Publ. Math. Debrecen
64/1-2 (2004), 101–106

Non-planar simplices are not reduced

By HORST MARTINI (Chemnitz) and KONRAD J. SWANEPOEL (Pretoria)

Abstract. A convex body in $\mathbb{R}^n$ which does not properly contain a convex body of the same minimum width is called a reduced body. It is not known whether there exist reduced n-dimensional polytopes for $n \geq 3$. We prove that no n-dimensional simplex is reduced if $n \geq 3$.

1. Introduction

Due to E. Heil [4], a convex body $K \subset \mathbb{R}^n$ is called *reduced* if there is no convex body L properly contained in K such that the *minimum width* $\Delta(L)$ (=minimal distance between two different parallel supporting hyperplanes) of L is equal to $\Delta(K)$. Reduced bodies are interesting in view of several extremal problems, for example regarding the long-standing question: Which convex body of given minimum width has minimal volume? The extremal body has obviously to be reduced. Every body of constant width in $\mathbb{R}^n$ is reduced, but there are many further examples. For instance,

Mathematics Subject Classification: 52A20, 52B12.
Key words and phrases: body of constant width, convex polytope, difference body, projection body, reduced body, simplex.
This work was supported by a grant from the cooperation between the Deutsche Forschungsgemeinschaft (Germany) and the National Research Foundation (South Africa).
Martini acknowledges the hospitality of Unisa during his visits to its Department of Mathematics, Applied Mathematics and Astronomy during Jan–Feb 2002 and March 2003.

all regular m-gons in $\mathbb{R}^2$ with m odd are reduced, as well as the intersection of the unit ball of $\mathbb{R}^n$ with an orthant of the respective Cartesian coordinate system (for $n = 2$ yielding a quarter of the unit disk). Many geometric properties of reduced bodies were found by M. Lassak [5]. In his paper also the following problem was posed: *Do there exist reduced n-dimensional polytopes for $n \geq 3$?*

Although this question was repeated in [6], the answer is still unknown. Using special geometric properties of tetrahedra (that no longer hold for n-simplices if $n \geq 4$), the authors of [9] proved that there is no reduced 3-simplex. It is our goal to extend this observation to higher dimensions.

2. The result and its proof

For an n-dimensional simplex $S \subset \mathbb{R}^n$, $n \geq 2$, we will use the following notions and abbrevations. The *vertex set* of S is given by $\{x_1, \ldots, x_{n+1}\}$ and, for any $i \in \{1, \ldots, n+1\}$, F_i denotes the unique $(n-1)$-face of S which is opposite to the vertex x_i. We also use some functions defined on the unit sphere S^{n-1} of $\mathbb{R}^n$. Most of their properties considered here hold for arbitrary convex bodies (see [3]), but we introduce them only for simplices. For an arbitrary unit vector $u \in S^{n-1}$ the *width* $w(S, u)$ of S in direction u is the distance of the two different parallel supporting hyperplanes of S which are orthogonal to u. The minimum of the function $w(S, u)$, $u \in S^{n-1}$, is called the *minimum width* or *thickness* of S, and is denoted by $\Delta(S)$. There exists a chord of S parallel to the direction of that minimum and having length $\Delta(S)$ (see [2, §§ 33]). Such a chord is said to be a *thickness chord* of S. Thus, if a segment $[a, b] \subset S$ is a thickness chord of S, then there are different supporting hyperplanes H_1, H_2 of S which are both orthogonal to $[a, b]$ and satisfy $a \in H_1, b \in H_2$. In other words, denoting by $V_1(S, u)$, $u \in S^{n-1}$, the function describing the maximal chord length of S for any direction u, we have

$$\min_{u \in S^{n-1}} V_1(S, u) = \Delta(S). \qquad (1)$$

The *brightness function* $V_{n-1}(S, u)$, $u \in S^{n-1}$, of an n-simplex S is the $(n-1)$-volume of the orthogonal projection of S onto the $(n-1)$-subspace orthogonal to u.

