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Pexider generalization of a functional equation of multiplicative symmetry

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Abstract. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. The problem of finding the continuous solutions $f, g, h : \mathbb{K} \to \mathbb{K}$ of the functional equation:

$$f(xg(y)) = h(x)h(y) \qquad (x, y \in \mathbb{K}) \tag{1}$$

may be reduced to the problem of finding the continuous solutions $F, G : \mathbb{K} \to \mathbb{K}$ of the functional equation:

$$F(xG(y)) = F(x)F(y) \qquad (x, y \in \mathbb{K}). \tag{2}$$

In the present paper, we obtain the continuous solutions $F : H \to \mathbb{K}$ and $G : H \to H$ of (2) when H is a nontrivial connected subset of $\mathbb{K}$ satisfying $H^2 \subseteq H$.

1. Introduction

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. In [3] we obtained the continuous solutions $f : \mathbb{K} \to \mathbb{K}$ of the functional equations of multiplicative symmetry:

$$f(xf(y)) = f(yf(x)) \qquad (x, y \in \mathbb{K}) \tag{3}$$

$$f(xf(y)) = f(x)f(y) \qquad (x, y \in \mathbb{K}) \tag{4}$$

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under the hypothesis that $f(\mathbb{C})\setminus\{0\}$ is connected if f is not constant and if $\mathbb{K}=\mathbb{C}$.

In the present paper we consider a Pexider generalization of the functional equation (4). More precisely, we look for the continuous solutions $f,g,h:\mathbb{K}\to\mathbb{K}$ of the functional equation:

$$f(xg(y)) = h(x)h(y) \qquad (x,y\in\mathbb{K}) \tag{1}$$

We have the following result:

Proposition 1. *All the continuous solutions $f,g,h:\mathbb{K}\to\mathbb{K}$ of the functional equation:*

$$f(xg(y)) = h(x)h(y) \qquad (x,y\in\mathbb{K}) \tag{1}$$

are the following:

(i) *either $g\equiv 0$ and*

- *if $\mathbb{K}=\mathbb{R}$, f is arbitrary but $f(0)\geq 0$ and either $h\equiv\sqrt{f(0)}$ or $h\equiv-\sqrt{f(0)}$*
- *if $\mathbb{K}=\mathbb{C}$, f is arbitrary and either $h\equiv\sqrt{f(0)}$ or $h\equiv-\sqrt{f(0)}$ where $\sqrt{f(0)}$ is one of the square roots of $f(0)$.*

(ii) *or $g\not\equiv 0$ and*

$$f(x)=\beta^2F\left(\frac{x}{\alpha}\right),\ h(x)=\beta F(x),\ g(y)=\alpha G(y) \quad (x,y\in\mathbb{K}) \tag{5}$$

where α and β are arbitrary elements of $\mathbb{K}$ such that $\alpha\neq 0$ and $F,G:\mathbb{K}\to\mathbb{K}$ are continuous solutions of the functional equation:

$$F(xG(y)) = F(x)F(y) \qquad (x,y\in\mathbb{K}). \tag{2}$$

Proof. The case (i) is obvious. If g is not identically zero, there exists y_0 in $\mathbb{K}$ such that $g(y_0)=\alpha\neq 0$. Letting $y=y_0$ in (1), we get: $f(\alpha x)=h(x)h(y_0)$ $(x\in\mathbb{K})$.

The case $h(y_0)=0$ leads to $f\equiv 0$ and $h\equiv 0$, which are obviously solutions of (1) for an arbitrary function g.

So, we suppose now $h(y_0)=\beta\neq 0$ and we get: $h(x)=\frac{1}{\beta}f(\alpha x)$ $(x\in\mathbb{K})$. If we define: $F(x)=\frac{1}{\beta^2}f(\alpha x)$ $(x\in\mathbb{K})$, $G(y)=\frac{1}{\alpha}g(y)$ $(y\in\mathbb{K})$, F and G are solutions of (2).

Conversely, if F and G are solutions of (2) and if f, g, h are defined by (5) with $\alpha\neq 0$, f, g, h are solutions of (1). $\square$

In the sequel we will solve the functional equation (2) on a subset of $\mathbb{K}$.

In the whole paper, U will denote $\{z \in \mathbb{C};\ |z| = 1\}$.

2. Problem of finding the continuous solutions of (2)

Let H be a nontrivial connected subset of $\mathbb{C}$ satisfying $H^2 \subseteq H$, where $H^2 = \{xy;\ x \in H,\ y \in H\}$. We look for the continuous solutions $F : H \to \mathbb{K}$ and $G : H \to H$ of the functional equation (2). $F \equiv 0$ and $F \equiv 1$ are the only constant solutions of (2) for an arbitrary continuous function G.

From now on, we suppose that F is a nonconstant continuous solution of (2). This implies that *G is not constant.*

We have by (2):

$$\begin{aligned} F(x)F(y)F(z) &= F(xG(y))F(z) = F(xG(y)G(z)) \\ &= F(x)F(yG(z)) = F(xG(yG(z))). \end{aligned}$$

We deduce:

$$F(xG(yG(z))) = F(xG(y)G(z)) \qquad (x, y, z \in H). \tag{6}$$

Since F is not identically zero, there exists x_0 in H such that $F(x_0) \neq 0$ and we have from (2):

$$F(y) = \varphi(G(y)) \qquad (y \in H) \tag{7}$$

where $\varphi : H \to K$ is the continuous function defined by:

$$\varphi(y) = \frac{F(x_0 y)}{F(x_0)} \qquad (y \in H). \tag{8}$$

Using (2), (6), (7) and (8), we get:

$$\begin{aligned} F(xG(y)) &= \varphi(G(xG(y)) = \varphi(G(x)G(y)) \\ &= F(x)F(y) = \varphi(G(x))\varphi(G(y)). \end{aligned}$$

We deduce:

$$\varphi(G(x)G(y)) = \varphi(G(x))\varphi(G(y)) \qquad (x, y \in H). \tag{9}$$

So, if we determine the range of G, we can deduce $\varphi : G(H) \to K$ by using the continuous solutions of the Cauchy's power functional equation, and we get F from (7).

3. Case where H contains 0 and $H \setminus \{0\}$ is a multiplicative group

We have first the following result whose method of proof has been used in [3].

Lemma 1. *The only closed connected subsets H of $\mathbb{C}$ containing 0 such that $H \setminus \{0\}$ is a multiplicative group are : $H = \mathbb{C}$ and $H = \Gamma \cup \{0\}$ with $\Gamma = \{e^{\lambda a + nb};\ n \in \mathbb{Z},\ \lambda \in \mathbb{R}\}$ where $a, b \in \mathbb{C}$, $\operatorname{Re} a \neq 0$ and either $b = 0$ or $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$.*

Proof. The mapping h defined by: $h(x) = e^x \quad (x \in \mathbb{C})$ is a continuous homomorphism from the additive group $(\mathbb{C}, +)$ onto the multiplicative group $(\mathbb{C} \setminus \{0\}, .)$. Since $M = H \setminus \{0\}$ is a closed subgroup of $(\mathbb{C} \setminus \{0\}, .)$, $h^{-1}(M)$ is a closed additive subgroup of $(\mathbb{C}, +)$. We deduce that we have the following possibilities (cf. [2]):

(i) $h^{-1}(M) = a\mathbb{R}$;

(ii) $h^{-1}(M) = a\mathbb{Z}$ where a is some nonzero complex number;

(iii) $h^{-1}(M) = a\mathbb{Z} + b\mathbb{Z}$;

(iv) $h^{-1}(M) = a\mathbb{R} + b\mathbb{Z}$ where $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$;

(v) $h^{-1}(M) = \mathbb{C}$;

(vi) $h^{-1}(M) = \{0\}$.

