 3 a_i's with $1 \le i \le 10$ or $2 \le i \le 11$ composed only of 2, 3, 5, 7.

Now we use SIMATH to solve the equations (24), (25), (26), (27) and (28). This was used for the first time in a similar context by Filakovszky and Hajdu [5]. Further we describe how to obtain Table 1 from the above solutions. Let $2 \le g \le 7$ be given and we restrict to (24), (25), (26), (27), (28) according as $g = 2$, $g = 3$, $g = 4, 5$, $g = 6$ and $g = 7$, respectively. We observe that $n + i_0 = X/b$ is an integer. Further $k \le [\sqrt{X/b}\,]$ since $n > k^2$. Let $K = \min\{[\sqrt{X/b}\,], k_0(g)\}$. Thus

$$k \le K. \tag{29}$$

Now $i_0 + q \le k - 1 \le K - 1$ implying that $0 \le i_0 \le K - q - 1$. Therefore

$$n \in [X/b - K + q + 1, X/b]. \tag{30}$$

For n, k satisfying (29) and (30), we include (g, n, k) in Table 1 if and only if the number of i with $0 \le i \le k-1$ such that $n+i$ is divisible by a prime $> k$ to odd power is exactly equal to g. We explain the above argument in the case $g = 2$. Thus we need to solve (24). For example, we consider (24) with $p = 3$, $q = 4$ and $b = 15$. We have $X = 15(n + i_0) > 15k^2$ and $k \ge 5$. Now we conclude by SIMATH that $X = 675$. Then $K = 6$ and $k = 5, 6$ by (29). Further $n = 44, 45$ by (30). Thus we need to consider only the pairs $(n, k) = (44, 5), (44, 6), (45, 5), (45, 6)$. The first two pairs are excluded since $g = 3$ for both. On the other hand, we find that $g = 2$ for the last two pairs. Hence the values of n and k corresponding to these pairs are included in Table 1 against $g = 2$.

For a composite k, it remains to show that $g(n, k) \neq 2, 3, 4, 5, 6, 7$ according as k exceeds 7, 11, 11, 11, 13, 17, respectively. Let $k' > 7$ be composite. Let $k_1 \geq 7$ and k_2 be consecutive primes such that $k_1 < k' < k_2$. As shown above, $g(n, k_1) \neq 2$ and $g(n, k) \neq 1$ for every $k \geq k_1$ by (6). Therefore the assumptions of Lemma 1 with $t = 1$ are satisfied. Hence we derive from Lemma 1 with $t = 1$ that $g(n, k') \neq 2$. Thus $g(n, k) \neq 2$ for every $k \geq 7$. As proved above, $g(n, k) \neq 3$ whenever $k \geq 11$ is prime. Now we conclude from Lemma 1 with $t = 2$ that $g(n, k) \neq 3$ for every $k \geq 11$. Further we apply Lemma 1 inductively with $t = 3, 4, 5, 6$ to complete the proof of Theorem 2. □

4. Proof of Theorem 3

Let $2 \leq t \leq 7$, $k \geq 2 + t$ and we assume (14). Let $k = 2 + t$. We may assume that all the solutions of (14) are given by $1 \cdot 4 = 2^2$, $1 \cdot 9 = 3^2$, $2 \cdot 8 = 4^2$, $4 \cdot 9 = 6^2$, $9 \cdot 16 = 12^2$. The last one is covered by (17) and the remaining ones by (16). Thus we may assume that $k > 2 + t$. Further we observe from (7) that $g \leq t$.

Let $n > k^2$. Then $g \geq 2$ by (6) and the assertion of Theorem 2 holds. If $t = 7$, then $k \geq 10$ and $g \geq 5$ by Table 1. Similarly $g \geq 4$ if $t = 5, 6$ and $g \geq 3$ if $t = 4$. Further we check whether every possible product of $k - t$ distinct integers out of $k - g$ integers $n + i$ with $0 \leq i \leq k - 1$ and $i \notin G$ is a square. We find that all the solutions of (14) are given by (15). Thus we may assume that $n \leq k^2$.

Let $n > k$. Then $g' = g \leq t$. Now we apply (1) to derive $k \leq \kappa_1(t)$ where

$$\kappa_1(2) = 8,\ \kappa_1(3) = 12,\ \kappa_1(4) = 22,$$

$$\kappa_1(5) = 36,\ \kappa_1(6) = 46,\ \kappa_1(7) = 60.$$

Further we apply (3) to sharpen $k \leq \kappa_1(t)$ to $k \leq \kappa_2(t)$ with $t \geq 4$ where

$$\kappa_2(2) = 8,\ \kappa_2(3) = 12,\ \kappa_2(4) = 18,$$

$$\kappa_2(5) = 28,\ \kappa_2(6) = 30,\ \kappa_2(7) = 36.$$

While applying (3), we check that the exceptions given in (4) with $\kappa_1(t) < k \le \kappa_2(t)$, $t \ge 5$ are excluded by noting that $\pi(2k) - \pi(k) - 1 > t$ and (5) holds. Further the exceptions given in (4) with $\kappa_1(4) < k \le \kappa_2(4)$ are excluded by direct computations. Finally we conclude (16) from $k \le \kappa_2(t)$ by computations as above in the case $n > k^2$.

Therefore we may suppose that $n \le k$. Then $n \le (n+k)/2 < n+k-1$ and we see from (14) that

$$\pi(n+k-1) - \pi\left(\frac{n+k}{2}\right) \le t.$$

This implies that $n+k \le 122$ by Lemma 4. This is improved to $n+k \le 66$ by using exact values of π function. Thus $k \le 65$ which we sharpen to (17) by checking whether a product of $k-t$ distinct integers $n+i$ with $0 \le i \le k-1$ such that $n+i$ is composite whenever $i > (k-n)/2$, is a square. $\square$

5. Proof of Corollary 3

Assume (14) with $t = 2$. Then we conclude from Theorem 3 that $n \le k^2$. Further $k \le 8$ if $n > k$, $k \le 11$ if $n \le k$ and the assertion of Corollary 3 follows by computations as in the proof of Theorem 3. $\square$

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