In [8] it was shown that for the *volume* $V_n(S)$ of an arbitrary n-simplex S and *any* direction $u \in S^{n-1}$ the relation

$$V_n(S) = \frac{1}{n} \cdot V_{n-1}(S,u) \cdot V_1(S,u) \tag{2}$$

holds. With (1) this implies in particular

$$V_n(S) = \frac{1}{n} \cdot \max_{u \in S^{n-1}} V_{n-1}(S,u) \cdot \Delta(S), \tag{3}$$

i.e., the maximum brightness and the minimum width of S occur in the same direction.

Now we are ready to prove our

Theorem. *No n-dimensional simplex $S \subset \mathbb{R}^n$, $n \geq 3$, is reduced.*

Proof. We will prove that statement by contradiction. Assuming that S is reduced, it follows firstly that S has to be *equiareal*, i.e., that each $(n-1)$-face F_i must have the same $(n-1)$-volume $V_{n-1}(F_i)$, $i = 1, \ldots, n+1$. Indeed, in the classical formula

$$V_n(S) = \frac{1}{n} \cdot V_{n-1}(F_i) \cdot h_i, \qquad i \in \{1, \ldots, n+1\}, \tag{4}$$

where h_i denotes the length of the i-th altitude of S orthogonal to the affine hull of F_i, h_i is equal to $w(S, u_i)$ with u_i as (outer) normal direction of F_i. If we had $h_i \neq \Delta(S)$ for some $i \in \{1, \ldots, n+1\}$, the corresponding vertex x_i would not belong to a thickness chord of S and could be cut off to get from S a convex body L properly contained in S and satisfying $\Delta(L) = \Delta(S)$, a contradiction to the assumed reducedness of S. Thus we must have $h_i = \Delta(S)$ for all $i \in \{1, \ldots, n+1\}$, implying by (4) that S is equiareal.

Moreover, combining (4) and (3), we obtain

$$V_{n-1}(F_i) = \max_{u \in S^{n-1}} V_{n-1}(S,u), \qquad i = 1, \ldots, n+1. \tag{5}$$

From [3, § 4.1] we read off that the brightness function of S has the representation

$$V_{n-1}(S,u) = \frac{1}{2} \sum_{i=1}^{n+1} |\langle v_i, u \rangle|, \qquad u \in S^{n-1}, \tag{6}$$

where $v_i := V_{n-1}(F_i) \cdot u_i$. Due to $\sum_{i=1}^{n+1} v_i = o$ (Minkowski's existence theorem, cf. [3, Appendix A]) this can also be written in the form

$$V_{n-1}(S,u) = \sum_{i \in I(u)} \langle v_i, u \rangle, \qquad u \in S^{n-1}, \tag{7}$$

where $I(u) := \{j \in \{1,\dots,n+1\} : \langle v_j, u\rangle \geq 0\}$. From (7) it follows that

$$\max_{u \in S^{n-1}} V_{n-1}(S,u) = \Big\| \sum_{i \in I^*} v_i \Big\|, \tag{8}$$

where the nonempty index set $I^* \subseteq \{1,\dots,n+1\}$ is determined by

$$\Big\| \sum_{i \in I^*} v_i \Big\| = \max_{I \subseteq \{1,\dots,n+1\}} \Big\| \sum_{i \in I} v_i \Big\|.$$

Without loss of generality, we may consider $\{v_1,\dots,v_{n+1}\}$ as a system of *unit vectors* since S is assumed to be equiareal. Therefore we can continue with the following

Lemma. *Given $m > 3$ unit vectors $v_1,\dots,v_m$ in $\mathbb{R}^n$. Then there exist distinct indices i, j such that $\|v_i + v_j\| > 1$.*

Proof. Suppose that $\|v_i + v_j\| \leq 1$ for all $1 \leq i < j \leq m$. Squaring we obtain $\|v_i\|^2 + 2\langle v_i, v_j\rangle + \|v_j\|^2 \leq 1$, implying $2\langle v_i, v_j\rangle \leq -1$. Hence,

$$\Big\| \sum_{i=1}^m v_i \Big\|^2 = \sum_{i=1}^m \Big\| v_i \Big\|^2 + 2 \sum_{i<j} \langle v_i, v_j \rangle \leq m - \binom{m}{2},$$

yielding $m - \binom{m}{2} \geq 0$. Thus $m \leq 3$, contradicting the hypothesis. □