Since H is connected, the cases (ii), (iii) and (vi) do not occur. The cases (v), (i) and (iv) lead to the result. □

Remark. The only closed connected subsets H of $\mathbb{C}$ containing 0 and included in $\mathbb{R}$, such that $H \setminus \{0\}$ is a multiplicative group, are $H = [0, +\infty)$ and $H = \mathbb{R}$, which correspond respectively to the cases $a \in \mathbb{R}$, $\frac{\operatorname{Im} b}{2\pi} \in \mathbb{Z}$ and $a \in \mathbb{R}$, $\frac{\operatorname{Im} b}{\pi} \in 2\mathbb{Z} + 1$.

If H is given by Lemma 1, we have the following result concerning (2).

Lemma 2. *Let us suppose that H is given by Lemma 1. If $F : H \to \mathbb{K}$, $G : H \to H$ are continuous solutions of (2) such that F is not constant, G is a nonconstant solution of the following functional equation:*

$$|G(yG(z))| = |G(y)|\, |G(z)] \qquad (y, z \in H). \tag{10}$$

Proof. If F, G are continuous solutions of (2), they satisfy (6).

If $G(y)G(z) \neq 0$, we define: $h(y,z) = \frac{G(yG(z))}{G(y)G(z)}$ which belongs to H. With x replaced by $\frac{x}{G(y)G(z)}$ in (6), we get: $F(xh(y,z)) = F(x)$ $(x \in H)$. Since F is not constant, we have $h(y,z) \neq 0$ and

$$F(x(h(y,z))^n) = F(x) \qquad (x \in H, n \in \mathbb{Z}). \tag{11}$$

If $|h(y,z) \neq 1$, (11) and the continuity of F at 0 would imply that F is constant, which is not the case. Therefore, we have $|h(y,z)| = 1$ i.e. (10).

If $G(y)G(z) = 0$, we have by (6): $F(xG(yG(z)) = F(0)$ $(x \in H)$. Since F is not constant, we have $G(yG(z)) = 0$ which implies (10). □

From the functional equation (10), we will first determine G and then we get F with (9) and (7).

Let $F : H \to \mathbb{K}$ and $G : H \to H$ be nonconstant continuous solutions of (2). We denote $N = G^{-1}(0)$ and *we suppose that $H \setminus N$ is connected if H is not included in* $\mathbb{R}$.

By (10), the function $h : (H \setminus N) \times (H \setminus N) \to H \setminus \{0\}$ defined by: $h(y,z) = \frac{G(yG(z))}{G(y)G(z)}$ $(y, z \in H \setminus N)$ takes its values in $U \cap H$.

If $H = \Gamma \cup \{0\}$ with $b = 0$, we have: $h(y,z) = 1 \quad (y, z \in H \setminus N)$.

If $H = \mathbb{R}$, $h(y,z)$ belongs to $\{-1, 1\}$ for all y and z in $H \setminus N$.

If $H = \Gamma \cup \{0\}$ is not included in $\mathbb{R}$ and $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$, we have: $h(y,z) = e^{\lambda a + nb} = e^{in(\beta' - \beta\frac{\alpha'}{\alpha})}$ $(y, z \in H \setminus N)$ with $a = \alpha + i\alpha'$, $b = \beta + i\beta'$, $\beta' - \beta\frac{\alpha'}{\alpha} \neq 0$. Since $H \setminus N$ is connected, there exists some n_0 in $\mathbb{Z}$ such that

$$h(y,z) = e^{in_0(\beta' - \beta\frac{\alpha'}{\alpha})} \qquad (y, z \in H \setminus N). \tag{12}$$

Moreover, if y belongs to $H \setminus N$, we have $G(y) = e^{\lambda a + nb}$ and $|G(y)| = e^{\lambda\alpha + n\beta}$, which implies: $\lambda = \frac{1}{\alpha}(\ln|G(y)| - n\beta)$. We deduce: $G(y) = |G(y)|^{\frac{a}{\alpha}} e^{in(\beta' - \beta\frac{\alpha'}{\alpha})}$. Since $H \setminus N$ is connected, $\{G(y).|G(y)|^{-\frac{a}{\alpha}};\ y \in H \setminus N\}$ is a connected subset of $\{e^{in(\beta' - \beta\frac{\alpha'}{\alpha})};\ n \in \mathbb{Z}\}$ which is a discrete set of points of U. Therefore, there exists some m_0 in $\mathbb{Z}$ such that we have: $G(y) = |G(y)|^{\frac{a}{\alpha}} e^{im_0(\beta' - \beta\frac{\alpha'}{\alpha})}$ $(y \in H \setminus N)$. The definition of $h(y,z)$, (10) and (12) imply: $e^{in_0(\beta' - \beta\frac{\alpha'}{\alpha})} = e^{-im_0(\beta' - \beta\frac{\alpha'}{\alpha})}$. We deduce:

$$G(y) = |G(y)|^{\frac{a}{\alpha}} e^{-in_0(\beta' - \beta\frac{\alpha'}{\alpha})} \qquad (y \in H \setminus N). \tag{13}$$

In a first case, we have the following result:

Proposition 2. *Let us suppose that H is given by Lemma 1. Let $F : H \to \mathbb{K}$ and $G : H \to H$ be nonconstant continuous solutions of* (2) *such that $H \setminus G^{-1}(0)$ is connected in the case where H is not included in $\mathbb{R}$.*
We suppose that there exists y and z in $H \setminus G^{-1}(0)$ such that
if $H = \mathbb{R}$, $h(y, z) = -1$
if H is not included in $\mathbb{R}$, $h(y, z)$ is not a root of 1.
Then, if $H = \mathbb{R}$, $G(x) = -d|x| \qquad (x \in \mathbb{R})$,
if H is not included in $\mathbb{R}$, we have $H = \mathbb{C}$ and

$$G(x) = \begin{cases} 0 & \textit{if } x = 0 \\ dx\theta(x) & \textit{if } x \neq 0 \end{cases} \tag{14}$$

where d is some positive real number and $\theta : \mathbb{C} \setminus \{0\} \to U$ is some nonconstant continuous function and, in both cases,

$$F(x) = \begin{cases} (d|x|)^{\gamma} & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{cases}$$

where γ is some element of $\mathbb{K}$ such that $\operatorname{Re} \gamma > 0$.

Proof. In the case $H = \mathbb{R}$, we deduce from (11) that F is an even function.

In the case where H is not included in $\mathbb{R}$, $\{(h(y, z)^n\}_{n \in \mathbb{Z}}$ is dense in U. The fact that H is closed, the continuity of F and (11) imply : $U \subset H$ and $F(\lambda x) = F(x)$ $(x \in H, \lambda \in U)$. Therefore, we have: $|x| \in H$ if $x \in H$ and $F(x) = F(|x| \frac{x}{|x|}) = F(|x|)$ $(x \in H \setminus \{0\})$.

So, in both cases, we have:

$$F(x) = F(|x|) \qquad (x \in H). \tag{15}$$

By (8), we deduce:

$$\varphi(y) = \varphi(|y|) \qquad (y \in H). \tag{16}$$

Therefore, we have by (9):

$$\varphi(|G(x)|\,|G(y)|) = \varphi(|G(x)|)\varphi(|G(y)|) \qquad (x, y \in H). \tag{17}$$

We first determine the range of $|G|$. (15) and (2) imply:

$$F(x|G(y)|) = F(x)F(y) \qquad (x, y \in H) \tag{18}$$

The functional equation (18) is of the form (2) where G is replaced by $|G|$. By Lemma 2, $|G| : H \to [0, +\infty)$ is a nonconstant continuous solution of the following functional equation:

$$|G(y.|G(z)|)| = |G(y)||G(z)| \qquad (y, z \in H) \tag{19}$$

which is nothing but the functional equation (4). We deduce from Theorem 1 of [3] and from the Remark following this theorem:

- in the case $H = \mathbb{R}$, $|G(x)| = \mathrm{Sup}(-cx, dx)$ $(x \in \mathbb{R})$ where c and d are nonnegative real numbers satisfying $d > -c$
- in the case $H = \mathbb{C}$ either
  (i) $|G(x)| = d|x| (x \in \mathbb{C})$ where d is some positive real number, or
  (ii) $|G(x)| = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \neq 0 \text{ and } \dfrac{x}{|x|} \in \mathcal{N} \\ |x|\psi\left(\dfrac{x}{|x|}\right) & \text{if } x \neq 0 \text{ and } \dfrac{x}{|x|} \notin \mathcal{N} \end{cases}$

  where $\psi : U \to [0, +\infty)$ is some continuous function with $\mathcal{N} = \psi^{-1}(0)$ such that $U \setminus \mathcal{N}$ is connected.

Let us now consider the case where $H = \Gamma \cup \{0\}$ is not included in $\mathbb{R}$ and $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$. In this case, we have (12) with $n_0 \neq 0$ and $\beta' - \beta\frac{\alpha'}{\alpha} \notin 2\pi\mathbb{Q}$. By (13), for $y \in H \setminus N$, we have $G(y) = e^{\lambda a + nb}$ with $(n + n_0)(\beta' - \beta\frac{\alpha'}{\alpha}) \in 2\pi\mathbb{Z}$. This implies: $n = -n_0$ and $G(y) = e^{\lambda a - n_0 b}$. Therefore, we have: $|G(y)| = e^{\mu\alpha}$ with $\mu = \lambda - n_0\frac{\beta}{\alpha}$, and so $|G|(H \setminus N) \subset e^{\alpha\mathbb{R}}$. Since $|G|$ is not constant, (19) implies: $G(0) = 0$. By the connectedness of H, we get: $|G|(H) = e^{\alpha I} \cup \{0\}$ where $I = (-\infty, \delta|$ for some δ ($|$ means either) or]). Using (19), we have by induction: $(|G|(H))^n \subset |G|(H)$ $(n \in \mathbb{N})$. We deduce: $|G|(H) = e^{\alpha\mathbb{R}} \cup \{0\} = [0, +\infty)$.

So, in all cases, we have $|G|(H) = [0, +\infty)$. From (17) we see, by letting $u = |G(x)|$, $v = |G(y)|$, that $\varphi : [0, +\infty) \to \mathbb{K}$ is a continuous solution of the Cauchy's power functional equation: $\varphi(uv) = \varphi(u)\varphi(v)$. Since F is not constant, φ is not constant and we get (cf. [1]):

$\varphi(u) = \begin{cases} u^\gamma & \text{if } u > 0 \\ 0 & \text{if } u = 0 \end{cases}$ where γ is some element of $\mathbb{K}$ such that $\mathrm{Re}\,\gamma > 0$.

We deduce from (7) and (16):

$$F(x) = \begin{cases} |G(x)|^{\gamma} & \text{if } x \notin N \\ 0 & \text{if } x \in N \end{cases}.$$

This implies with (15): $x \in N \iff |x| \in N$ and

$$|G(x)| = |G(|x|)| \qquad (x \in H) \tag{20}$$

Finally, we determine G.

In the case $H = \mathbb{R}$, (20) implies: $|G(x)| = c|x| (x \in \mathbb{R})$ with $c > 0$ and we get either $G(x) = dx (x \in \mathbb{R})$ or $G(x) = d|x|$ $(x \in \mathbb{R})$ where d is some nonzero real number. Since we assume $h(y, z) = -1$ for some y and z in $\mathbb{R}$, we have: $G(x) = -d|x|$ $(x \in \mathbb{R})$ where d is some positive real number.

In the case $H = \mathbb{C}$, the form (ii) of $|G|$ satisfies (20) if, and only if, $\psi \equiv d$ where d is some positive real number. Therefore, $|G|$ has the form (i) and we deduce (14) in this case. The hypothesis that $h(y, z)$ is not a root of 1 for some y and z in $\mathbb{C}$ implies that θ is not constant.

Let us finally consider the case where $H = \Gamma \cup \{0\}$ is not included in $\mathbb{R}$ and $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$. If we suppose $G(y_0) = 0$ for some y_0 in Γ, we have by (19): $G(\lambda y_0) = 0$ $(\lambda \geq 0)$. We get from (20): $G(\lambda |y_0|) = 0$ $(\lambda \geq 0)$ which implies: $G(\lambda) = 0$ $(\lambda \geq 0)$. (20) implies that G is identically zero which is not the case. Therefore, we have $N = \{0\}$. So, $H \setminus \{0\}$ is connected and is therefore of the form: $H \setminus \{0\} = \{z \in \mathbb{C} : z = |z|^{\frac{a}{\alpha}} e^{ip_0(\beta' - \beta\frac{\alpha'}{\alpha})}\}$ for some $p_0 \in \mathbb{Z}$. This brings a contradiction with the fact that $|G|(H) = [0, +\infty) \subset H$. Therefore, this case does not occur. □

In the other case, we have the following result:

Proposition 3. *Let us suppose that H is given by Lemma 1. Let $F : H \to \mathbb{K}$ and $G : H \to H$ be nonconstant continuous solutions of* (2) *such that $H \setminus G^{-1}(0)$ is connected in the case where H is not included in $\mathbb{R}$.*

We suppose that, for all y and z in $H \setminus G^{-1}(0)$, $h(y, z)$ is either equal to 1 in the case $H = \mathbb{R}$, or a root of 1 in the case where H is not included in $\mathbb{R}$.