In view of (8), this lemma says that for an equiareal n-simplex S, $n \geq 3$, the quantity $\max_{u \in S^{n-1}} V_{n-1}(S,u)$ cannot be equal to the $(n-1)$-volume of an $(n-1)$-face, i.e., (5) is not satisfied. By (3) it follows that no such simplex has its mimimum width in the normal direction of an $(n-1)$-face, i.e., its vertices are not contained in thickness chords and can be cut off without decreasing $\Delta(S)$. Thus, there is no reduced n-simplex for $n \geq 3$. □

3. Concluding remarks

(1) Our theorem might be considered as a starting point to solve M. Lassak's problem for all convex n-polytopes (e.g. by some inductional approach based on the cardinality of the vertex set). However, the method presented here can no longer be used. Namely, the function $V_{n-1}(S,u)$, $u \in S^{n-1}$, considered above is known to be the support function of the so-called *projection body* ΠS of the simplex S, and $V_1(S,u)$, $u \in S^{n-1}$, is the radius function of the *difference body* $DS = S+(-S)$ of S. In these terms, relation (2) says that ΠS and DS are polar reciprocal with respect to the sphere of radius $\sqrt{n \cdot V_n(S)}$ which is centred at the origin. (For definitions and many properties of the bodies ΠS and DS, associated with S, the reader should consult [2, §§ 30 and §§ 33] and [3, § 4.1 and § 3.2].) It was proved in [7] that for all convex n-polytopes which are not simplices such a polarity (even with respect to spheres of arbitrary radii) does no longer hold. Thus our conclusion from (2) to (3) is, in general, no longer true.

(2) To get a dualization of the famous Jung theorem (cf. [2], §§ 44), W. Blaschke erroneously assumed that the minimum width of a *regular* n-simplex in $\mathbb{R}^n$ is attained at the normal directions of its $(n-1)$-faces, see [1]. (Blaschke's assumption is true only for $n = 2$, and his statement for higher dimensions was corrected by P. Steinhagen [10].) From our considerations it follows that *no equiareal n-simplex*, $n \geq 3$, has the property assumed by Blaschke.

References

[1] W. Blaschke, Über den größten Kreis in einer konvexen Punktmenge, *Jahresber. Deutsche Mathematikervereinigung* **23** (1914), 369–374.

[2] T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, *Springer*, *Berlin*, 1934, English translation: Theory of Convex Bodies, BCS Associates, Moscow, ID, 1987.

[3] R. J. Gardner, Geometric Tomography, *Cambridge University Press*, *Cambridge*, 1995.

[4] E. Heil, Kleinste Körper gegebener Dicke., *Fachbereich Mathematik, TH Darmstadt*, 1978, Preprint No. 453.

[5] M. Lassak, Reduced convex bodies in the plane, *Israel J. Math.* **70** (1990), 365–379.

[6] M. Lassak, Contributed Problem No. 12, in: Polytopes – Abstract, Convex and Computational, NATO ASI Series, Ser. C., Vol. 440, (T. Bisztriczky, P. McMullen, R. Schneider, and A. Ivić-Weiss, eds.), *Kluwer*, *Dordrecht*, 1994, 495.

[7] H. Martini, Convex polytopes whose projection bodies and difference sets are polars, *Discrete Comput. Geom.* **6** (1991), 83–91.

[8] H. Martini, A new view on some characterizations of simplices, *Arch. Math.* **55** (1990), 389–393.

[9] H. Martini and W. Wenzel, Tetrahedra are not reduced, *Appl. Math. Letters* **15** (2002), 881–884.

[10] P. Steinhagen, Über die größte Kugel in einer konvexen Punktmenge, *Abh. Math. Sem. Hamburg* **1** (1921), 15–26.

HORST MARTINI
FAKULTÄT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT CHEMNITZ
D-09107 CHEMNITZ
GERMANY

KONRAD J. SWANEPOEL
DEPARTMENT OF MATHEMATICS, APPLIED MATHEMATICS AND ASTRONOMY
UNIVERSITY OF SOUTH AFRICA
P.O. BOX 392, PRETORIA 0003
SOUTH AFRICA

(Received December 7, 2002; revised April 1, 2003)