Then, • *in the the case where* $H = \Gamma \cup \{0\}$ *is not included in* $\mathbb{R}$ *and* $\{a, b\}$ *is a basis of the real vector space* $\mathbb{C}$*,*

$$G(x) = \begin{cases} dx & \text{if } x \in e^{a\mathbb{R}+p_0 b} \\ 0 & \text{if } x \notin e^{a\mathbb{R}+p_0 b} \end{cases} \tag{21}$$

where p_0 *is some integer in* $\mathbb{Z}$ *and* d *is some element of* $e^{a\mathbb{R}-p_0 b}$*,*

• *in the cases* $H = \mathbb{R}$*,* $H = \Gamma \cup \{0\}$ *with* $b = 0$*,* $H = \mathbb{C}$*, either*

$$G(x) = dx \qquad (x \in H) \tag{22}$$

where d *is some element of* $H \setminus \{0\}$ *or, in the case* $H = \mathbb{R}$ *only,*

$$G(x) = \mathrm{Sup}(-cx, dx) \qquad (x \in \mathbb{R}) \tag{23}$$

where c *and* d *are some nonnegative real numbers satisfying* $d > -c$*,*
or, in the case $H = \mathbb{C}$ *only,*

$$G(x) = \begin{cases} 0 \text{ if } x = 0 \text{ or} & \text{if } x \neq 0 \text{ and } \dfrac{x}{|x|^{1+i\delta}} \in \mathcal{N} \\ e^{\frac{2ip\pi}{n}} \left(|x| \psi\left(\dfrac{x}{|x|^{1+i\delta}} \right) \right)^{1+i\delta} & \text{if } x \neq 0 \text{ and } \dfrac{x}{|x|^{1+i\delta}} \notin \mathcal{N} \end{cases} \tag{24}$$

where p *is some integer in* $\mathbb{Z}$*,* n *is some positive integer,* δ *is some real number,* $\psi : U \to [0, +\infty[$ *is some continuous function which satisfies:* $\psi(e^{\frac{i\pi}{n}} x) = \psi(x)(x \in U)$ *in the case* $p \neq kn$ $(k \in \mathbb{Z})$*,* $\mathcal{N} = \psi^{-1}(0)$ *and* $U \setminus \mathcal{N}$ *is connected,*
and, in all cases,

$$F(x) = \begin{cases} |G(x)|^{\gamma} & \text{if } G(x) \neq 0 \\ 0 & \text{if } G(x) = 0 \end{cases} \tag{25}$$

where γ *is some element of* $\mathbb{K}$ *such that* $\mathrm{Re}\,\gamma > 0$*,*
in the case $H = \mathbb{R}$ *and* (22)*,*

$$F(x) = \begin{cases} |G(x)|^{\gamma} \operatorname{sign} G(x) & \text{if } G(x) \neq 0 \\ 0 & \text{if } G(x) = 0 \end{cases} \tag{26}$$

where γ *is some element of* $\mathbb{K}$ *such that* $\mathrm{Re}\,\gamma > 0$*,*
in the case $H = \mathbb{C}$ *and* (22)*,*

$$F(x) = \begin{cases} |G(x)|^{\gamma} (G(x))^{k} & \text{if } G(x) \neq 0 \\ 0 & \text{if } G(x) = 0 \end{cases} \tag{27}$$

where k belongs to $\mathbb{Z}$ and γ is some element of $\mathbb{K}$ such that $\operatorname{Re}\gamma > -k$.

Proof. We noticed already that we have $h(y,z) = 1$ $(y,z \in H \setminus N)$ in the case $H = \Gamma \cup \{0\}$ with $b = 0$. So, in the cases $H = \mathbb{R}$ and $H = \Gamma \cup \{0\}$ with $b = 0$, $G : H \to H$ is a nonconstant continuous solution of the functional equation:

$$G(xG(y)) = G(x)G(y) \qquad (x, y \in H). \tag{4bis}$$

If $H = \mathbb{R}$, (4bis) is nothing but (4) and Theorem 1 of [3] implies that G has either the form (22) or the form (23).

We have in this case either $G(H) = [0, +\infty)$ or $G(H) = \mathbb{R}$. By (9), $\varphi : H \to \mathbb{K}$ is a nonconstant continuous solution of the Cauchy's power functional equation. We deduce (cf. [1]): either

$$\varphi(u) = \begin{cases} |u|^\gamma & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases} \quad \text{or} \quad \varphi(u) = \begin{cases} |u|^\gamma \text{sign} u & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$$

where $\operatorname{sgn} u = \frac{u}{|u|}$ and γ is some element of $\mathbb{K}$ such that $\operatorname{Re}\gamma > 0$. From (7), we deduce the forms (25) and (26) for F.

In the case $H = \Gamma \cup \{0\}$ with $b = 0$, since $G(H)$ is a connected part of H containing 0, we have $G(H) = e^{aI} \cup \{0\}$ where $I = (-\infty, \delta|$ for some real number δ. We get from (4bis) by induction: $(G(H))^n \subset G(H)$ $(n \in \mathbb{N})$ which implies $G(H) = H$. If $G(y_0) = 0$ for $y_0 \in \Gamma$, we get: $G(y_0 G(y)) = 0$ $(y \in H)$, which contradicts $G \not\equiv 0$. We deduce $N = \{0\}$. The function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying: $G(e^{\lambda a}) = e^{a\phi(\lambda)}(\lambda \in \mathbb{R})$ is defined by: $\phi(\lambda) = \frac{1}{\alpha} \ln |G(e^{\lambda a})| (\lambda \in \mathbb{R})$, and, by (4bis), ϕ is a nonconstant continuous solution of the functional equation:

$$\phi(\lambda + \phi(\mu)) = \phi(\lambda) + \phi(\mu) \qquad (\lambda, \mu \in \mathbb{R}). \tag{28}$$

We get from [5]: $\phi(\lambda) = \lambda + \eta$ $(\lambda \in \mathbb{R})$ where η is an arbitrary real number and we deduce that G has the form (22).

In the case $H = \mathbb{C}$, the hypothesis, the connectedness of $\mathbb{C} \setminus N$ and the continuity of G imply that the range of h is a connected part of $e^{2i\pi\mathbb{Q}}$, which is totally disconnected. Therefore, h is a constant function and there exists λ_0 in $e^{2i\pi\mathbb{Q}}$ such that we have with (11):

$$\begin{cases} G(xG(y)) = \lambda_0 G(x)G(y) \\ F(\lambda_0 x) = F(x) \end{cases} \qquad (x, y \in \mathbb{C}). \tag{29}$$

In this case, $G : \mathbb{C} \to \mathbb{C}$ is a nonconstant continuous solution of the functional equation (3) which satisfies (29) and such that $\mathbb{C} \setminus N$ is connected. Therefore, it has one of the forms given in Theorem 2 of [3].

G of the form (22), where d is some nonzero complex number, satisfies (29) if, and only if, $\lambda_0 = 1$. In this case, we have $G(\mathbb{C}) = \mathbb{C}$. Therefore, by (9), $\varphi : \mathbb{C} \to \mathbb{K}$ is a nonconstant continuous solution of the Cauchy's power functional equation: $\varphi(uv) = \varphi(u)\varphi(v)$. If u and v belong to $\mathbb{C} \setminus \{0\}$, we see, by letting $u = e^x$, $v = e^y$, that the function $\phi : \mathbb{C} \to \mathbb{K}$ defined by: $\phi(x) = \varphi(e^x)$ $(x \in \mathbb{C})$ is a nonconstant continuous solution of the Cauchy's exponential functional equation: $\phi(x + y) = \phi(x).\phi(y)$ $(x, y \in \mathbb{C})$. We get from [1]: $\phi(x) = e^{\gamma x + \delta \bar{x}}$ $(x \in \mathbb{C})$ where γ and δ are some complex numbers. This implies: $\varphi(u) = u^{\gamma} \bar{u}^{\delta}$ $u \in \mathbb{C} \setminus \{0\})$. Such a function is continuous on $\mathbb{C} \setminus (-\infty, 0]$. φ is continuous on $(-\infty, 0)$ if, and only if, $e^{i\pi(\gamma-\delta)} = e^{-i\pi(\gamma-\delta)}$ i.e. $\gamma - \delta = k \in \mathbb{Z}$. We deduce: $\varphi(u) = |u|^{\gamma} \bar{u}^{-k}$ $(u \in \mathbb{C} \setminus \{0\})$. This function is continuous at 0 if, and only if, $\operatorname{Re} \gamma > k$. From (7), we get the form (27) for F.

G of the form: $G(x) = \begin{cases} d|x|\theta(|x|) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, where d is some positive real number and $\theta : (0, +\infty) \to U$ is some continuous function, satisfies (29) if and only if: $\theta(d|x|\,|y|) = \lambda_0 \theta(|x|)\theta(|y|)$ $(x, y \in \mathbb{C} \setminus \{0\})$. By letting $u = d|x|$, $v = d|y|$, we see that the function $\tau : (0, +\infty) \to U$ defined by: $\tau(u) = \lambda_0 \theta(\frac{u}{d})$ $(u > 0)$ is a continuous solution of the Cauchy's power functional equation: $\tau(uv) = \tau(u)\tau(v)$. We deduce (cf. [1]): $\tau(u) = u^{i\delta}$ $(u > 0)$ where δ is some real number, and we get: $G(x) = \frac{1}{\lambda_0}(d|x|)^{1+i\delta}$ $(x \in \mathbb{C} \setminus \{0\})$. Therefore, in this case, G is of the form (24) with $\psi \equiv d$.

If G is of the form (24), we have: $G(\mathbb{C}) = e^{\frac{2ip\pi}{n}} e^{(1+i\delta)I} \cup \{0\}$ where $I = (-\infty, \delta|$ for some real number δ. Using (10), we have by induction: $(|G|(x))^n \in |G|(\mathbb{C})$ $(n \in \mathbb{N}, x \in \mathbb{C})$. We deduce:

$$G(\mathbb{C}) = e^{\frac{2ip\pi}{n}} e^{(1+i\delta)\mathbb{R}} \cup \{0\} = \frac{1}{\lambda_0} e^{(1+i\delta)\mathbb{R}} \cup \{0\}.$$

Let us consider now the case where $H = \Gamma \cup \{0\}$ is not included in $\mathbb{R}$ and $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$. We have (12) and (13) with $\beta' - \beta\frac{\alpha'}{\alpha} \in 2\pi\mathbb{Q}$. Since $G(H)$ is a connected part of H containing 0, we get from (13): $G(H) = e^{aI} e^{-in_0(\beta' - \beta\frac{\alpha'}{\alpha})} \cup \{0\}$ where $I = (-\infty, \delta|$ for some real number δ. By using (10), we prove as before that $I = \mathbb{R}$ and we deduce: $G(H) = \frac{1}{\lambda_0} e^{a\mathbb{R}} \cup \{0\}$.

In the three cases: $H = \mathbb{C}$ and G of the form (24), $H = \Gamma \cup \{0\}$ with $b = 0$, $H = \Gamma \cup \{0\}$ not included in $\mathbb{R}$ and $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$, we have: $G(H) = \frac{1}{\lambda_0} e^{\eta\mathbb{R}} \cup \{0\}$ where η is respectively $(1 + i\delta)$ and a. Since by (29) we have: $\varphi(\lambda_0 x) = \varphi(x)$ $(x \in H)$, the function $\phi : \mathbb{R} \to \mathbb{K}$ defined by: $\phi(x) = \varphi(e^{\eta x})$ $(x \in \mathbb{R})$ is by (9) a nonconstant continuous solution of the Cauchy's exponential functional equation: $\phi(x + y) = \phi(x) \cdot \phi(y)$. We get from [1]: $\phi(x) = e^{\gamma x}$ $(x \in \mathbb{R})$ where γ is some element of $\mathbb{K}$. Since φ is continuous at 0 and $\varphi(0) = 0$, we deduce: $\varphi(u) = \begin{cases} |u|^\gamma & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$ where γ is some element of $\mathbb{K}$ such that $\operatorname{Re}\gamma > 0$. We obtain the form (25) for F.

Let us now determine G in the case where $H = \Gamma \cup \{0\}$ is not included in $\mathbb{R}$ and $\{a, b\}$ is a basis of the real vector space $\mathbb{C}$. Since $H \setminus N$ is connected, in the same way as we proved (13), we can prove that there exists some p_0 in $\mathbb{Z}$ such that we have: $x = |x|^{\frac{a}{\alpha}} e^{ip_0(\beta' - \beta\frac{\alpha'}{\alpha})}$ $(x \in H \setminus N)$. Therefore, we have: $H \setminus N = e^{aI} e^{ip_0(\beta' - \beta\frac{\alpha'}{\alpha})}$ where I is an interval of $\mathbb{R}$. Let us suppose $I \neq \mathbb{R}$. Then, there exists y_0 in $e^{a\mathbb{R}} e^{ip_0(\beta' - \beta\frac{\alpha'}{\alpha})}$ such that $G(y_0) = 0$. (10) and (13) imply: $G(y_0 e^{a\mathbb{R}} e^{-in_0(\beta' - \beta\frac{\alpha'}{\alpha})}) = 0$. However, by (11), (12) and (25), we have also:

$$|G(x)| = |G(e^{in_0(\beta' - \beta\frac{\alpha'}{\alpha})} x)| \qquad (x \in H). \tag{30}$$

We deduce: $G(y_0 e^{a\mathbb{R}}) = G(e^{a\mathbb{R}} e^{ip_0(\beta' - \beta\frac{\alpha'}{\alpha})}) = 0$ which brings the contradiction. So, we have obtained:

$$H \setminus N = e^{a\mathbb{R}} e^{ip_0(\beta' - \beta\frac{\alpha'}{\alpha})} = e^{a\mathbb{R} + p_0 b} \tag{31}$$

By (30), if x belongs to $H \setminus N$, $e^{ikn_0(\beta' - \beta\frac{\alpha'}{\alpha})} x$ belongs also to $H \setminus N$ for all k in $\mathbb{Z}$. (31) implies: $n_0(\beta' - \beta\frac{\alpha'}{\alpha}) \in 2\pi\mathbb{Z}$. We deduce: $h(y, z) = 1$ and, by (13), $G(H) = e^{a\mathbb{R}} \cup \{0\}$. The definition of $h(y, z)$ implies that $G : H \to \mathbb{K}$ is a nonconstant continuous solution of (4bis). The function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying: $G(e^{\lambda a} e^{ip_0(\beta' - \beta\frac{\alpha'}{\alpha})}) = e^{\phi(\lambda) a} (\lambda \in \mathbb{R})$ is defined by: $\phi(\lambda) = \frac{1}{\alpha} \ln |G(e^{\lambda a} e^{ip_0(\beta' - \beta\frac{\alpha'}{\alpha})})|$ $(\lambda \in \mathbb{R})$, and (4bis) implies that ϕ is a nonconstant continuous solution of the functional equation (28). We get from [5]: $\phi(\lambda) = \lambda + \eta$ ($\lambda \in \mathbb{R}$) where η is an arbitrary real number. We deduce that G has the form (21). □

Since all the forms of G and F given in the Propositions 2 and 3 are solutions of (2), we have got the following result.

Theorem 1. *Let us suppose that H is given by Lemma 1. All continuous solutions $F : H \to K$ and $G : H \to H$ of the functional equation* (2) *such that $H \setminus G^{-1}(0)$ is connected in the case where H is not included in $\mathbb{R}$, are given by: either*

(i) $F \equiv 0$ *or* $F \equiv 1$, G *arbitrary, or*

(ii) *in the cases* $H = \mathbb{R}$, $H = \Gamma \cup \{0\}$ *with* $b = 0$, $H = \mathbb{C}$, G *is given by* (22), *or*

(iii) *in the case* $H = \mathbb{R}$ *only , either* G *is of the form* (23), *or* $G(x) = -d|x|$ ($x \in \mathbb{R}$) *where* d *is an arbitrary positive real number, or*

(iv) *in the case where* $H = \Gamma \cup \{0\}$ *is not included in* $\mathbb{R}$ *and* $\{a, b\}$ *is a basis of the real vector space* $\mathbb{C}$, G *is of the form* (21), *or*

(v) *in the case* $H = \mathbb{C}$ *only,* G *is either of the form* (14) *or of the form* (24),

and, in all cases F *is given by* (25), *in the case* $H = \mathbb{R}$ *and* (22) *only* F *is given by* (26), *in the case* $H = \mathbb{C}$ *and* (22) *only* F *is given by* (27).

4. Case where H does not contain 0 or $H \setminus \{0\}$ is a not multiplicative group.

We shall restrict ourselves to the case where H is an interval of $\mathbb{R}$. The only possibilities for H such that H does not contain 0 or $H \setminus \{0\}$ is a not multiplicative group, but $H^2 \subseteq H$, are:

$$H = (0, +\infty);\ H = |a, +\infty), \quad a \geq 1;$$

$$H = |b, a|, \quad -1 \leq b \leq 0 < a \leq 1,\ b^2 \leq a$$

where | means either (or [or) or].

In order to get φ, and so F, from (9), we shall use the following result that we can obtain from [6] or [1]:

Lemma 3. *If I is a subinterval of $(0,+\infty)$, all the continuous solutions $\varphi : I \cup I^2 \to \mathbb{K} \setminus \{0\}$ of the Cauchy's power functional equation: $\varphi(uv) = \varphi(u)\varphi(v)(u, v \in I)$ are given by:*

$$\varphi(u) = \begin{cases} Au^{\gamma} & (u \in I) \\ A^2u^{\gamma} & (u \in I^2) \end{cases}$$

where A and γ are arbitrary elements of K such that $A \neq 0$ ($A = 1$ if $I^2 \cap I \neq \emptyset$).

In order to apply this Lemma to (9), we shall study $F^{-1}(0)$ and $G^{-1}(0)$ when F and G are continuous solutions of (2) and F is not constant.

We first remark that, if 0 belongs to H, we have $F(0) = 0$, since $x = 0$ in (2) gives: $F(0) = F(0)F(y)$ $(y \in H)$.

We have the following result.

Lemma 4. *Let $F : H \to \mathbb{K}$ and $G : H \to H$ be continuous solutions of (2) such that F is not constant. Then, we have: $F^{-1}(0) = G^{-1}(0)$.*

If $0 \notin H$, F and G do not vanish.

If $0 \in H$, either F and G vanish only at 0, or, in the case $H = |b, a|$ with $b < 0$ only, we may have:

$$\begin{cases} F(x) = G(x) = 0 & \forall x \geq 0 \text{ (resp. } x \leq 0) \\ F(x) \neq 0, G(x) > 0 & \forall x < 0 \text{ (resp. } x > 0). \end{cases}$$

Proof. We denote $I = G(H)$, which is a nontrivial interval of $\mathbb{R}$ included in H.

Let us suppose that there exists y_0 in H such that $F(y_0) = 0$.

First, in the case $H = (0, +\infty)$, we have $G(y_0) > 0$ and (2) with x replaced by $\frac{x}{G(y_0)}$ and $y = y_0$ implies that F is identically zero, which is not the case. Therefore, F does not vanish in this case.

In the other cases, we have by (2):

$$\text{either} \qquad G(y_0) \neq 0 \quad \text{and} \quad F(x) = 0 \quad (x \in G(y_0).H) \tag{32}$$

$$\text{or} \qquad G(y_0) = 0 \quad \text{and} \quad F(x) = 0 \quad (x \in y_0.I). \tag{33}$$

Since F is not identically zero, the continuity of F implies that there exists z_0 in $H \setminus \{0\}$ such that $0 < |F(z_0)| < 1$. Letting $x = z_0.(G(z_0))^{n-1}$, $y = z_0$

in (2), we get by induction:

$$F(z_0.(G(z_0))^n) = (F(z_0))^{n+1} \neq 0 \qquad (n \in \mathbb{N}). \tag{34}$$

Since $0 < |F(z_0)| < 1$, we have $\lim_{n\to+\infty}(F(z_0))^{n+1} = 0$ and therefore $|G(z_0)| \neq 1$.

If $H = |a, +\infty)$, $a \geq 1$, we have $G(z_0) > 1$ and there exists n in $\mathbb{N}$ such that $z_0.(G(z_0))^n$ belongs to $G(y_0).H$. This is impossible by (32) and (34). Therefore, F does not vanish in this case.

If $H = |b, a|$ with $-1 \leq b \leq 0 < a \leq 1$, we have $|G(z_0)| < 1$. Let us suppose first $G(y_0) \neq 0$. Then, there exists n in $\mathbb{N}$ such that $z_0.(G(z_0))^n$ belongs to $G(y_0).H$. This is impossible by (32) and (34). We deduce first that, if $0 \notin H$, F does not vanish.

Then, if $0 \in H$, we must have $G(y_0) = 0$. By (2), we get:

$$\text{if} \quad 0 \in H, \qquad F(y_0) = 0 \Longleftrightarrow G(y_0) = 0. \tag{35}$$

In particular, we have in this case $F(0) = G(0) = 0$. Suppose now that there exists $y_0 \neq 0$ in H such that $F(y_0) = 0$. We have $G(y_0) = 0$. But, there exists n in $\mathbb{N}$ such that $z_0.(G(z_0))^n$ belongs to $y_0.I$, except maybe in the case where b is negative and I is included in $[0, +\infty)$. This is impossible by (33) and (34). Therefore, by (35), except in the latter case, if $0 \in H$, F and G vanish only at 0.

Let us consider now the case $H = |b, a|$, $b < 0$, $I \subset [0, +\infty)$. Let us suppose that $y_0 > 0$ satisfies $F(y_0) = 0$. Then, by (35), we have $G(y_0) = 0$. If F is not identically zero on $[0, a|$, there exists z_0 in $(0, a)$ such that $0 < |F(z_0)| < 1$ and there exists n in $\mathbb{N}$ such that $z_0.(G(z_0))^n$ belongs to $y_0 I$. This is impossible by (33) and (34). Therefore, F is identically zero on $[0, a|$. Similarly, if there exists y_0 in $|b, 0)$ such that $F(y_0) = 0$, F is identically zero on $|b, 0]$. Using (35), we deduce that we may have in this case:

$$\text{either} \quad F(x) = G(x) = 0 \ \forall x \leq 0; \quad F(x) \neq 0, \quad G(x) > 0 \quad \forall x > 0$$

$$\text{or} \quad F(x) = G(x) = 0 \ \forall x \geq 0; \quad F(x) \neq 0, \quad G(x) > 0 \quad \forall x < 0. \qquad \square$$

Using Lemmas 3 and 4, we shall now prove the following result.

Theorem 2. *If $H = (0, +\infty)$ or $H = |a, +\infty)$ with $a \geq 1$ or $H = |b, a|$ with $-1 \leq b \leq 0 < a \leq 1$, $b^2 \leq a$, all continuous solutions $F : H \to K$ and $G : H \to H$ of the functional equation (2) are given by:*

- *$F \equiv 0$ or $F \equiv 1$, G arbitrary,*
- $$G(x) = cx \qquad (x \in H) \tag{36}$$

where $c > 0$ if $H = (0, +\infty)$, $c \geq 1$ if $H = |a, +\infty)$, $0 < c \leq 1$ if $H = (0, a|$, $\mathrm{Sup}(\frac{b}{a}, \frac{a}{b}) \leq c \leq 1$, $c \neq 0$ if $H = |b, a|$, $b < 0$, ($\mathrm{Sup}(\frac{b}{a}, \frac{a}{b}) < c \leq 1$ if $H = (b, a]$, $a \geq |b|$, or $H = [b, a)$, $a \leq |b|$)

and, in the case $H = |b, a|$ with $b < 0$ only,

$$G(x) = -c|x| (x \in H) \text{ with } 0 < c \leq \mathrm{Inf}\left(1, \frac{|b|}{a}\right)$$
$$\left(c < \mathrm{Inf}\left(1, \frac{|b|}{a}\right)\right) \text{ if } H = (b, a],\ |b| \leq a) \tag{37}$$

$$G(x) = \mathrm{Sup}(-c_2 x, c_1 x)\ (x \in H) \text{ with } 0 \leq c_1 \leq 1,\ 0 \leq c_2 \leq \frac{a}{|b|},$$
$$c_1 + c_2 \neq 0 \qquad \left(c_2 < \frac{a}{|b|} \text{ if } H = [b, a)\right), \tag{38}$$

- *if $0 \notin H$, $F(x) = (G(x))^\gamma$ $(x \in H)$ where γ is an arbitrary nonzero element of $\mathbb{K}$,*

if $0 \in H$,

$$F(x) = \begin{cases} |G(x)|^\gamma & \text{if } G(x) \neq 0 \\ 0 & \text{if } G(x) = 0 \end{cases} \tag{39}$$

where γ is some element of $\mathbb{K}$ such that $\mathrm{Re}\,\gamma > 0$,
and, in the case where $H = |b, a|$ and G is of the form (36) only,

$$F(x) = \begin{cases} |G(x)|^\gamma \,\mathrm{sign}\, G(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{40}$$

where γ is some element of $\mathbb{K}$ such that $\mathrm{Re}\,\gamma > 0$.

Proof. Let $F : H \to \mathbb{K}$ and $G : H \to H$ be continuous solutions of (2) such that F is not constant.

1. We consider first the cases: $H = (0, +\infty)$, or $H = |a, +\infty)$, $a \geq 1$, or $H = (0, a|$, $a \leq 1$.

By Lemma 4, F and G do not vanish. So, by (9) and with $I = G(H)$, $\varphi : I \cup I^2 \subseteq H \subseteq (0, +\infty) \to \mathbb{K} \setminus \{0\}$ is a nonconstant continuous solution of the Cauchy's power functional equation. By Lemma 3, we have:
$\varphi(u) = Au^\gamma (u \in I)$ where A and γ are some nonzero elements of $\mathbb{K}$. By (7), we get: $F(x) = A(G(x))^\gamma$ $(x \in H)$. With (2) we obtain:

$$G(xG(y))^\gamma = A(G(x))^\gamma (G(y))^\gamma \qquad (x, y \in H). \tag{41}$$

If $\operatorname{Re}\gamma \neq 0$, we get: $G(xG(y)) = BG(x)\ G(y)$ $(x, y \in H)$ with $B = |A|^{\frac{1}{\operatorname{Re}\gamma}}$. If $\operatorname{Re}\gamma = 0$, we have $\gamma = e^{ic}$, $c \neq 0$. (41) implies that $A = e^{ic'}$ and, by the continuity of G, there exists n in $\mathbb{Z}$ such that: $G(xG(y)) = e^{\frac{2\pi n + c'}{c}} G(x)G(y)$ $(x, y \in H)$.

So, in all cases, there exists $B > 0$ such that:

$$G(xG(y)) = BG(x)G(y) \qquad (x, y \in H). \tag{42}$$

This implies:

$$G(xt) = BG(x)t \qquad (x \in H, t \in I). \tag{43}$$

Let us now determine I. By (42) we have by induction:

$$G(x(G(y))^n) = G(x)(BG(y))^n \qquad (x \in H, t \in I). \tag{44}$$

Since G is not constant, there exists y in I such that $BG(y) \neq 1$.

In the case $H = (0, +\infty)$, since the formula (44) is true for all n in $\mathbb{Z}$, we have with for example $BG(y) > 1$: $\lim_{n\to+\infty}(BG(y))^n = +\infty$ and $\lim_{n\to-\infty}(BG(y))^n = 0$. We deduce from (44): $I = H = (0, +\infty)$.

In the other cases: if $BG(y) > 1$, we have $\lim_{n\to+\infty}(BG(y))^n = +\infty$, and (44) implies: $I = |a_1, +\infty)$, $H = |a, +\infty)$ with $a_1 \geq a$,
if $BG(y) < 1$, we have $\lim\limits_{n\to+\infty}(BG(y))^n = 0$, and (44) implies: $I = (0, a_1|$, $H = (0, a|$ with $a_1 \leq a$.

If $I = H = (0, +\infty)$, (43) with $x = 1$ implies the expression (36) for G.

In the other cases, we have from (43): $G(x) = \frac{G(xt)}{Bt}$ $(x \in H,\ t \in I)$. Since at belongs to H, the continuity of G on H implies that
$\delta = \lim_{x\to a, x\in H} G(x)$ exists and is positive. We have:

$$G(at) = B\delta t \qquad (t \in I). \tag{45}$$

Now, for all x and y in H, $xG(y)$ belongs to aI and we have by (42) and (45): $G(x) = \frac{G(xG(y))}{BG(y)} = \frac{\delta}{a}x$. So, in all these cases, we have the expression (36) for G. The conditions on c are given by the fact that G takes its values in H. (41) implies now $A = 1$, and therefore $F(x) = (G(x))^\gamma (x \in H)$.

2. Let us consider now the case $H = [0, a|$, $a \le 1$.

By Lemma 4 F and G vanish only at 0. Therefore, $F : (0, a| \to \mathbb{K} \setminus \{0\}$ and $G : (0, a| \to (0, a|$ are nonconstant continuous solutions of (2). Using the continuity of F and G at 0, we deduce from the previous case that G has the form (36) and F has the form (39).

3. Finally let us consider the case $H = |b, a|$, $-1 \le b < 0 < a \le 1$, $b^2 \le a$.

3.1. We shall first investigate the case $I = [0, \alpha| \subseteq [0, a|$.

By Lemma 4 and (7) we have: $F(y) = \varphi(G(y)) = 0 \iff G(y) = 0$. Therefore, φ does not vanish on $(0, \alpha|$. By (9), Lemma 3 and the fact that $(0, \alpha^2| \subset (0, \alpha|$, we have: $\varphi(u) = u^\gamma$ $(u \in (0, \alpha|)$ where γ is some nonzero element of $\mathbb{K}$. Lemma 4, (7), (8) and the continuity of F imply that F has the form (39) with $\operatorname{Re}\gamma > 0$. Using (2), we get:

$$G(xG(y)) = G(x)G(y) \qquad (x, y \in H). \tag{46}$$

This implies: $G(xt) = G(x).t$ $(x \in H,\ t \in I)$. Let us fix $t > 0$ in I. Since at and bt belong to H, the continuity of G on I implies that $\delta_1 = \lim_{x \to a-0} G(x) = \frac{G(at)}{t}$ and $\delta_2 = \lim_{x \to b+0} G(x) = \frac{G(bt)}{t}$ exist and are nonnegative. We deduce:

$$G(x) = \begin{cases} c_1 x & \text{if } x \in [0, a\alpha| \\ c_2 x & \text{if } x \in |b\alpha, 0] \end{cases} \qquad \text{with} \quad c_1 \ge 0 \quad \text{and} \quad c_2 \le 0 \tag{47}$$

Now, for all x and y in H with $G(y) \neq 0$, $xG(y)$ belongs to αH and we have by (46) and (47):

$$G(x) = \frac{G(xG(y))}{G(y)} = \begin{cases} c_1 x & \text{if } x \in [0, a| \\ c_2 x & \text{if } x \in |b, 0]. \end{cases}$$

We deduce that G has the form (38). The conditions on c_1 and c_2 come from the fact that G is not identically zero and takes its values in H.

3.2. Let us investigate now the case $I = |\beta, 0] \subseteq |b, 0]$.

Like in §3.1. φ does not vanish on $|\beta, 0)$ and satisfies: $\varphi(uv) = \varphi(u)\varphi(v)$ $(u, v \in |\beta, 0))$. Let us denote: $\lambda = \varphi(u_0) \neq 0$ for some fixed u_0 in $|\beta, 0)$. The function $\phi : (0, \frac{\beta}{u_0}| \to \mathbb{K} \setminus \{0\}$ defined by: $\phi(x) = \frac{1}{\lambda^2}\varphi(u_0^2 x)$ $(x \in (0, \frac{\beta}{u_0}|)$ is a nonconstant continuous solution of the Cauchy's power functional equation. We get from Lemma 3: $\phi(x) = x^\gamma (x \in (0, \frac{\beta}{u_0}|)$ where γ is some element of $\mathbb{K}$. We deduce: $\varphi(u) = \dfrac{1}{\lambda}\varphi(u_0 u) = \lambda\phi(\dfrac{u}{u_0}) = A|u|^\gamma$ $(u \in |\beta, 0))$ where A is some element of $\mathbb{K} \setminus \{0\}$. The continuity of F at 0 and (7) imply:

$$F(x) = \begin{cases} A|G(x)|^\gamma & \text{if } G(x) \neq 0 \\ 0 & \text{if } G(x) = 0 \end{cases}$$

with $\operatorname{Re}\gamma > 0$. Using (2), we see that there exists $B = |A|^{\frac{1}{\operatorname{Re}\gamma}} > 0$ such that:

$$G(xG(y)) = -BG(x)G(y) \qquad (x, y \in H). \tag{48}$$

With the same argument as in 3.1., we can prove that G has the form:

$$G(x) = \begin{cases} c_1 x & \text{if } x \in [0, a| \\ c_2 x & \text{if } x \in |b, 0] \end{cases} \qquad \text{with} \quad c_1 \leq 0 \quad \text{and} \quad c_2 \geq 0. \tag{49}$$

Now, if $x \in (0, a|$ and $G(y) \neq 0$, $x\, G(y)$ belongs to $|b, 0)$ and we have by (48) and (49): $G(xG(y)) = c_2 xG(y) = -BG(x)G(y) = -Bc_1 xG(y)$ which implies $c_2 = -Bc_1$. Similarly with $x \in |b, 0)$ we prove $c_1 = -Bc_2$. This implies $B = 1$ and $c_1 = -c_2$. From (2) we have $A = 1$. Therefore, G has the form (37) and F has the form (39).The conditions on c are imposed by the fact that G takes its values in H.

3.3. Let us finally investigate the case $I = |\beta, \alpha| \subseteq |b, a|$, $\beta < 0 < \alpha$.

By Lemma 4 F and G vanish only at 0. By (9) φ is a nonconstant continuous solution of the restricted Cauchy's power functional equations:

$$\varphi(uv) = \varphi(u)\varphi(v) \quad (u, v \in (0, \alpha|) \qquad (\text{resp. } (u, v \in |\beta, 0))).$$

As in 3.1. and in 3.2. there exist A, γ_1, γ_2 in $\mathbb{K}$, with $A \neq 0$, $\operatorname{Re}\gamma_1 > 0$, $\operatorname{Re}\gamma_2 > 0$ such that $\varphi(u) = u^{\gamma_1}$ $(u \in (0, \alpha|)$ and $\varphi(u) = A|u|^{\gamma_2}$ $(u \in |\beta, 0))$.

Now, if $u \in |\beta, 0)$, there exists $0 < \alpha' < \alpha$ such that $uv \in |\beta, 0)$ for all $v \in (0, \alpha')$ and we have: $\varphi(uv) = A|uv|^{\gamma_2} = A|u|^{\gamma_2} v^{\gamma_1}$. This implies: $\gamma_2 = \gamma_1 = \gamma$.

Furthermore, if $u \in |\beta, 0)$, there exists $v \in |\beta, 0)$ such that $uv \in (0, \alpha|$ and we have: $\varphi(uv) = uv^{\gamma} = A^2|u|^{\gamma}|v|^{\gamma}$, which implies $A = \pm 1$. So, we have obtained:

$$\text{either} \quad \varphi(u) = \begin{cases} |u|^{\gamma} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases} \quad \text{or} \quad \varphi(u) = \begin{cases} |u|^{\gamma} \operatorname{sign} u & \text{if } u \neq 0 \\ 0 & \text{if } u = 0. \end{cases}$$

We deduce that F has either the form (39) or the form (40). (2) implies now: $|G(xG(y))| = |G(x)|\,|G(y)|$ $(x, y \in H)$, which implies:
$|G(xt)| = |G(x)|\,|t|$ $(x \in H, t \in I)$. With the same argument as in 3.1., we can prove that $|G|$ has the form:

$$|G(x)| = \begin{cases} c_1|x| & \text{if } x \in aI \\ c_2|x| & \text{if } x \in bI \end{cases} \qquad \text{where } c_1 \text{ and } c_2 \text{ are positive.}$$

Since $aI \cap bI$ is a nontrivial interval, we have $c_1 = c_2 = c$. Now, for all x and y in H with $y \neq 0$, $xG(y)$ belongs to $aI \cup bI$ and we have: $|G(x)| = \frac{|G(xG(y))|}{|G(y)|} = c|x|$. Since G is neither always nonpositive nor always nonnegative, we deduce that G is of the form (36). The conditions on c are obtained from the fact that G takes its values in H. □

Corollary (cf. [3]). *If $H = (0, +\infty)$ or $H = |a, +\infty)$ with $a \geq 1$ or $H = |b, a|$ with $-1 \leq b \leq 0 < a \leq 1$, $b^2 \leq a$, all continuous solutions $f : H \to H$ of the functional equation: $f(xf(y)) = f(x)f(y)$ $(x, y \in H)$ are given by $f = \phi_{|H}$ where $\phi : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous solution of the functional equation* (4) *satisfying $\phi(H) \subseteq H$. ϕ is unique in the case $H = |b, a|$, $b < 0$.*

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